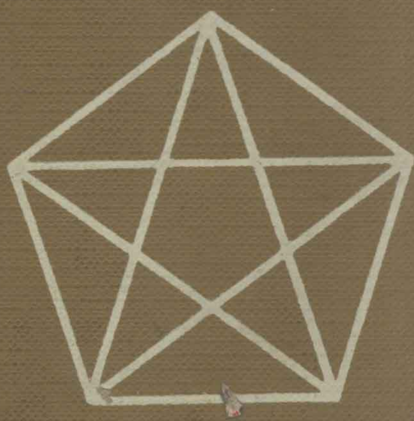
Introduction to

GEOMETRY



Coxeter

Introduction to

GEOMETRY

second edition

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to Rien

without whose patient encouragement
the work could never have been completed

Preface to the second edition

I am grateful to the readers of the first edition who have made suggestions for improvement. Apart from some minor corrections, the principal changes are as follows.

The equation connecting the curvatures of four mutually tangent circles, now known as the *Descartes Circle Theorem* (p. 14), is proved along the lines suggested by Mr. Beecroft on pp. 91–96 of "The Lady's and Gentleman's Diary for the year of our Lord 1842, being the second after Bissextile, designed principally for the amusement and instruction of Students in Mathematics: comprising many useful and entertaining particulars, interesting to all persons engaged in that delightful pursuit."

For *similarity* in the plane, a new treatment (pp. 73–76) was suggested by A. L. Steger when he was a sophomore at the University of Toronto. For similarity in space, a different treatment (p. 103) was suggested by Professor Maria Wonenburger. A new exercise on p. 90 introduces the useful concept of *inversive distance*. Another has been inserted on p. 127 to exhibit R. Krasnodębski's drawings of symmetrical loxodromes.

Pages 203–208 have been revised so as to clarify the treatment of *affinities* (which preserve collinearity) and *equiaffinities* (which preserve area). The new material includes some challenging exercises. For the discovery of *finite* geometries (p. 237), credit has been given to von Staudt, who anticipated Fano by 36 years.

Page 395 records the completion, in 1968, by G. Ringel and J. W. T. Youngs, of a project begun by Heawood in 1890. The result is that we now know, for every kind of surface the minimal number of colors that will suffice for coloring every map on the surface, though for anyone dissatisfied with a computer-generated proof, there remains a modicum of doubt in the case of the sphere (or plane).

Answers are now given for practically all the exercises; a separate booklet is no longer needed. One of the prettiest answers (p. 453) was kindly supplied by Professor P. Szász of Budapest.

H.S.M. Coxeter

Toronto, Canada
December, 1980

Preface to the first edition

For the last thirty or forty years, most Americans have somehow lost interest in geometry. The present book constitutes an attempt to revitalize this sadly neglected subject.

The four parts correspond roughly to the four years of college work. However, most of Part II can be read before Part I, and most of Part IV before Part III. The first eleven chapters (that is, Parts I and II) will provide a course for students who have some knowledge of Euclid and elementary analytic geometry but have not yet made up their minds to specialize in mathematics, or for enterprising high school teachers who wish to see what is happening just beyond their usual curriculum. Part III deals with the foundations of geometry, including projective geometry and hyperbolic non-Euclidean geometry. Part IV introduces differential geometry, combinatorial topology, and four-dimensional Euclidean geometry.

In spite of the large number of cross references, each of the twenty-two chapters is reasonably self-contained; many of them can be omitted on first reading without spoiling one's enjoyment of the rest. For instance, Chapters 1, 3, 6, 8, 13, and 17 would make a good short course. There are relevant exercises at the end of almost every section; the hardest of them are provided with hints for their solution. (Answers to some of the exercises are given at the end of the book. Answers to many of the remaining exercises are provided in a separate booklet, available from the publisher upon request.) The unifying thread that runs through the whole work is the idea of a group of transformations or, in a single word, *symmetry*.

The customary emphasis on analytic geometry is likely to give students the impression that geometry is merely a part of algebra or of analysis. It is refreshing to observe that there are some important instances (such as the Argand diagram described in Chapter 9) in which geometrical ideas are needed as essential tools in the development of these other branches of mathematics. The scope of geometry was spectacularly broadened by Klein in his *Erlanger Programm* (Erlangen program) of 1872, which stressed the fact that, besides plane and solid Euclidean geometry, there are many other geometries equally worthy of attention. For instance, many of Euclid's own propositions belong to the wider field of *affine* geometry, which is valid not only in ordinary space but also in Minkowski's space-time, so successfully exploited by Einstein in his special theory of relativity.

Geometry is useful not only in algebra, analysis, and cosmology, but also in kinematics and crystallography (where it is associated with the theory of groups), in statistics (where finite geometries help in the design of experiments), and even in botany. The subject of topology (Chapter 21) has been developed so widely that it now stands on its own feet instead of being regarded as part of geometry; but it fits into the Erlangen program, and its early stages have the added appeal of a famous unsolved problem: that of deciding whether every possible map can be colored with four colors.

The material grew out of courses of lectures delivered at summer institutes for school teachers and others at Stillwater, Oklahoma; Lunenburg, Nova Scotia; Ann Arbor, Michigan; Stanford, California; and Fredericton, New Brunswick, along with several public lectures given to the Friends of Scripta Mathematica in New York City by invitation of the late Professor Jekuthiel Ginsburg. The most popular of these separate lectures was the one on the golden section and phyllotaxis, which is embodied in Chapter 11.

Apart from the general emphasis on the idea of transformation and on the desirability of spending some time in such unusual environments as affine space and absolute space, the chief novelties are as follows: a simple treatment of the orthocenter (§ 1.6); the use of dominoes to illustrate six of the seventeen space groups of two-dimensional crystallography (§ 4.4); a construction for the invariant point of a dilative reflection (§ 5.6); a description of the general circle-preserving transformation (§ 6.7) and of the spiral similarity (§ 7.6); an "explanation" of phyllotaxis (§ 11.5); an "ordered" treatment of Sylvester's problem (§ 12.3); an economical system of axioms for affine geometry (§ 13.1); an "absolute" treatment of rotation groups (§ 15.4); an elementary treatment of the horosphere (§ 16.8) and of the extreme ternary quadratic form (§ 18.4); the correction of a prevalent error concerning the shape of the monkey saddle (§ 19.8); an application of geodesic polar coordinates to the foundations of hyperbolic trigonometry (§ 20.6); the classification of regular maps on the sphere, projective plane, torus, and Klein bottle (§ 21.3); and the suggestion of a statistical honeycomb (§ 22.5).

I offer sincere thanks to M. W. Al-Dhahir, J. J. Burckhardt, Werner Fenchel, L. M. Kelly, Peter Scherk, and F. A. Sherk for critically reading various chapters; also to H G. Forder, Martin Gardner, and C. J. Scriba for their help in proofreading, to S. H. Gould, J. E. Littlewood, and J. L. Synge for permission to quote certain passages from their published works, and to M. C. Escher, I. Kitrosser, and the Royal Society of Canada for permission to reproduce the plates.

H.S.M. Coxeter

Toronto, Canada
March, 1961

Mathematics possesses not only truth, but supreme beauty
—a beauty cold and austere, like that of sculpture,
without appeal to any part of our weaker nature . . .
sublimely pure, and capable of a stern perfection
such as only the greatest art can show.

BERTRAND RUSSELL (1872 - 1970)

Introduction to Geometry

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