# FUNDAMENTAL CONCEPTS OF MATHEMATICS

*by*

R. L. GOODSTEIN,

SECOND EDITION

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## PREFACE TO THE SECOND EDITION

The fifteen years which have elapsed since the publication of the first edition have been noteworthy for the solution of some remarkable problems which David Hilbert posed in a famous lecture at the beginning of the century, and in this new edition I have taken the opportunity to draw attention to these problems and to make the consequent changes in the text which their solution makes necessary.

Amongst these problems is the tenth in Hilbert's list of 23, the problem of determining which integral polynomials have integral solutions. Another is the problem of showing that four colours suffice to colour any map.

The existing chapter on sentence logic and informal set theory has been supplemented by a new chapter on predicate logic and axiomatic set theory.

A further major change is the introduction of numerous sets of exercises with detailed solutions.

# PREFACE

THE title of this book *Fundamental Concepts of Mathematics* correctly describes one aim of the book, to give an account of some of the notions which play a fundamental part in modern mathematics. The book makes no claim to be exhaustive and what has been omitted is not thereby judged to be of less importance; I have written about the ideas which interest me most at the moment. The class of readers for whom the book is intended is rather more difficult to specify. Certainly I am not writing for the professional mathematician, who does not need the help I try to give the reader. Nor am I writing for the student who is seeking training in a technique. The cultivated amateur is one whose needs I had in mind, but above all this account is intended for teachers of mathematics who feel that their background knowledge is out of date, and for teachers in training. There is a great hunger in the world for mathematicians and a great hunger for mathematics, and both these needs can be met only by means of an immense increase in the number of teachers of mathematics with a thorough comprehension of fundamental concepts.

R. L. GOODSTEIN

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## CHAPTER 1

# NUMBERS FOR COUNTING

THE whole structure of mathematics, rich in its three thousand years of development and with an almost bewildering variety of growing points, has both its foundation and its origin in the numbers with which we count.

What do we do when we count the objects in a collection? Our first answer might be that we use the numbers as labels, and give each object in turn a label, one, two, three, and so on, just as we name our children. To make the situation more definite let us suppose we have a stock of number labels, from which we draw the numbers to be attached to the objects counted. When each of the objects has been assigned its label, what has been accomplished, apart, that is, from just affixing the labels? Can we say that we have now found out the number of objects in the collection, that this number is that on the last label we used? For instance, let me count a row of squares:

I place a numeral in each square, thus:

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |

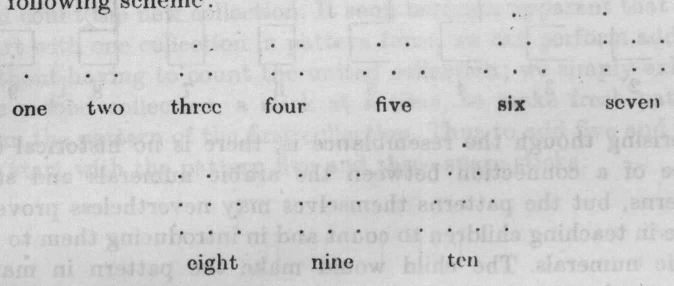
and say that there are seven squares in the row. But why did I place just *these* numerals in the squares? Why did I not perhaps fill the squares in this way:

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | 5 | 3 | 7 | 9 | 4 | 1 |

Counting therefore cannot be simply a process of naming. Our next suggestion may be that the labels must be kept, and used, in a definite order. That the label 1 must be used first, then the label 2, and so on. This would safeguard us against the error we committed above, but how is the *order* of the labels to be determined? We might of course rely upon an order established by custom, as with the letters of the alphabet; in fact both the Greeks and the Hebrews used the letters of their alphabets as number labels in this way. Thus we might label the row of squares with letters:

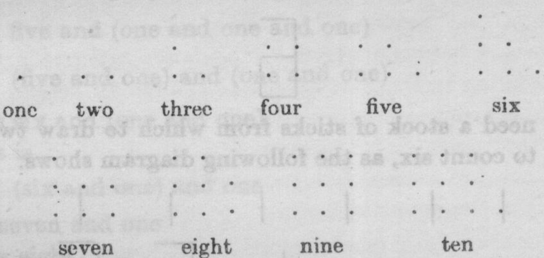
| | | | | | | |
|---|---|---|---|---|---|---|
| *a* | *b* | *c* | *d* | *e* | *f* | *g* |

and say that we have *g* squares, the established order of the letters of the alphabet guarding us against error. For counting sufficiently small collections such a procedure would be adequate, but it does not take us to the heart of the matter. Our stock of labels is necessarily limited, yet we know that we can number any collection, however great. And the need to commit to memory, or record in some other form, an established order of number names becomes increasingly burdensome as the collection of number labels increases. Concealed beneath our counting process there is in fact a remarkable mechanism for generating numbers as large as we like and for ordering them automatically. The familiar process of counting is really a combination of two operations. One of these is a number generating process and the other a process of translation or abbreviation. To separate these two aspects of counting, let us start by recognising the number names: "two", "three", "four", and so on, as abbreviations for "one and one", "two and one", i.e. "one and one and one", "three and one", and so on. Thus one part of counting consists in reciting the definitions "one and one is two", "two and one is three", "three and one is four", and so on, only, in counting, we do not repeat the whole of the definition, we just say "one, two, three, ..." omitting the "and one" or rather replacing it by looking at the object counted, or by touching it. The other part of the process of counting is a process of *copying* the objects counted, or matching them, by dots, or strokes, or match sticks, or words. Thus counting may be said to have originated in the practice of making a copy of a collection, perhaps in pebbles on the ground, or in strokes on sand, or cuts in a length of wood; but this making a copy is by no means the whole of counting. The essential step lies in organising the copy into a readily identifiable whole, as we do when we translate "one and one" into "two", "one and one and one" into "three", and so on. It is instructive to look at this organising process in other settings. Instead of using number words, let us work with dot patterns, named according to the following scheme:

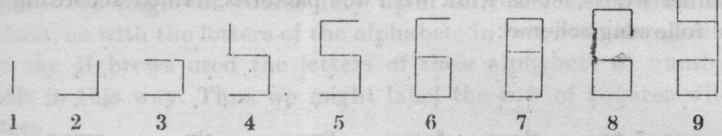



Faced with a row of dots

we seek to organize the dots into one of the named patterns to determine its number. This bears a certain relationship to counting, but lacks one essential feature; counting is systematic, and is entirely free from trial and error, but with the above dot patterns we must try each pattern afresh, to see which we can make. This is because the dot patterns have no internal connections, we do not pass from one to the next by adding a fresh dot, as we pass from "one and one" to "two" and from "two and one" to "three" and so on by adding one. We can remedy this by redesigning the patterns:

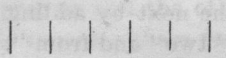



Each pattern is now contained in its successor, and to count a row of dots we *form the patterns in turn*, adding one dot at a time until the collection of dots is exhausted. If we work with stick patterns instead of dots we can produce patterns which bear some resemblance to the arabic numerals themselves. Probably the set which bears the closest resemblance is this:



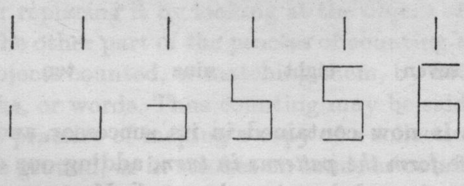

Surprising though the resemblance is, there is no historical evidence of a connection between the arabic numerals and stick patterns, but the patterns themselves may nevertheless prove of value in teaching children to count and in introducing them to the arabic numerals. The child would make the pattern in match sticks, and name it, before learning to write the arabic numeral. Stick pattern making would provide an activity to accompany the learning of the definitions "one and one is two", "two and one is three", and so on, and would provide both a visual and a tactile aid.

The use of stick patterns shows how curious an operation counting would be if it really consisted in assigning a pattern (number name) not to the whole collection but to each separate object. Thus to count the row of sticks



instead of organising the sticks into the single pattern

we should need a stock of sticks from which to draw twenty-one sticks just to count six, as the following diagram shows.

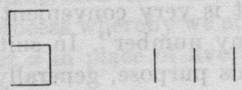


**Addition**

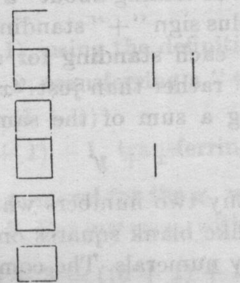
After counting, the next operation of arithmetic is *addition*. Addition of numbers may be thought of as uniting two collections. Thus to add two and three, we unite the collections . ., . . . to form

. . . . .

and count the new collection. It soon becomes apparent that if we start with one collection in pattern form, we can perform addition without having to count the united collection; we simply exhaust the second collection, a stick at a time, to make fresh patterns from the pattern of the first collection. Thus to add five and three we start with the pattern five and three spare sticks:



and form in turn



showing that the sum is eight. Translated into words the addition of five and three consists in the steps:

five and (one and one and one)

is

(five and one) and (one and one)

which is six and (one and one)

and this is

(six and one) and one

that is seven and one

which is eight.

I have used brackets to help the eye follow the transfer of a 'one' from one collection to the other.

The fact that the order of addition is irrelevant may most readily be seen by setting out the objects of the collections to be united in a row. For instance to show that the result of adding three to five is the same as the result of adding five to three, we set out collections of five and three side by side:

× × × × × • • •

reading from left to right we have five to which three is added, but reading from right to left we have three to which five is added. The fact that the order in which addition is performed is irrelevant is usually expressed by saying that addition is *commutative*.

In stating general properties of numbers, like the commutative property of addition, it is very convenient to have a symbol to express the idea of "any number". In suitable contexts we can use single letters for this purpose, generally letters from the end of the alphabet. Instead of talking about "a sum of two numbers" we write "$x + y$", the plus sign "+" standing as usual for addition and the "$x$" and "$y$" each standing for any numeral. We use two letters "$x$" and "$y$" rather than just "$x$", to avoid suggesting that we are considering a sum of the same two numbers. The expression

$$x + y$$

stands for the sum of any two numbers whatever.

The $x$ and the $y$ are like blank squares on an income tax form, which are to be filled by numerals. The commutative property of addition may now be expressed by the equation

$$x + y = y + x$$

the equals sign "=" between "$x + y$" and "$y + x$" affirming that the two numbers between which it stands (or rather the numbers obtained by filling in the blanks) are the same. Of course "$x$" and "$y$" are not themselves numerals; their role is the same as that of words like "he" and "she" in language, which stand in place of names, but are not themselves names. Since "he" is a pronoun, "$x$" and "$y$" in the equation

$$x + y = y + x$$

may be called *pronumerals*. But in fact the parallel is not an exact one, for in the equation

$$x + y = y + x$$

we may replace "$x$" and "$y$" by any numerals we please, whereas in common usage, "he" refers to a particular person in a particular context. The equation

$$x + y = y + x$$

summarizes all such instances of the commutative property of addition as

$$2 + 3 = 3 + 2,\ 4 + 5 = 5 + 4,\ 2 + 5 = 5 + 2, \text{ etc.}$$

The equation $2 + 3 = 3 + 2$ does not of course say that $2 + 3$ and $3 + 2$ are the same sign — they obviously are not — but that each denotes the same number, or in a sense we shall later explain, that each may be transformed into the other.

Let us repeat the process whereby we added three to five, taking this time any number $x$ in place of five and writing numerals in place of words:

$$\begin{aligned} &x + 3 \\ &= x + (2 + 1), \text{ using the definition } 3 = 2 + 1, \\ &= (x + 1) + 2, \text{ transferring a "}+ 1\text{" from 2 to } x, \\ &= (x + 1) + (1 + 1) \\ &= ((x + 1) + 1) + 1, \text{ transferring "}+ 1\text{" again.} \end{aligned}$$

If we now fill in any numeral for the $x$, we have before us the sum of that numeral and 3. For instance, with "6" for "$x$":

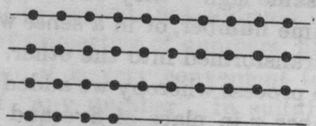
$$6 + 3 = ((6 + 1) + 1) + 1.$$

If we now use the definitions $6 + 1 = 7$, $7 + 1 = 8$, $8 + 1 = 9$ we arrive at the result

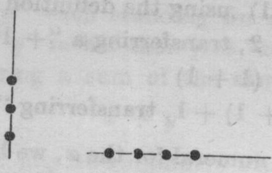
$$6 + 3 = 9.$$

Of course the chain of definitions $1 + 1 = 2, \ldots, 8 + 1 = 9$, may be continued as far as we please, but, as is well known, the numerals after 9 are not individual signs but are compounded of the numerals 1 to 9, by means of a most ingenious device, *positional notation*, according to which the number a numeral denotes depends upon the position the numeral occupies. Positional notation is most easily described in terms of the bead frames from which it originated.

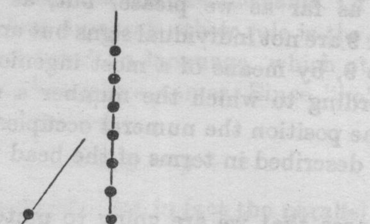
Let us suppose that we are going to match some collection in beads, storing the beads on a wire. We choose a wire which can hold exactly ten beads. At first we might use a great many wires in exactly the same way; we fill the wires and store each full wire until the whole collection has been matched. This is very uneconomical both in beads and wires, and one day it occurred to some one that it was not really necessary to store the full wires, provided that we matched the full wires by beads on another wire. Now we should need only two wires, one to hold the beads which we match with the given collection, and another to hold the beads which match the full wires. Instead of recording a collection of thirty-four by means of three full wires, and four over



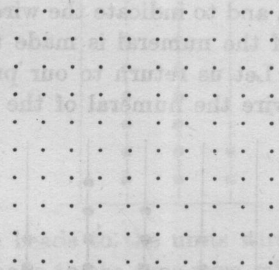
we introduce an upright wire on which we match the full wires by additional beads, and the record now takes the form



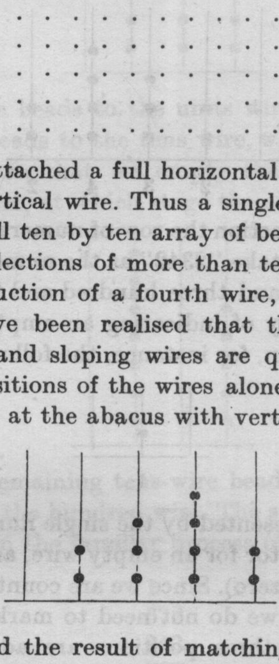
which records 3 full wires and 4. This two-wire device would be adequate for quite small collections, but faced with a large collection (of several hundred, say) we should find the single upright wire insufficient, and be obliged to introduce a third wire on which we placed beads to match the full upright wires. Thus for instance, using only wires which hold ten beads, a collection of two hundred and seventy-four would be recorded in the form



Each bead on the sloping wire represents a full upright wire, and since each bead on an upright wire represents a full horizontal wire, it follows that a bead on the sloping wire represents the following array of beads



in which we have attached a full horizontal row of beads to each bead on the full vertical wire. Thus a single bead on the sloping wire represents a full ten by ten array of beads, a hundred beads in all. Matching collections of more than ten hundred would necessitate the introduction of a fourth wire, and by this time no doubt it would have been realised that the special devices of horizontal, upright and sloping wires are quite unnecessary and that the relative positions of the wires alone serve to distinguish them, and we arrive at the abacus with vertical wires.



We have represented the result of matching a collection of two and four-tens and three-ten-tens and two-ten-ten-tens (i.e. two thousand, three hundred and forty-two). In talking about the abacus, the right-hand end wire is called the *unit* wire, then from right to left the successive wires are known as the *tens* wire, *hundreds* wire, *thousands* wire, *ten-thousands* wire. Of course the number words one, two, three, four themselves provide a more logical nomenclature (a fact which we shall exploit when we come to the study of indices) the tens wire being one from the end, the