Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Topological Properties of Spaces of Continuous Functions

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INTRODUCTION

Spaces of functions have been used since the late 19th century to form a framework in which convergence of sequences of functions could be studied. Since then several natural topologies have been frequently used to study function spaces. The purpose of this book is to bring together the techniques used in studying the topological properties of such function spaces and to organize and present the theory in a general setting. In particular, a study is made of C(X,R), the space of all continuous functions from a topological space X into a topological space R.

For almost any natural topology imposed on C(X,R), the topological properties of X and R interact with the topological properties of C(X,R). One of the things which is emphasized is the study of these interactions, especially the deduction of the topological properties of C(X,R) from those of X and R. The two major classes of topologies on C(X,R) which are studied are the set-open topologies and the uniform topologies. Each chapter has a number of exercises, not only about these two classes of topologies, but about other kinds of function space topologies found in the literature. Chapters I, II and III contain basic properties and techniques, as well as classical theory. Chapters IV and V have the characterizations of many topological properties of function spaces. Those in Chapter IV are given in the more general setting of cardinal functions.

The range space throughout this book is denoted by R, and whenever the properties of R are not important for the discussion, C(X,R) is abbreviated as C(X). in order to eliminate pathologies and ensure that C(X,R) is large enough, all spaces are assumed to be completely regular Hausdorff spaces, and R is assumed to contain a nontrivial path. The symbol ω denotes the first infinite ordinal number (which is the set of all natural numbers), and $\mathbb{R}$ is used to indicate the space of real numbers with the usual topology.

Chapter I

FUNCTION SPACE TOPOLOGIES

A concept which plays an important role is that of a network on a space. Let α be a family of subsets of X. A nonempty family β of nonempty subsets of X is an α-network on X provided that for each $A \in \alpha$ and open neighborhood U of A there exists a $B \in \beta$ such that $A \subset B \subset U$. A network on X is an α-network on X where α consists of the singleton subsets of X. A network on X is called a closed (or compact) network on X provided each member is closed (or compact). Similarly a closed (or compact) neighborhood base for X is a neighborhood base for X such that each member is closed (or compact).

1. Set-open Topologies. If $A \subset X$ and $B \subset R$, then the notation $[A,B]$ is defined by

$$[A,B] = \{f \in C(X,R): \ f(A) \subset B\}.$$

It is straightforward to check that

$$[A,B_1 \cap B_2] = [A,B_1] \cap [A,B_2], \text{ and}$$

$$[A_1 \cup A_2, B] = [A_1,B] \cap [A_2,B].$$

If $x \in X$ and $B \subset R$, then $[\{x\},B]$ is abreviated as $[x,B]$.

A topology on $C(X,R)$ is called a set-open topology provided there is some closed network α on X such that

$$\{[A,V]: A \in \alpha \text{ and } V \text{ is open in } R\}$$

is a subbase for the topology. In this case the function space having this topology is denoted by $C_\alpha(X,R)$ or $C_\alpha(X)$. In addition, if Y is a subspace of X, then $C_\alpha(Y,R)$ denotes $C_\beta(Y,R)$ where $\beta = \{A \cap Y : A \in \alpha\}$.

For topological spaces X and Y, the notation $X \leq Y$ means that X and Y have the same underlying set and the topology on Y is finer than or equal to the topology on X. With this notation, the following can be established.

Theorem 1.1.1. If α and β are closed networks on X, then $C_\alpha(X) \leq C_\beta(X)$ if and only if every member of α is contained in a finite union of members of β.

Proof. Let p: $[0,1] \to R$ be a path (continuous function) in R such that $p(0) \neq p(1)$, let f_0 be the constant function taking X to p(0), and let $V = R \setminus \{p(1)\}$. Take any $A \in \alpha$. Then $[A,V]$ is a neighborhood of f_0 in $C_\beta(X)$, so that there exists a basic neighborhood $W = [B_1,V_1] \cap ... \cap [B_n,V_n]$ of f_0 in $C_\beta(X)$ which is contained in $[A,V]$. Let $B = B_1 \cup ... \cup B_n$. To show that $A \subset B$, suppose on the contrary that there exists some $x \in A \setminus B$. Since X is completely regular, there exists a $\phi \in C(X,[0,1])$ such that $\phi(B) = \{0\}$ and $\phi(x) = 1$. Then $p \circ \phi \in W$ while $p \circ \phi \notin [A,V]$, which is a contradiction. This establishes the necessity; the sufficiency is immediate. ■

There are two well-studied examples of set-open topologies. One is the point-open topology, or topology of pointwise convergence, where the closed network on X is the family of all nonempty finite subsets of X. This function space is denoted by $C_p(X,R)$ or $C_p(X)$. On the other hand, the other commonly used set-open topology is the compact-open topology, or topology of compact convergence, where the closed network on X is the family of all nonempty compact subsets of X. This function space is denoted by $C_k(X,R)$ or $C_k(X)$.

The next couple of theorems give facts about the topology of pointwise convergence. The proof of the first fact follows immediately from the definition of the product topology, and the second fact follows from Theorem 1.1.1.

Theorem 1.1.2. The space $C_p(X,R)$ is a dense subspace of R^X with the Tychonoff product topology.

Theorem 1.1.3. If α is any closed network on X, then $C_p(X) \leq C_\alpha(X)$.

Therefore the topology of pointwise convergence is the smallest set-open topology. The largest set-open topology may be obtained by taking the family of all nonempty closed subsets for the closed network. The function space having the largest set-open topology is denoted by $C_w(X,R)$ or $C_w(X)$. These special set-open topologies are then related by

$$C_p(X) \leq C_k(X) \leq C_w(X).$$

These inequalities are only equalities for special X, as given by the following corollary to Theorem 1.1.1.

Theorem 1.1.4. The space $C_p(X) = C_k(X)$ if and only if every compact subset of X is finite; and $C_k(X) = C_w(X)$ if and only if X is compact.

The next theorem establishes the separation properties of set-open topologies.

Theorem 1.1.5. If α is a closed network on X, then $C_\alpha(X)$ is a Hausdorff space. Furthermore, if α is a compact network on X, then $C_\alpha(X)$ is completely regular.

Proof. The first part is immediate, so to show the second part, let $f \in [A,V]$ (a subbasic set suffices since a finite minimum of continuous functions is continuous). Now there exists a $\psi \in C(R,[0,1])$ such that $\psi(f(A)) = \{0\}$ and $\psi(R \setminus V) = \{1\}$. Then define $\phi \in C(C_\alpha(X), [0,1])$ by $\phi(h) = \sup\{\psi(h(a)): a \in A\}$ for each $h \in C_\alpha(X)$. It follows that $\phi(f) = 0$ and $\phi\{C_\alpha(X) \setminus [A,V]) = \{1\}$. ∎

Sometimes it is more convenient to work with basic open subsets of the range space rather than with arbitrary open subsets. If some additional assumptions are made about the closed network on X, then it is sufficient to use basic open sets in R to generate the topology on $C_\alpha(X,R)$. A closed network is called hereditarily closed provided that every closed subset of a member is a member.

Theorem 1.1.6. If α is a hereditarily closed, compact network on X and σ is a subbase for R, then $\{[A,S]: A \in \alpha \text{ and } S \in \sigma\}$ is a subbase for $C_\alpha(X,R)$.

Proof. Let $A \in \alpha$, let V be open in R, and let $f \in [A,V]$. For each $a \in A$, there exists a finite subset $\sigma_a \subset \sigma$ such that $f(a) \in \cap\{S: S \in \sigma_a\} \subset V$, and there exists a neighborhood U_a of a in X such that $\overline{U}_a \subset f^{-1}(\cap\{S: S \in \sigma_a\})$. Since A is compact, there exists a finite subset A' of A such that $S \subset \cup\{U_a: a \in A'\}$. Then define

$$W = \cap\{[A\cap\overline{U}_a,S]: a \in A' \text{ and } S \in \sigma_a\},$$

which clearly contains f. To show that $W \subset [A,V]$, let $g \in W$ and let $x \in A$. Then for some $a \in A'$, $x \in U_a$, so that $g(x) \in \cap\{S: S \in \sigma_a\} \subset V$. ■

Additional algebraic structures on R induce corresponding structures on C(X,R). For example, if R is a group with operation +, then for each $f,g \in C(X,R)$, $f + g$ is defined by $(f + g)(x) = f(x) + g(x)$ for each $x \in X$. This defines the induced group structure on C(X,R).

Whenever α is a hereditarily closed, compact network on X and R is a locally convex topological vector space, then $C_\alpha(X,R)$ is also a locally convex topological vector space. Part of the proof of this is incorporated in the next theorem.

Theorem 1.1.7. If α is a hereditarily closed, compact network on X and R is a topological group, then $C_\alpha(X,R)$ is a topological group.

Proof. Let the group operation be denoted by + as above, and start with $f - g \in [A,V]$. Then for each $a \in A$, there exist neighborhoods V_a and W_a of f(a) and g(a) such that $V_a - W_a \subset V$. Also for each $a \in A$, there exists a closed neighborhood N_a of a in

X such that $f(N_a) \subset V_a$ and $g(N_a) \subset W_a$. Since A is compact, there exists a finite subset A' of A such that $A \subset \cup\{N_a: a \in A'\}$. Then define

$$S = \cap\{[A\cap N_a, V]: a \in A'\} \text{ and}$$

$$T = \cap\{[A\cap N_a, W_a]: a \in A'\},$$

which are neighborhoods of f and g in $C_\alpha(X,R)$. Now it is easy to check that $S - T \subset [A,V]$. ■

As a result of Theorem 1.1.7, if α is a hereditarily closed, compact network on X and R is a topological group, then $C_\alpha(X,R)$ is homogeneous. In this case it generally suffices to work only with neighborhoods of the zero function f_0. Furthermore, if α is closed under finite unions and if $B = [A_1,V_1] \cap \dots \cap [A_n,V_n]$ is a basic neighborhood of f_0, then $A = A_1 \cup \dots \cup A_n \in \alpha$ and $V = V_1 \cap \dots \cap V_n$ contains 0, so that $f_0 \in [A,V] \subset B$. Therefore in this case it suffices to work with sets of the form [A,V] which contain the zero function. This discussion includes two of the most commonly used function spaces, $C_p(X,R)$ and $C_k(X,R)$.

<u>2.</u> <u>Uniform Topologies.</u> Let α be a closed network on X, and let μ be a compatible (diagonal) uniformity on R. The topology induced on C(X,R) by the uniform structure which is about to be defined on C(X,R) is the same whether a diagonal uniformity is used on R or whether its corresponding covering uniformity is used. So all uniformities are taken as diagonal uniformities.

For each $A \in \alpha$ and $M \in \mu$, define

$$\hat{M}(A) = \{(f,g) \in C(X) \times C(X): \text{ for each } x \in A, (f(x), g(x)) \in M\}.$$

In the case that A = X, set $\hat{M} = \hat{M}(X)$. It is straightforward to check that the family $\{\hat{M}(A): A\in\alpha, M\in\mu\}$ is a subbase for a uniformity on C(X). In fact if α is closed under finite unions, then this family is a base for a uniformity on C(X). The space with the topology induced by the uniformity generated by $\{\hat{M}(A): A\in\alpha, M\in\mu\}$ is denoted by $C_{\alpha,\mu}(X,R)$ or $C_{\alpha,\mu}(X)$. The topology induced in this manner is called the uniform topology on α (with respect to μ) or the topology of uniform convergence on α (with respect to μ). The open sets in $C_{\alpha,\mu}(X)$ can be described as the family of all subsets W of C(X) such that for all f $\in$ W, there exist $A_1,\ldots, A_n \in \alpha$ and $M_1,\ldots,M_n \in \mu$ with

$$\hat{M}_1(A_1)[f] \cap \ldots \cap \hat{M}_n(A_n)[f] \subset W,$$

where for each A $\in$ α and M $\in$ μ, $\hat{M}(A)[f]$ is defined by

$$\hat{M}(A)[f] = \{g \in C(X): (f,g) \in \hat{M}(A)\}.$$

In the case that $\alpha = \{X\}$, then set $C_\mu(X) = C_{\alpha,\mu}(X)$. The topology on $C_\mu(X)$ is called the uniform topology (with respect to μ) or the topology of uniform convergence (with respect to μ). In this case, $\{\hat{M}: M \in \mu\}$ is a base for the uniformity inducing this topology, and a subset W of $C_\mu(X)$ is open provided that for each f $\in$ W there is some M $\in$ μ such that $\hat{M}[f] \subset W$.

As an illustration of these concepts, the proof is given for the sufficiency of the following theorem. The necessity can be established in a manner similar to the proof of Theorem 1.1.1.

Theorem 1.2.1. If α and β are closed networks on X and μ is a compatible uniformity on R, then $C_{\alpha,\mu}(X) \leq C_{\beta,\mu}(X)$ if and only if every member of α is contained in a finite union of members of β.

Proof. (of sufficiency). Let A $\in$ α, M $\in$ μ, and f $\in$ C(X). Then there exist $B_1,\ldots,B_n \in \beta$ such that $A \subset B_1 \cup \ldots \cup B_n$. But $\hat{M}(B_1 \cup \ldots \cup B_n) = \hat{M}(B_1) \cap \ldots \cap \hat{M}(B_n)$, so that $\hat{M}(B_1)[f] \cap \ldots \cap \hat{M}(B_n)[f] = (\hat{M}(B_1) \cap \ldots \cap \hat{M}(B_n))[f] = \hat{M}(B_1 \cup \ldots \cup B_n)[f] \subset \hat{M}(A)[f]$. ■

The next fact follows immediately from definition.

Theorem 1.2.2. If α is a closed network on X and μ is a compatible uniformity on R, then $C_{\alpha,\mu}(X) \leq C_{\mu}(X)$.

The relation between set-open topologies and uniform topologies is given by the next fundamental result.

Theorem 1.2.3. If α is a compact network on X and μ is a compatible uniformity on R, then $C_{\alpha}(X) \leq C_{\alpha,\mu}(X)$. If, in addition, α is hereditarily closed, then $C_{\alpha}(X) = C_{\alpha,\mu}(X)$.

Proof. Let $A \in \alpha$, let V be open in R, and let $f \in [A,V]$. For each $a \in A$, there exists an $M_a \in \mu$ such that $M_a[f(a)] \subset V$; choose $N_a \in \mu$ such that $N_a \circ N_a \subset M_a$. Now f(A) is compact, so there exists a finite subset A' of A such that $f(A) \subset \cup\{N_a[f(a)]: a \in A'\}$. Then define $N = \cap\{N_a: a \in A'\}$. To show that $\hat{N}(A)[f] \subset [A,V]$, let $g \in \hat{N}(A)$ and let $x \in A$. There exists some $a \in A'$ with $f(x) \in N_a[f(a)]$, so that $(f(a),f(x)) \in N_a$. Since $(f(x),g(x)) \in N \subset N_a$, then $(f(a),g(x)) \in N_a \circ N_a \subset M_a$. Therefore $g(x) \in M_a[f(a)] \subset V$, so that $g \in [A,V]$.

For the reverse inequality, let $A \in \alpha$, let $M \in \mu$, and let $f \in C(X)$. Let N be a closed and symmetric element of μ such that $N \circ N \circ N \subset M$. Again since f(A) is compact, there exists a finite subset A' of A so that $f(A) \subset \cup\{N[f(a)]: a \in A'\}$. For each $a \in A'$, define $A_a = A \cap f^{-1}(N[f(a)])$, which is in α since α is hereditarily closed; also define V_a to be the interior of $(N \circ N)[f(a)]$. Finally define $W = \cap\{[A_a,V_a]: a \in A'\}$, which is open in $C_{\alpha}(X)$. Since V_a contains $N[f(a)]$ for each $a \in A'$, then $f \in W$. To see that $W \subset \hat{M}(A)[f]$, let $g \in W$ and let $x \in A$. There exists some $a \in A'$, with $f(x) \in N[f(a)]$, so that $(f(a), f(x)) \in N$. Also $g(x) \in V_a \subset (N \circ N)[f(a)]$, so that $(f(a),g(x)) \in N \circ N$. Then since N is symmetric, $(f(x),g(x)) \in N \circ N \circ N \subset M$, and thus $g \in \hat{M}(A)[f]$. ■

In particular, it follows from Theorem 1.2.3 that the compact-open topology is the same as the topology of uniform convergence on compact sets (independent of the uniformity used). Also Theorems 1.2.2 and 1.2.3 give the following.

Theorem 1.2.4. If α is a compact network on X and μ is a compatible uniformity on R, then $C_\alpha(X) \leq C_\mu(X)$.

It is of interest to know when the inequality in Theorem 1.2.4 is an equality. One answer is given by the next theorem.

Theorem 1.2.5. A space X is compact if and only if $C_\mu(X) = C_k(X)$ for every compatible uniformity μ on R.

Proof. If X is compact and $\alpha = \{X\}$, then by Theorem 1.2.1, $C_\mu(X) = C_{\alpha,\mu}(X) \leq C_{k,\mu}(X)$. Also $C_{k,\mu}(X) \leq C_\mu(X)$ by Theorem 1.2.2, so that $C_\mu(X) = C_{k,\mu}(X)$. But Theorem 1.2.3 says that $C_{k,\mu}(X) = C_k(X)$. The converse follows from Theorem 1.2.1. ■

A special kind of uniform topology is the supremum metric topology. In this case the range space R must have a compatible metric ρ, which can be chosen to be bounded. This metric on R induces a metric $\hat{\rho}$ on C(X) defined by

$$\hat{\rho}(f,g) = \sup\{\rho(f(x),g(x)) : x \in X\},$$

which is called the supremum metric. If ρ is complete, then $\hat{\rho}$ is also complete.

The ε-balls in R with respect to metric ρ are denoted by $B_\rho(x,\varepsilon)$ or $B(x,\varepsilon)$, while those in C(X) with respect to metric $\hat{\rho}$ are denoted by the similar notation $B_\rho(f,\varepsilon)$ or $B(f,\varepsilon)$. Then $\{B_\rho(f,\varepsilon): f \in C(X) \text{ and } \varepsilon > 0\}$ is a base for some topology on C(X) called the supremum metric topology. The resulting topological space is denoted by $C_\rho(X,R)$ or $C_\rho(X)$.

Every metric naturally induces a uniformity. It turns out that the supremum metric topology is equal to the uniform topology with respect to the uniformity induced by this metric. The next theorem makes this precise.

Theorem 1.2.6. For any space X, if ρ is a compatible bounded metric on R and if μ is the uniformity on R induced by ρ, then $C_\rho(X,R) = C_\mu(X,R)$.

Proof. Let $f \in C(X)$ and $\varepsilon > 0$ be given. For each $\delta > 0$, let $M_\delta = \{(s,t) \in R\times R: \rho(s,t) < \delta\}$. Then the family $\{M_\delta: \delta>0\}$ is a base for μ. To show that $\hat{M}_{\varepsilon/2}[f] \subset B(f,\varepsilon)$, let $g \in \hat{M}_{\varepsilon/2}[f]$. Then $(f,g) \in \hat{M}_{\varepsilon/2}$, so that for every $x \in X$, $(f(x),g(x)) \in M_{\varepsilon/2}$; or $\rho(f(x),g(x)) < \varepsilon/2$. But then $\hat{\rho}(f,g) \leq \varepsilon/2 < \varepsilon$, so that $g \in B(f,\varepsilon)$. This establishes that $C_\rho(X,R) \leq C_\mu(X,R)$.

For the reverse inequality, let $f \in C(X)$ and $0<\varepsilon<1$. To show that $B(f,\varepsilon) \subset \hat{M}_\varepsilon[f]$, let $g \in B(f,\varepsilon)$. Then $\hat{\rho}(f,g) < \varepsilon$, so that $\rho(f(x),g(x)) < \varepsilon$ for all $x \in X$. But then $(f(x),g(x)) \in M_\varepsilon$ for all $x \in X$, so that $(f,g) \in \hat{M}_\varepsilon$; and thus $g \in \hat{M}_\varepsilon[f]$. ∎

If α is a closed network on X and ρ is a compatible bounded metric on R, then $C_{\alpha,\rho}(X,R)$ is defined as $C_{\alpha,\mu}(X,R)$, where μ is the uniformity on R induced by ρ. Then for a hereditarily closed, compact network α on X, $C_{\alpha,\rho}(X,R) = C_\alpha(X,R)$. This means that for such α, sets of the following form are basic open sets. For each $A \in \alpha$, $f \in C(X,R)$ and $\varepsilon>0$, define

$$<A,f,\varepsilon> = \{g\in C(X,R): \text{ for each } a\in A,\ \rho(f(a),g(a))<\varepsilon\}.$$

When R is metrizable, use of this kind of basic open set in $C_\alpha(X,R)$ is sometimes more convenient.

For a metric space R, the topology on $C_\rho(X,R)$ is dependent on the choice of compatible metric ρ on R. That is, different compatible metrics on R may generate different supremum metric topologies on C(X,R). This is illustrated by the following example.

Example 1.2.7. Let $R = \mathbb{R}$ and let ρ be the usual metric on $\mathbb{R}$ bounded by 1. That is, $\rho(s,t) = \min(1, |s-t|\}$. Also let σ be the metric on $\mathbb{R}$ defined by

$$\sigma(s,t) = \left|\frac{s}{1+|s|} - \frac{t}{1+|t|}\right|,$$

which is compatible with the usual topology. To prove that $C_\rho(\mathbb{R}) \neq C_\sigma(\mathbb{R})$, let $f \in C(\mathbb{R})$ be the identity function, and for each $n \in \omega$ let $f_n \in C(\mathbb{R})$ be defined by $f_n(x) = x$ if $x < n$ and $f_n(x) = n$ if $x \geq n$. Then for each n, $\hat{\rho}(f,f_n) = 1$; while if $x \geq n$,

$$\sigma(f(x),f_n(x)) = \left|\frac{x}{1+x} - \frac{n}{1+n}\right| = \frac{x-n}{(1+n)(1+x)} < \frac{1}{1+n},$$

so that $\hat{\sigma}(f,f_n) \leq \frac{1}{1+n} < \frac{1}{n}$. Therefore for every n, $B_\sigma(f,1/n)$ is not contained in $B_\rho(f,1)$.

This example also shows that different compatible uniformities on R may generate different uniform topologies on C(X,R). A natural question is: when do compatible uniformities (or metrics) on R generate the same topology on C(X,R)? If X is compact, then by Theorem 1.2.5, all compatible uniformities on R generate the compact-open topology on C(X,R). In particular, if X is compact and ρ is a compatible bounded metric on R, then $C_\rho(X,R) = C_k(X,R)$. On the other hand, if R is compact, then there is only one compatible uniformity on R, so that all compatible uniformities on R (and hence by Theorem 1.2.6, all compatible bounded metrics on R) generate the same topology on C(X,R). Although in this latter case, the topology generated on C(X,R) may not be the compact-open topology.

For a compatible uniformity μ on R, $C_\mu(X,R)$ is homogeneous only in special cases.