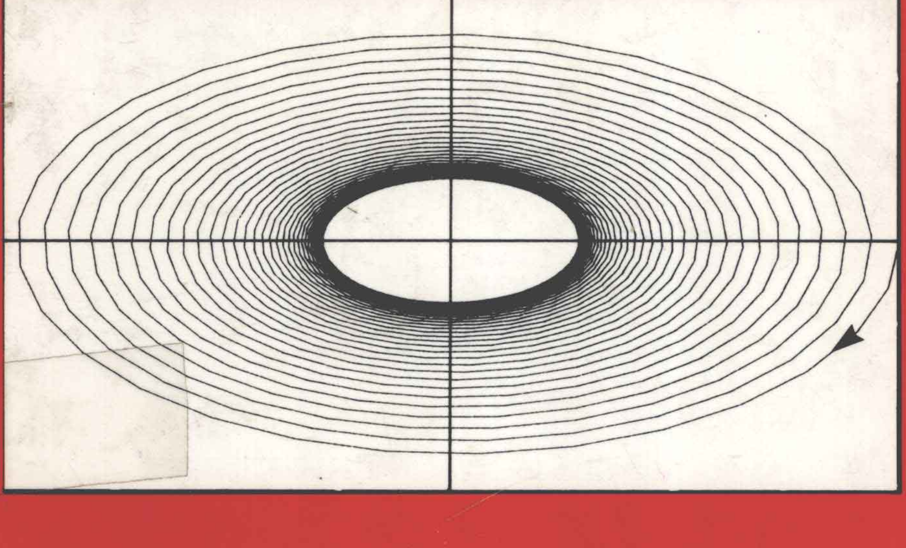
MATTHEW CARTMELL

Introduction to Linear, Parametric and Nonlinear Vibrations



Chapman and Hall

Introduction to Linear, Parametric and Nonlinear Vibrations

MATTHEW CARTMELL

B.Sc., Ph.D., C.Eng., M.I.Mech.E.
Lecturer
University of Aberdeen

CHAPMAN AND HALL

LONDON • NEW YORK • TOKYO • MELBOURNE • MADRAS

UK	Chapman and Hall, 11 New Fetter Lane, London EC4P 4EE
USA	Chapman and Hall, 29 West 35th Street, New York NY10001
JAPAN	Chapman and Hall Japan, Thomson Publishing Japan, Hirakawacho Nemoto Building, 7F, 1-7-11 Hirakawa-cho, Chiyoda-ku, Tokyo 102
AUSTRALIA	Chapman and Hall Australia, Thomas Nelson Australia, 480 La Trobe Street, PO Box 4725, Melbourne 3000
INDIA	Chapman and Hall India, R. Sheshadri, 32 Second Main Road, CIT East, Madras 600 035

First edition 1990

Typeset in 10/12 Times by
KEYTEC, Bridport, Dorset
Printed in Great Britain by
T. J. Press (Padstow) Ltd, Padstow, Cornwall

ISBN 0 412 30730 8

British Library Cataloguing in Publication Data
Cartmell, Matthew
Introduction to linear, parametric and nonlinear vibrations.
1. Vibration
I. Title
531.32

ISBN 0-412-30730-8

Library of Congress Cataloging-in-Publication Data
Cartmell, Matthew, 1958–
Introduction to linear, parametric, and nonlinear vibrations/Matthew Cartmell.
p. cm.
Includes bibliographical references.
ISBN 0-412-30730-8
1. Vibration. 2. Vibration, Parametric. I. Title.
TA355.C34 1990
620.3–dc20 89-71225
CIP

Acknowledgements

I would like to thank Ms Anne Shipley and Mr Stuart Cutts of the University of Aberdeen for their respective efforts in the typing of the manuscript and the production of the illustrations. I have had much useful input from my academic colleagues within the Department of Engineering, and would therefore like to thank Professor Allan D. S. Barr, Dr David Watt, Dr Richard D. Neilson, and also Dr J. W. Roberts of the Department of Mechanical Engineering in the University of Edinburgh for his help and comments. I am also greatly indebted to Dr Watt for his informed and perceptive comments on the manuscript as a whole. I have incorporated acknowledgements within the text where I have leant heavily on the work of others. I would also like to acknowledge the very considerable debt of gratitude that I owe to Raymond G. Fielding for inspiring an enthusiasm for all things mechanical and electrical. Finally I wish to thank my wife Fiona for her interest and support for the project, and for her patience with the many crises of confidence that beset me.

Matthew P. Cartmell

Preface

This book is intended to provide an introduction to some of the interesting, and perhaps surprising, phenomena often encountered in systems which vibrate and which do so under the influence of parametric and/or nonlinear effects. Therefore the bias of the book is predominantly towards the phenomenological, but mainly in the context of mechanical engineering. I have, in places, attempted to bring in applications or circumstances outside mechanical engineering within which particular effects have been observed. However, in the main, the book is directed towards mechanical engineers. There is no shortage of excellent texts in the general area of parametric and nonlinear vibrations but in almost all cases there is an assumption either of prior knowledge and experience or of a very considerable facility with the appropriate forms of analysis. Clearly having the former will only serve to clarify and add to the picture obtained from studying advanced texts, but since many aspiring students in the field will probably not be experienced researchers a more gentle introduction, such as may be found within this book, should be of some help. In addition to this, the second point raised above on analytical skill merits some attention. I hope that by virtue of the reasonably detailed examples given in the book some of the subtleties of certain parametric instability problems, and of aspects of nonlinear vibrations, are brought out without getting too lost in the mathematics. In this context it might be appropriate to draw the reader's attention to certain specific sections within the text and to offer a brief explanation for the length of the discussion at those points. Section 1.7.4 deals with Virtual Work and Lagrange's Equations in the standard manner with minimal abbreviation. This is to allow the interested reader to work through this important analysis and perhaps to gain a clearer understanding of what it means in the process. On the other hand, the result may be noted without going through the derivation; either option is available. A similar approach has been taken in Section 2.1 in which a reasonably concise, yet formal, appraisal of the stability of Mathieu–Hill type equations is presented. There is a brief review of electronic applications of parametric systems in Section 2.4.4(b) and the aim here is to draw the reader's attention to these and to attempt to show that parametric amplifiers are deliberately stabilized by means of specifically introduced nonlinear circuit elements. The discussion is relatively superficial given that this is not mainstream material; references for further reading are given. Sections 3.1.3(a) and (b) treat the kinematics and the derivation of the equations of motion respectively for a parametrically excited cantilever beam where combined bending and torsional motion is possible. Again a reasonably full treatment of the problem is given so that the source of the nonlinearities that are highlighted is made clear. Dealing with a 'simple' problem in this way hopefully goes some way towards showing the potential complexity of nonlinear vibrations without being completely inpenetrable. In the final chapter a section on chaos is included (Section 5.3). I have not attempted to deal with the subject in any great depth but have presented some principals and definitions that I consider important. Several excellent books on chaos have recently been published and the reader is strongly advised to investigate these for further details.

The general intention behind writing this book has been to provide a starting-off point for those who wish to extend their appreciation of mechanical vibrations into the parametric and nonlinear domains. I have provided a list of references which is fairly broad in coverage in order to enable the student to get started, hopefully without being put off by the immensity of the available literature. I therefore strongly recommend that the references to other texts are followed up so that further insight is obtained.

Matthew P. Cartmell
Department of Engineering
King's College
University of Aberdeen

Contents

1

Linear vibrations in mechanical engineering

INTRODUCTION

It is intended that this first chapter should serve as a general and broad-based revision of the concepts of mechanical and structural vibrations, specifically those which are conventionally classified as being 'linear'. Although it is assumed that the reader has some familiarity with vibrations theory and appropriate areas of applied dynamics, a substantial awareness of the subject is not a prerequisite for gainfully reading this, or further chapters. It is suggested, however, that appreciation of the central theme of the book, namely parametric and non-linear vibrations, will be enhanced by referring to this section.

1.1 CLASSIFICATION OF VIBRATION PROBLEMS

There are several ways in which we can attempt to classify a system, or a problem in vibrations; for instance it may be conservative, where the total energy of the system remains constant during motion, or non-conservative in which case energy is expended in overcoming some form of dissipation, and where there may also be a kind of forcing or excitation acting on the system. An alternative approach would be to relegate these criteria to a different level of classification and to adopt the terms linear and non-linear as more fundamental categories of problem definition.

Linear vibrating systems will contain mass (or inertia), stiffness, and damping (to some degree) and so, as long as these quantities are themselves linear in their behaviour and are not varying with time, a mathematical model employing a linear ordinary differential equation with constant coefficients should adequately portray the motion of the system. Since damping, or dissipation, is never totally absent in practice we generally accept that a non-conservative linear model will often prove to be a realistic starting point in our explorations. Further assumptions enter into our conceptualization of linear vibrating systems in the form of inelastic mass and inertia elements, massless springs and dampers which possess neither elasticity nor mass but introduce dissipatory forces proportional to the relative velocity across the element. An important and unique identifier of linear action is the principle of superposition which simply states that resultant oscillations may be composed of two, or more, separately excited motions which combine linearly to generate one response motion. The principle can be used to determine the complete solution to a forced vibration problem by combining the particular solution due to the forcing and the complementary solution arising from the initial conditions.

It will probably be apparent, even at this stage, that a truly linear system is unlikely to be found except on paper and that practical questions must surely threaten our convention of linearity, and in particular the notion of easily obtained solutions to manageable and well-behaved equations of motion. Therefore we must be confident that the type of linear model proposed is adequate in the sense that the distributive nature of mass, elasticity and damping, as pertaining to the specific problem, is properly understood. To do this we can additionally categorize the problem as being discrete or continuous in form.

A discretized system may contain one element of mass, connected to attendant massless springs and damping devices, or several lumped masses, or inertias, joined together by massless springs and dampers. A model of this type will rarely be wholly representative of the system under investigation, since it is often difficult to break down real machine and structural problems in the certainty that one is creating an accurate mathematical portrayal. On the other hand, a continuous formulation of the problem, where the properties of mass and elasticity are each considered to be inseparable along the axis of action, will avoid such hazardous simplifications but this will be at the expense of subsequent analytical ease and convenience.

In the case of discrete models one needs to consider the number of degrees of freedom of the problem. This is the number of independent coordinates that are needed to fully describe the possible motion of the system. It does not always follow that each mass or inertia has merely one degree of freedom since its motion may consist of a variety of different translations or rotations, or both, however it does follow that any discrete system will have a finite number of independent coordinates and therefore a finite number of degrees of freedom. A continuous model must by definition have an infinite number of degrees of freedom as its mass is continuously distributed within the structure.

In addition to the above a system may appear to be linear only under certain conditions, and a commonly occurring example of this is the case of small oscillations about the equilibrium point. The requirement that a spring obeys Hooke's law is only adequately satisfied for relatively small deflections, after which its linearity may be in question. Similarly, classical linear viscous damping, as so often called upon in engineering problems, may only hold over a restricted range of velocities, and even then it may not be a realistic indication of that which actually exists within the system.

In conclusion, therefore, it can be said that our interpretation of vibration problems in engineering is of prime importance, and that the answers obtained from analysis will generally involve some form of approximation, so it is crucial for the engineer to develop a logical as well as an instinctive facility with the domains of linear modelling, modelling of nonlinear problems where linearization may under certain conditions be acceptable, and entirely nonlinear problems, where alternative techniques must be applied.

1.1.1 Classification in terms of constraints

Because all problems in dynamics involve formulating relationships between forces as well as amongst displacements it is obvious that forms of constraint must exist, which, in a practical sense, serve to restrict or contain the operation of the system by means of boundary effects or internal interactions of some description. This implies that if we can represent such constraints mathematically then we have a means of specifying relationships between coordinates as well as between forces and torques. This in turn points to the possibility of a distinction between those coordinates that can be related in this way and those that cannot. Coordinates that do not feature in equations of constraint must therefore be independent and will equate to the numbers of degrees of freedom active in the system. These can be translational or rotational, relative or absolute. We can now see that each constraint equation that exists for a system will reduce the number of truly independent coordinates by one (and hence the number of degrees of freedom).

Many cases of structural and machine vibrations allow for the identification of a set of independent or generalized coordinates by considering the constraints acting, and thus the constrained coordinates which result. We use the term holonomic to apply to these. Conversely this process of elimination of constrained coordinates to render a set of generalized coordinates is not always possible, for example we can consider instances in which the active constraint equations contain, say, velocity dependent forces, such as those that give rise to relative sliding velocities in structural joints as a case in point. These constraint equations are unlikely to be integrable, and as a result the system is said to possess non-holonomic constraints.

Within the domain of holonomic constraints two supplementary constraint types are available and these refer directly to the presence of time dependency, so for the case of a time variant holonomic constraint system we use the term rheonomic holonomic. The time-invariant case is called scleronomic holonomic.

By considering and briefly discussing the ideas of conservation, linearity, discrete and continuous distributions of mass and elasticity, and constraints, we have mapped out ways in which we can begin to identify our problem and then we can proceed to devise a useful solution. Application of the tools of classification may not always be a wholly conscious step in the process, however implementation of these ideas to some extent (and perhaps not always explicitly) will always be necessary to derive descriptive equations of motion.

1.2 FUNDAMENTALS OF VIBRATING MECHANICAL SYSTEMS

All mechanical engineering structures which undergo oscillatory motion will contain, and be expressible in terms of, mass, stiffness, and damping. Useful related parameters are inertia (synonymous with mass but found in rotating systems), flexibility which can be shown to be the inverse of stiffness, and dissipatory forces which embrace 'damping' but also describe other non-conservative force effects such as dry friction.

The way in which we choose to identify the significant effects of these mechanical parameters is of great importance since their effect can be straightforwardly direct, or alternatively seemingly indirect in the case of effectives where we attempt to model a system in the context of remote parameters. Therefore such remote effects may have a principal point of action removed from the point of interest, however their influence is still highly significant there.

1.2.1 Mass and inertia

Mass is an elementary property capable of acquiring or expending kinetic energy within the system due to imposed velocity gradients. Weight is a force quantity and is the effect of gravitational acceleration acting on mass. In the same way a mass will accelerate in the direction of (and under the influence of) a force, or resultant force in a force system. We derive the work quantity from the product of force and displacement in the direction of the force. Work on a mass is commensurate with the overall change in kinetic energy. Accordingly the concept of mass is central to Newton's laws of motion upon which the above statements are founded. We conventionally regard mass as inelastic, and add on our notions of elasticity and dissipation later as though they are physically separable from one another within the system. This simplistic view of things fades to some extent the nearer one gets to a continuous model, however it is a perfectly valid approach in lumped parameter representations.

Interpretation of effective mass in an engineering system depends on how we can break it down into discrete masses and stiffnesses. If several identifiable mass elements are clearly connected by spring-like elements, whose individual masses are negligible, then the problem is likely to be multi-degree of freedom in nature. Alternatively rigid interconnections between isolated lumped masses compel us to evaluate an effective mass quantity which can then be regarded as acting at any convenient point in the system. It is usual to refer the effective mass to a particular point of interest. Some well-known illustrations of how we use effective mass are, for example, natural frequency evaluation in a single degree of freedom mass and spring system where the spring mass is not negligible, or similar analysis of an oscillating motion conversion linkage as found in an engine valve gear configuration. Many such cases can be found in the literature (Tse, Morse and Hinkle, 1978; Rao, 1984) and involve expressing the system kinetic energy in a form equal to an equivalent kinetic energy at the chosen point of interest. This approach works equally well for translating and rotating systems (or combinations of these).

Discrete systems with many degrees of freedom naturally require a rather more complicated mathematical model than single degree of freedom problems, and the conventional approach here is to express the mass constituents of the system in the form of a matrix. In its simplest form the mass matrix does not depict any form of coordinate coupling, since in such cases all non-leading diagonal terms will be zero. More complicated problems will exhibit non-zero terms in other elements of the matrix; this points towards dynamic or inertia coupling between the coordinates. It is worth mentioning that identical considerations also apply to stiffness, and any coordinate coupling in this sense is called static or elastic coupling. An absence of both dynamic and static coupling indicates a situation where the resulting equations of motion can be solved independently. Coordinates in uncoupled problems are termed principal coordinates. Mass considerations in the domain of continuous systems introduce the idea of generalized mass, a quantity which is calculable for any particular mode of vibration. So, in a continuous system the mass which is operative in the system is a function of the mode shape and as such the resulting 'mass' will be seen to vary substantially between the modes.

1.2.2 Stiffness

All vibrating systems possess some form of elasticity, and in its most elementary form this is regarded as a simple linear proportionality relating displacement to applied load. Springs may assume a variety of physical forms and helical, torsional and flat springs are commonplace. The mass of a spring, or spring-like element, is often neglected in elementary analysis, however it can frequently be found to be contributory to a sizeable correction in natural frequency estimation.

Deformation of a spring arises because of an applied force so that there is a relative displacement between the extremities of the spring. The magnitudes of the force and displacement in a linear spring are related by a simple constant of proportionality sometimes known as the rate or stiffness. The units of stiffness are conventionally N/m. A deformed spring stores up potential or strain energy, this is equal to the work done in deformation. Stiffness is an additive quantity in the sense that several connected stiffnesses will have an overall combined effect. Simple rules apply, and a restatement and elaboration of these is unnecessary here; substantial coverage of simple spring equivalence problems is attributed to many authors, (Den Hartog, 1956; Timoshenko *et al.*, 1974; Tse *et al.*, 1978; Thomson, 1981; Rao, 1984; Meirovitch, 1986). Potential energy considerations also call for an appraisal of mass positioning, whereby an additional mg term can sometimes be added to the strain energy term.

Linearity in springs is only guaranteed for small deflections and after a specific point on the load deflection plane we find that the simple proportionality rule no longer applies. It is possible to linearize large deflection problems, however the errors introduced can be excessively high. Further discussion on the subject of nonlinear springs will be found in the second section of this book.

1.2.3 Damping

Damping is a very complicated and specialized subject in itself but one to which we must devote a certain amount of effort in understanding if we are to competently incorporate it in our analysis. Relatively few texts are available which deal with the subject in the depth it deserves, but the reader is referred to two authoritative books (Lazan, 1968; Nashif *et al.*, 1985), which are exceptions to this rule and which deal with the many different facets of structural and machine damping in a practical and useful manner. There is also much useful information on damping and its mechanisms in the excellent recent text on nonlinear oscillations (Nayfeh and Mook, 1979).

It is well known that lightly damped structures dissipate their energy more slowly than their heavier damped counterparts and therefore continue in motion for longer, with a correspondingly slower decay in the magnitude of motion. Damping as a phenomenon does not appear only in oscillatory problems, but in all instances in which mechanical energy is expended in inciting a system into motion. Nashif *et al.*(1985) mention the indisputable but perhaps not immediately obvious example of the 'efficient' golfball, where the highly elastic inner material of the ball is specially designed to instantaneously absorb a huge amount of energy on impact with the club and then to release it as quickly as possible so that almost all this energy is utilized in propulsion. Clearly this is an example of minimal damping.

Classroom theoretical damping models usually assume the form of the classical linear viscous damper where the damping forces generated are proportional to the velocity gradient across the device. This oil–piston–dashpot model is rather contrived and not indicative of 'real' engineering. The nearest practical instance of the simple dashpot damper is the automotive shock absorber, but in most cases additional non-viscous effects and supplementary stiffnesses are introduced which take us further away from the simple $C\dot{x}$ model (where C is the coefficient of linear viscous damping and $\dot{x}$ the velocity across the damper). The measurement method that relates to viscous damping with its exponential decay characteristics, is the logarithmic decrement, by which we relate the amplitudes of two cycles n cycles apart in the form of

$$\delta = \frac{1}{n} \ln\left(\frac{x_1}{x_2}\right). \tag{1.1}$$

It is a comparatively easy matter to evaluate the damping coefficient C, or alternatively the damping ratio ξ, where $\xi = C/C_c$, given that C_c is the critical damping constant, and we can show that

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} = \frac{\pi C}{m\omega_d}, \tag{1.2}$$

where m and ω_d represent the mass and the damped natural frequency of free vibration. Viscous damping serves to limit resonant motion and is easily incorporated into most mathematical models.

Other definable linear damping varieties are dry friction, hysteretic and acoustic damping. Dry friction, or Coulomb damping, can be found in situations where relative motion between two contacting surfaces introduces a frictional force so that a single degree of freedom problem could be modelled by

$$m\ddot{x} + \mu N \operatorname{sgn}\dot{x} + kx = F(t), \tag{1.3}$$

where we assume $F(t)$ is a harmonically varying force and μ and N are the coefficient of friction and some prescribed static normal force respectively. If $F < \mu N$ then there will clearly be no motion of mass m, however when we reach $F > \mu N$ continuous oscillatory motion will occur (and the frictional force will change sign with $\dot{x}$). This type of damping does not necessarily always provide a limitation on resonant amplitude and the reader is referred to the books of Lazan (1968) and Nashif *et al.* (1985) for further insight.

Hysteretic damping is a form of material damping and this differs from the damping forms discussed previously since these are essentially the result of structural configuration. All elastic materials will give rise to a 'loading–unloading' loop in the force–displacement or stress–strain plane. This loop is called the hysteresis loop and the exact geometrical nature of the loop depends on the material being investigated. The important behavioural difference between viscous and hysteretic damping is that the former dissipates its cyclic energy linearly with frequency, whereas the latter is independent of frequency and the damping is seen as a complex stiffness constituent. So for complicated problems one can assume a complex modulus which contains both a stiffness part (real) and a damping part (imaginary), without having to differentiate clearly between them in the physical problem.

This form of damping may be found in elastomeric and polymeric compounds, and in a structural sense it can occur through impact in gapped joints and also in some cases of joint slippage.

The fundamental damping quantity for the hysteretic case is the loss factor η, so we write the complex stiffness–damping modulus as

$$k' = k(1 + i\eta) \tag{1.4}$$

and in the case of a single degree of freedom problem we can put

$$m\ddot{x} + k'x = F(t). \tag{1.5}$$

We can compare the frequency independence of hysteretic damping with the strongly frequency dependent behaviour of the viscous case, as modelled by the equation

$$m\ddot{x} + c\dot{x} + kx = F(t). \tag{1.6}$$

By integrating the two cyclic products of force and displacement we can therefore arrive at the cyclic energy dissipation due to the two systems

$$D_V = \pi C \Omega A_1{}^2, \tag{1.7}$$

$$D_H = \pi k \eta B_1{}^2, \tag{1.8}$$

where A_1 and B_1 are assumed particular solution amplitudes for the viscous and the hysteretic cases respectively, and D_V and D_H are the viscous and hysteretic cyclic energies. Figure 1.1 (after Nashif *et al.*, 1985) shows how the hysteretically damped response curves for a single