Lecture Notes in Mathematics

Wolfgang Siegert

Local Lyapunov Exponents

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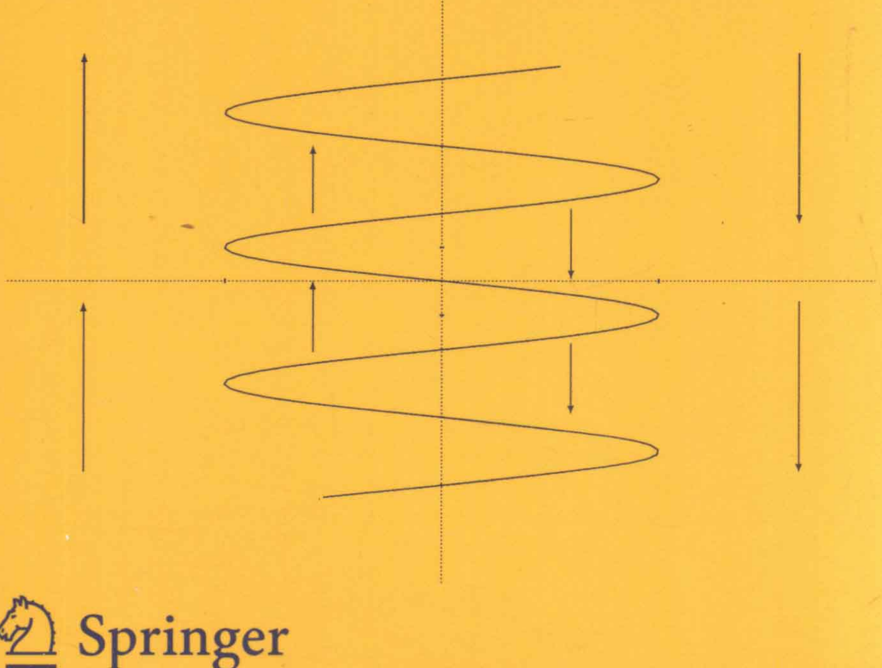
Sublimiting Growth Rates of Linear Random Differential Equations



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Sublimiting Growth Rates of Linear Random Differential Equations

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Preface

Establishing a new concept of local Lyapunov exponents, two separate theories are brought together, namely Lyapunov exponents and the theory of large deviations.

Specifically, for the stochastic differential system

$$\begin{aligned} dZ_t^\varepsilon &= \mathbf{A}\left(X_t^\varepsilon\right) Z_t^\varepsilon \, dt \\ dX_t^\varepsilon &= b\left(X_t^\varepsilon\right) dt + \sqrt{\varepsilon}\, \sigma\left(X_t^\varepsilon\right) \, dW_t \end{aligned}$$

the new concept is introduced. Due to stationarity, the Lyapunov exponents of Z_t^ε (which by Oseledets' Multiplicative Ergodic Theorem describe the exponential growth rates of Z_t^ε) do not depend on the initial position x of X^ε. Now the goal of this work is to provide a Lyapunov-type number for each regime of the drift b. As this characteristic number shall depend on the domain in which X^ε, a dynamical system perturbed by additive white noise, is starting, it yields a concept of locality for the Lyapunov exponents of Z_t^ε. Furthermore, the locality of such local Lyapunov exponents is to be understood as reflecting the quasi-deterministic behavior of X^ε which asserts that in the limit of small noise, $\varepsilon \to 0$, the process X^ε has *metastable states* depending on its initial value as well as on the time scale chosen (Freidlin-Wentzell theory).

Up to now local Lyapunov exponents have been defined as finite time versions of Lyapunov exponents by several authors, but here we target at investigating the large time asymptotics $t \to \infty$. So the goal is to connect the large parameters t and ε^{-1} in the customary definition of the Lyapunov exponents in order to approach the *sublimiting distributions* (Freidlin) which are supported by the metastable states of X^ε. The *local Lyapunov exponent* is then understood to be the exponential growth rate of Z^ε on the time scale chosen, subject to convergence in probability as $\varepsilon \to 0$. Notably, the system itself changes in the sense that the noise intensity converges to zero with the time horizon depending on the noise intensity parameter. In contrast to this new concept the Lyapunov exponents as obtained by the Multiplicative Ergodic Theorem reflect the information of *limit distributions*, i.e. of invariant measures, as time increases to infinity for the system parameter $\varepsilon > 0$ being fixed.

As a result we prove that the local Lyapunov exponent is bounded from above by the largest real part of the spectrum of the matrix **A** evaluated at the metastable state corresponding to the time scale; the respective bound from below holds true with the smallest real part of an eigenvalue of **A** at the corresponding metastable state.

Assuming that **A** takes its values in the diagonal matrices, it is shown that its eigenvalues at the respective metastable state are precisely the possible local Lyapunov exponents. Moreover, in a "strongly" hypoelliptic situation it can be proved that only the largest eigenvalue is observed under convergence in probability. The latter result is regarded as sublimiting Furstenberg-Khasminskii formula, since the resulting limit is obtained as a (trivial) integral which produces the top eigenvalue.

For the above tasks the prerequisites which fundamentally consist of knowing the exit probabilities of all the stochastic systems involved will be provided in detail: For this purpose, an integrated account of the theory for non-degenerate stochastic differential systems (Freidlin and Wentzell) and of the exit probabilities for degenerate stochastic systems (Hernández-Lerma) is given in chapters 2 and 3. The subsequent final chapter is the heart of the book. Here, all the results are proven and discussed.

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Berlin/Stuttgart *Wolfgang Siegert*
July 2008

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Introduction

During the last decades intense research has been devoted to dynamical systems subject to random perturbations: Considerable effort has been dedicated to investigate exit times and exit locations from given domains and how they relate to the respective deterministic dynamical system. Building upon considerations in physics and chemistry (see e.g. the classical paper by Kramers [Kr 40]) the theory of large deviations by Freidlin and Wentzell ([Fr-We 98], [Fr 00]) provides the correct mathematical framework for tackling these problems in case of Gaussian perturbations. This theory sets up the precise time scales for transitions of non-degenerate stochastic systems between certain regimes. The behavior of such systems is called *metastable*.

The theory of random dynamical systems, on the other hand, considers stochastic processes which satisfy a certain flow property, the cocycle-property. The main cornerstone here is the Multiplicative Ergodic Theorem by Oseledets [Os 68]; also see Arnold [Ar 98]: This theorem assigns Lyapunov exponents to linear random dynamical systems. These are the exponential growth rates as time grows large for fixed intensities of the underlying noise.

The following work now attempts to close the gap between these two stochastic disciplines: It does not study the exponential growth rate for a fixed noise intensity and large time (resulting in the Lyapunov exponents), but considers the exponential growth, if the time horizon depends on the noise intensity. Thus one considers the Lyapunov characteristics on time scales. Since these time scales correspond to metastable points, the Lyapunov exponents are *localized* by connecting the large time asymptotics to the limit of vanishing noise intensity.

What one usually does when dealing with "local Lyapunov exponents" is to replace the infinite time limit (characterizing the Lyapunov exponents) by a large, but finite time horizon; see Abarbanel et al. [Ab-Brw-Ke 91] and [Ab-Brw-Ke 92], Wolff [Wo 92], Pikovsky [Pk 93], Pikovsky and Feudel [Pk-Fe 95] and Bailey et al. [Ba-El-Ny 97]. A similar discussion in the same spirit is undertaken by Monahan [Mo 02]: In the case of the Maas model he describes the concept of a "local Lyapunov exponent" for which the infinite

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time limit is replaced by a large, but finite time horizon; this time then needs to be large enough for the system to sample the local attractor, but smaller than the average escape time of the regime. This rationale is also applied when calculating the respective exponents numerically. Now generally speaking, the problem in the case of an elliptic stochastic differential system is that switches to other regimes occur with strictly positive probability—no matter how small the time horizon is chosen. On the other hand, e.g. as computers can necessarily work with finite calculation horizons only, our concept of localizing Lyapunov exponents by means of time scales justifies the above finite-time procedure, if time scales are chosen appropriately. Let us further describe this rationale:

The base systems under consideration are dynamical systems with additive white noise perturbations; for simplicity let the process X^ε be defined by the stochastic differential equation (SDE) of gradient type

$$dX_t^\varepsilon \;=\; -\,\nabla U\left(X_t^\varepsilon\right)\,dt \;+\; \sqrt{\varepsilon}\;dW_t$$

for the moment. It describes the motion of a particle in a potential landscape which is derived from the real-valued, differentiable function U defined on $\mathbb{R}^d$. The *linearization* of X^ε is then given as the solution of the linear, real noise driven differential equation

$$dZ_t^\varepsilon \;=\; -\,H_U\left(X_t^\varepsilon\right)\,Z_t^\varepsilon\;\;dt\;,$$

the so called *variational equation*, in which $H_U(x)$ denotes the Hesse matrix of second derivatives of U at x. The variational equation governs the evolution of "infinitesimal disturbances" of X_0^ε. The *Lyapunov exponents* of the system are now defined as the exponential growth rates

$$\lim_{t\to\infty}\,\frac{1}{t}\,\log\left|\,Z_t^\varepsilon(\,.\,,x,z)\,\right|\,.$$

Their existence is assured by Oseledets' [Os 68] Multiplicative Ergodic Theorem and their number does not exceed the dimension d of the state space of X^ε. Due to the stationarity of the flow X^ε in Oseledets' theorem, the Lyapunov exponents do not depend on the initial condition $x = X_0^{\varepsilon,x}$.

Suppose that the potential function U has the qualitative shape as depicted in figure 1.

Moreover, let

$$\Lambda_1^0(x)\;\geq\;\ldots\;\geq\;\Lambda_d^0(x)$$

denote the (decreasingly indexed) eigenvalues of $-\,H_U(x)$. The goal of this contribution is to provide a Lyapunov-type number for each regime of the potential function U. As this characteristic number shall depend on the initial point (more precisely, on the well in which the stochastic solution X^ε is starting), it shall yield a concept of locality for Lyapunov exponents. The

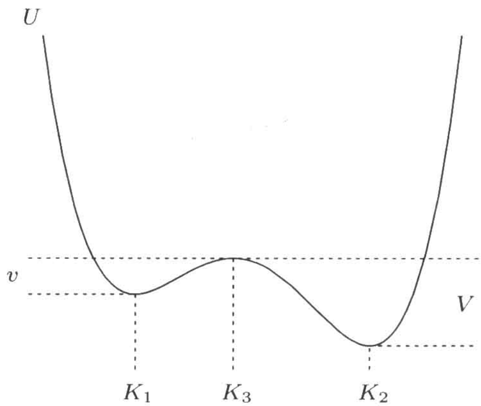

Fig. 1 A prototypical potential function $U : \mathbb{R}^d \to \mathbb{R}$ with two wells

main motivation for the concept thus defined is that in the small-noise-limit the particle X^ε stays in the initial shallow well near K_1 for an exponentially long time (Kramers' law, Freidlin-Wentzell theory) during which we can "see" the shallow well; afterwards the particle finally overcomes the barrier at K_3 and the deeper well around K_2 dominates the picture; mathematically this is made precise by observing that K_1 and K_2 are the supports of the *sublimiting distribution* on the corresponding time scales (Freidlin [Fr 77] and [Fr 00]). Furthermore, K_1 and K_2 are the *metastable points* of the potential. In order to capture this metastable behavior we connect the parameters t and ε by

$$\varepsilon(t) := \frac{\zeta}{\log t}$$

for a scaling parameter $\zeta > 0$ for approaching the sublimit distributions. Hence, we consider

$$\frac{1}{t} \log \left| Z_t^{\varepsilon(t)}(\,.\,, x, z) \right|$$

and we then conjecture that this random variable converges in probability to

$$\Lambda_j^0(K_k)$$

as $t \to \infty$, where $k \in \{1, 2\}$ depends on the initial position x and on the time scale parameter ζ of X_t^ε; more precisely, if x is in the K_1-regime and if also $\zeta < 2v$, then $k = 1$; otherwise, $k = 2$. This conjecture is proven in section 4.3 under the additional assumption that $H_\iota(\,.\,)$ only takes its values in the diagonal matrices; the index $j \in \{1, \ldots, d\}$ then, of course, depends on the initial direction z of Z. We call these limit numbers the *local Lyapunov exponents* of Z. In the general case, i.e. abstaining from the diagonality condition, section 4.4 gives conditions under which the above exponential growth rate converges in probability to the top eigenvalue of $-H_U(K_k)$,

$$\Lambda_1^0(K_k)\,,$$

where K_k again denotes the metastable state for the time scale chosen; this limit then is the *local Lyapunov exponent* of Z. In other words, defining

$$T(\varepsilon) \;:=\; e^{\zeta/\varepsilon}\,,$$

the previous discussion concerns the convergence in probability of

$$\frac{1}{T(\varepsilon)}\log\left|\,Z^{\varepsilon}_{T(\varepsilon)}(\,.\,,x,z)\,\right|$$

as $\varepsilon \to 0$. In comparison with the previously mentioned concept by Abarbanel et al. [Ab-Brw-Ke 91] and [Ab-Brw-Ke 92], Wolff [Wo 92], Pikovsky [Pk 93], Pikovsky and Feudel [Pk-Fe 95] and Bailey et al. [Ba-El-Ny 97] who would take the finite time growth rate

$$\frac{1}{T}\log\left|\,Z^{\varepsilon}_{T}(\,.\,,x,z)\,\right|$$

for fixed T and ε as "local Lyapunov exponent", one therefore obtains a rigorous explanation for how to correctly choose the time horizon depending on the underlying noise intensity ε.

Furthermore, we would like to comment on a second type of "local Lyapunov exponents" which can be found in the literature: Let $d = 1$, then the drift of Z^{ε} in the variational equation is

$$-\,H_U(x) \;\equiv\; -U''(x)\,,$$

the negative curvature of U at x. Several authors then call this number $-U''(x)$ the "local" or "local in phase space" or "instantaneous" Lyapunov exponent; see Fujisaka [Fu 83], van den Broeck and Nicolis [vB-Ni 93], Witt et al. [Wt-Ne-Kt 97] and Pikovsky and Feudel [Pk-Fe 95]. However, this is not in accordance with our understanding, since the corresponding stochastic system X^{ε} does not stay near an arbitrary initial point x, but is confined to one of the metastable points K_1, K_2 by the drift $-U'$. The system Z^{ε} mainly samples $-U'(K_k)$ for $k \in \{1,2\}$, but neglects contributions of some other $-U'(x)$. Therefore our result for the one-dimensional situation (see remark 4.1.4) only has the values

$$-U''(K_1) \qquad \text{and} \qquad -\,U''(K_2)$$

as local Lyapunov exponents in the sense of our definition in contrast to Fujisaka [Fu 83], van den Broeck and Nicolis [vB-Ni 93], Witt et al. [Wt-Ne-Kt 97] and Pikovsky and Feudel [Pk-Fe 95]. In other words, this expresses the fact that the instantaneous rates $-U''(x)$ do not have equal rights, but the Dirac measures δ_{K_1} and δ_{K_2} are the sublimiting distributions reflecting the preferences of X^ε on the time scales.

A third, deterministic concept of locality of Lyapunov exponents different from ours has been introduced by Eden, Foias and Temam [Ed-Fo-Tm 91].

It has already been indicated above that Kramers' law, made precise by Freidlin and Wentzell, plays a dominant role in the following considerations. It is interesting that already the classical Eyring-Kramers formula for the exit times of X^ε contains the eigenvalues of the Hesse matrix of U at K_1, K_2 and K_3. Namely let τ_{12}^ε (and τ_{21}^ε) denote the time at which X^ε enters the K_2-well when started in K_1 (and vice versa). Then in the limit as $\varepsilon \to 0$, under non-degeneracy assumptions, the following asymptotic expressions are known to hold for the mean exit times,

$$\mathbb{E}\,\tau_{12}^\varepsilon \approx \frac{2\pi}{\Lambda_1^0(K_3)} \sqrt{\left|\frac{\prod_{i=1}^d \Lambda_i^0(K_3)}{\prod_{i=1}^d \Lambda_i^0(K_1)}\right|}\; e^{2v/\varepsilon}$$

and

$$\mathbb{E}\,\tau_{21}^\varepsilon \approx \frac{2\pi}{\Lambda_1^0(K_3)} \sqrt{\left|\frac{\prod_{i=1}^d \Lambda_i^0(K_3)}{\prod_{i=1}^d \Lambda_i^0(K_2)}\right|}\; e^{2V/\varepsilon}$$

which are cited from Bovier at al. [Bv-Ec-Gd-Kn 04] in the above notation.

Abstracting the previous considerations one detects that the underlying diffusion does not have to stem from an SDE of gradient type; the Freidlin-Wentzell theory of large deviations and metastability also admits more general drift functions b and also state dependent noise coefficients σ under certain assumptions. Overall, the two characteristic features of the above stochastic differential system $(X^\varepsilon, Z^\varepsilon)$ are the following: Firstly, it is *degenerate* in the sense that in the equation for Z^ε there is no noise component but only a drift coefficient depending on X^ε; in other words, the differential equation of Z^ε is random, driven by the real noise process X^ε. Secondly, the differential system is linear with respect to Z^ε. Further abstracting the system matrix $\mathbf{A}$ is by no means restricted to be the negative of the Hesse matrix of some potential function; it can be any matrix valued function defined on the state space of X^ε. Moreover, we will also admit that the state spaces of X^ε and Z^ε can have different dimensions. Hence, the general object of the subsequent considerations is the real-noise driven, linear stochastic differential system

$$\begin{aligned} dZ_t^\varepsilon &= \mathbf{A}\,(X_t^\varepsilon)\; Z_t^\varepsilon\; dt \\ dX_t^\varepsilon &= b\,(X_t^\varepsilon)\; dt \;+\; \sqrt{\varepsilon}\;\sigma\,(X_t^\varepsilon)\; dW_t\;, \end{aligned} \tag{1}$$

where $\varepsilon \geq 0$, W is a Wiener process on $\mathbb{R}^d$, $\mathbf{A} \in C(\mathbb{R}^d, \mathbb{R}^{n\times n})$ (or $\mathbf{A} \in C(\mathbb{R}^d, \mathbb{C}^{n\times n})$), the drift $b : \mathbb{R}^d \to \mathbb{R}^d$ does not necessarily stem from a potential, but exhibits several regimes and the values of $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are close to $\mathrm{id}_{\mathbb{R}^d}$. All in all, X^ε is a diffusion in $\mathbb{R}^d$ with small noise intensity and Z^ε is a linear cocycle in $\mathbb{R}^n$. Generalizing the previously sketched idea the *local Lyapunov exponents* are the possible limits of

$$\frac{1}{t}\log\left| Z_t^{\varepsilon(t)}(\,.\,,x,z)\right|$$

as $t \to \infty$, where

$$\varepsilon(t) \equiv \frac{\zeta}{\log t}$$

for a scaling parameter $\zeta > 0$, depending on the initial condition x of X^ε and the initial direction z of Z^ε.

The goal of proving convergence in probability for this exponential growth rate on a time scale is organized as follows:

In the first chapter we collect known results on linear, real noise driven differential systems such as the Multiplicative Ergodic Theorem, their decomposition in spherical coordinates and the deterministic Hartman-Wintner-Perron theorem. The latter two subjects yield coordinate systems which will decisively come into play in chapter 4.

Beforehand, chapter 2 gives an account of the Freidlin-Wentzell theory as needed for describing the locality, metastability and sublimiting distributions. Since this theory is based on the fundamental exit time law for non-degenerate stochastic systems, this result is recalled in detail.

The system Z^ε of (1) is a degenerate stochastic differential system by definition, since there is no stochastic differential in the Z-component. This is in particular also true for the angle of Z^ε. As the behavior of the latter process needs to be investigated in chapter 4 chapter 3 recalls known results on its exit probabilities; more precisely, we give an account of the theorems by Hernández-Lerma concerning exit probabilities of degenerate systems which are not covered by Freidlin-Wentzell theory.

Chapter 4 finally investigates the exponential growth rates on the time scales: Firstly, it is proven that the top real part of an eigenvalue at the metastable state is an upper bound for the local Lyapunov exponent; likewise, the smallest real part of an eigenvalue at the metastable state is a lower bound for the local Lyapunov exponent; see section 4.1. Since $\det Z^\varepsilon$ is a process in $\mathbb{R}$ and the situation is quite tractable for a one-dimensional state space, a consequence for the exponential growth rate of $\det Z^\varepsilon$ can be drawn which is subject to section 4.2. The coordinates from the Hartman-Wintner-Perron theorem allow to explicitly calculate the exponential growth rate under the additional assumption that $\mathbf{A}$ solely takes diagonal values; see section 4.3. Finally, section 4.4 gives criteria under which one can obtain convergence in the general, two-dimensional case: Here the result by Hernández-Lerma comes into play as the decomposition of $|Z_t^\varepsilon|$ by means of spherical coordinates demands an assertion on the angle process of Z_t^ε. More precisely, a statement concerning the Lebesgue measure of the times which the angle process spends at the switching curves of its drift can be deduced; this result which is analogous to Freidlin's [Fr 00] metastability theorem then allows to calculate a sublimiting version of the Furstenberg-Khasminskii formula.

A final remark is necessary concerning the use of the notion of "time scales": There is now a very elaborate theory of "dynamic equations on time scales"; this theory has been invented by Hilger [Hi 88]; also see Bohner and Peterson [Boh-Pet 01]. This concept understands "time scales" (also called "measure chains") as certain time sets which are underlying to the systems under consideration. Hence, the investigation of an ordinary differential equation (ODE) means to work with the "time scale" $\mathbb{T} = \mathbb{R}_+$ or $\mathbb{R}$; an ordinary difference equation is understood as dynamic equation on the "time scale" $\mathbb{T} = \mathbb{N}_0$ or $\mathbb{Z}$. However, in this paper here the physical time set is always $\mathbb{R}_+$ and a *time scale* in our context is understood as a time horizon $T \equiv T(\varepsilon)$ which depends on the parameter $\varepsilon > 0$ of the base SDE of X^ε.

The symbol $\square$ will be used to mark the end of a proof. In order to avoid latent ambiguities, it will also be employed sporadically to finish remarks and examples where necessary.