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Abstract*

Let K be a cone with nonempty interior $\overset{\circ}{K}$ in a Banach space X and $f: \overset{\circ}{K} \to \overset{\circ}{K}$ a map. This paper treats questions like the following: Does f have a suitably normalized eigenvector u in the interior of K? Is the normalized eigenvector u unique? If $f(tu) = tu$ for every $t > 0$, is it true that for every $x \in \overset{\circ}{K}$ there exists $\lambda(x) > 0$ such that $\lim_{k\to\infty} f^k(x) = \lambda(x)u$? What can be said about the structure of the set of eigenvectors of f in $\overset{\circ}{K}$? The class of maps studied includes maps which are homogeneous of degree one and preserve the partial ordering induced by K. Applications are made to the theory of means and their iterates. In a subsequent paper applications are given to so-called D-A-D theorems and to some questions in mathematical biology.

Key words and phrases

Hilbert's projective metric, iterated nonlinear maps, order-preserving maps, means and their iterates.

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INTRODUCTION

By a cone K (with vertex 0) in a Banach space X we shall mean a closed, convex subset of X such that (a) $tK \subset K$ for all $t > 0$ and (b) if $x \in K-\{0\}$, then $-x \notin K$. Examples of cones are provided by

$$K = \{x \in \mathbb{R}^n : x_i \geq 0 \quad \text{for} \quad 1 \leq i \leq n\}$$

(which we shall call the standard cone in $\mathbb{R}^n$) and by the set of positive semidefinite, self–adjoint linear operators on a Hilbert space. Every cone K induces a partial ordering on X by $x \leq y$ if and only if $y-x \in K$.

If D is a subset of a cone K in a Banach space X, a map $f: D \to X$ is called "order–preserving" if for all $x, y \in D$ such that $x \leq y$ one has $f(x) \leq f(y)$. If $tD \subset D$ for all $t > 0$, f is called "homogeneous of degree one" if $f(tx) = tf(x)$ for all $t > 0$ and $x \in D$.

We shall be interested in a variety of questions about maps which are homogeneous of degree one and order–preserving. Actually, many of our theorems will treat more general classes of functions, but the basic difficulties are already apparent for certain order–preserving maps which are homogeneous of degree one.

In fact, it may be worthwhile to describe a concrete class of maps of the standard cone K in $\mathbb{R}^n$. Most of the questions we shall describe below are already nontrivial for this class. Recall that a "probability vector" $\sigma \in \mathbb{R}^n$ is a vector such that $\sigma_i \geq 0$ for $1 \leq i \leq n$ and

$$\sum_{i=1}^{n} \sigma_i = 1.$$

If r is a real number (possibly $r = 0$) and σ a probability vector, define $M_{r\sigma}: \overset{\circ}{K} \to \mathbb{R}$ by

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$$M_{r\sigma}(x) = \left[\sum_{j=1}^{n} \sigma_j x_j^r\right]^{\frac{1}{r}} .$$

If $r = 0$ and $x \in \overset{\circ}{K}$, define

$$M_{0\sigma}(x) = \prod_{j=1}^{n} x_j^{\sigma_j} = \lim_{r \to 0} M_{r\sigma}(x).$$

One can prove that $M_{r\sigma}$ extends continuously to K. For each i, $1 \le i \le n$, let Γ_i be a nonempty finite collection of ordered pairs (r,σ) (r a real number and σ a probability vector). For each $(r,\sigma) \in \Gamma_i$, let $c_{ir\sigma}$ be a positive real and define $f_i:K \to [0,\infty)$ by

$$f_i(x) = \sum_{(r,\sigma) \in \Gamma_i} c_{ir\sigma} M_{r\sigma}(x). \tag{0.1}$$

Define $f:K \to K$ to be a map whose i^{th} coordinate is given by (0.1). We shall say that $f \in M$ if $f:K \to K$ can be written in the form (0.1) and $f \in M_-$ if f can be written in the form (0.1) in such a way that $r < 0$ for all $(r,\sigma) \in \Gamma_i$, $1 \le i \le n$. Similarly, we shall write $f \in M_+$ if $r \ge 0$ for all $(r,\sigma) \in \Gamma_i$, $1 \le i \le n$. Note that linear maps L such that $L(\overset{\circ}{K}) \subset \overset{\circ}{K}$ lie in $M_+ \cap M_-$. Finally, we shall define $\mathcal{M}$ (or $\mathcal{M}_-$ or $\mathcal{M}_+$ respectively) to be the smallest set of maps which contains M (or M_- or M_+ respectively) and is closed under composition of functions, addition and multiplication by positive scalars. Questions about $\mathcal{M}$ are a primary impetus for this paper.

We are interested in this paper in the following questions (among others):

(1) If K is a cone with nonempty interior in a Banach space X and $f:\overset{\circ}{K} \to \overset{\circ}{K}$ is a continuous map (possibly homogeneous of degree one and order-preserving), does f have an eigenvector in $\overset{\circ}{K}$, i.e., does there exist $x \in \overset{\circ}{K}$ such that $f(x) = \lambda x$ for some $\lambda \ge 0$? Notice that if X is finite dimensional and f extends continuously to K, then an application of Brouwer's fixed point

theorem implies that for each $R > 0$ f has an eigenvector $y \in K$ with $\|y\| = R$. The problem is that y may not lie in $\overset{\circ}{K}$. In fact, even for functions $f \in M_-$, the question of whether f has an eigenvector in $\overset{\circ}{K}$ is frequently a delicate one. With some notable exceptions, this difficulty has not been recognized.

(2) Assuming that f has an eigenvector $u \in \overset{\circ}{K}$, normalized in some way, is this normalized eigenvector unique? For example, does f have a unique eigenvector of norm one in $\overset{\circ}{K}$? Notice that f may well have other eigenvectors in ∂K; uniqueness refers to eigenvectors in $\overset{\circ}{K}$. The map $f(x,y) = \left[\frac{x+y}{2}, \sqrt{xy}\right]$, familiar from the arithmetic–geometric mean of Gauss and Lagrange, has the eigenvector $(1,1)$ in $\overset{\circ}{K}$, which is unique to within scalar multiples; but $(1,0)$ is also an eigenvector in ∂K.

(3) If f has a unique eigenvector $u \in \overset{\circ}{K}$ of norm one and if one defines

$$g(x) = \frac{f(x)}{\|f(x)\|},$$

is it true that for all $x \in \overset{\circ}{K}$,

$$\lim_{k\to\infty} g^k(x) = u?$$

(4) Suppose that f is a map of $\overset{\circ}{K}$ to $\overset{\circ}{K}$ and that, for some $\lambda > 0$, $S = \{x \in \overset{\circ}{K}\colon f(x) = \lambda x\}$ is nonempty. What can one say about the structure of S? Does it have the same (trivial) homotopy type as $\overset{\circ}{K}$? If $g(x) = f(x)\|f(x)\|^{-1}$ and

$$S_1 = \{x \in \overset{\circ}{K}\colon \|x\| = 1 \text{ and } g(x) = x\},$$

what can one say about the structure of S_1?

(5) If g is as above, g has no fixed points in $\overset{\circ}{K}$, and $x \in \overset{\circ}{K}$, what

can one say about the behavior of $g^k(x)$ as k approaches infinity?

(6) Suppose that f is a map of $\overset{\circ}{K}$ to $\overset{\circ}{K}$ and that $S = \{u \in \overset{\circ}{K}: f(u) = u\}$ is nonempty. Is it true that for every $x \in \overset{\circ}{K}$ there exists $u = u_x \in S$ such that

$$\lim_{k\to\infty} f^k(x) = u_x?$$

If $S = \{tu: t > 0\}$ for some fixed $u \in \overset{\circ}{K}$, the question becomes whether, for each $x \in \overset{\circ}{K}$, there exists $\lambda(x) > 0$ such that

$$\lim_{k\to\infty} f^k(x) = \lambda(x)u.$$

This paper represents an attempt to answer the above questions and to begin applications of these answers to some problems of interest. The original motivation for our work came from some questions in population biology [25,26,39,49,60,66,73] and from problems concerning "means and their iterates" [2,3,8,9,12,13,22,27,28,29,35,50,63,65,79] and "D–A–D theorems" [4,15,16,30,32,36,43,48,53,54,55,76,77]. Although we shall say something about means and their iterates, considerations of length have forced us to defer other applications to a later paper [84].

Question one, or the problem of the existence of an eigenvector in $\overset{\circ}{K}$, is perhaps the central question in our work and the irreducible analytic difficulty. Of course the question is easily answered in some instances (see [84] for examples), but in general the problem appears highly nontrivial. In any event, for the most part we shall not consider question one here, but shall defer it to [84], which is an immediate sequel to this paper. Section two of [84] provides some conditions under which functions $f \in \mathcal{M}$ have an eigenvector in $\overset{\circ}{K}$; the theorems there generalize some preliminary results in Section four of [60]. Section three of [84] considers at length the question of whether a function $f \in \mathcal{M}_-$ (or in a similar class of maps) has an eigenvector in $\overset{\circ}{K}$. Again, in Section four of [84], which treats D–A–D theorems and their generalizations, the central question which is treated is the existence of an eigenvector in the interior of a cone for an appropriate nonlinear map.

If, however, the map f has an eigenvector in $\overset{\circ}{K}$, our theorems provide; quite general answers to questions two and three. In questions four, five and six, we essentially assume that the answer to question one is known; and while open problems remain, our theorems provide satisfactory answers in many cases.

There are at least two closely related questions which we shall not consider here. First, one can consider the analogues of questions one, two and three for ordinary differential equations. Indeed, some of the population biology literature [70, 73] is concerned with precisely this point. In his Rutger's Ph.D. dissertation [82], K. Wysocki has shown that the results of Section 2 of this paper can be used to study autonomous ordinary differential equations in cones. The following is a very special case of Wysocki's theorem: Suppose that K is the standard cone in $\mathbb{R}^n$ and that $f:\overset{\circ}{K} \to \mathbb{R}^n$ is C^1 and homogeneous of degree one. Assume that there exists $\alpha > 0$ such that $(f + \alpha I)(\overset{\circ}{K}) \subset \overset{\circ}{K}$ and $f'(x) + \alpha I$ is a primitive, non–negative matrix for all $x \in \overset{\circ}{K}$. Finally, suppose that f has an eigenvector u of norm one in $\overset{\circ}{K}$. Then for any $x_0 \in \overset{\circ}{K}$, if $x(t) = x(t;x_0)$ is the solution of

$$x'(t) = f(x(t))$$

$$x(0) = x_0,$$

$x(t)$ is defined and in $\overset{\circ}{K}$ for all $t \geq 0$ and

$$\lim_{t\to\infty} x(t)\|x(t)\|^{-1} = u.$$

A second question which will not be considered here is what are called "weak ergodic theorems" in the population biology literature (see [26], [39] and [38]). In weak ergodic theorems one considers the behavior of $F_n(x)$, $x \in \overset{\circ}{K}$, where $f_j:\overset{\circ}{K} \to \overset{\circ}{K}$, f_j lies in some given class S of maps, and

$$F_n = f_n f_{n-1} \cdots f_1.$$

If, for example, $S = M$, the existing literature is, in general, inapplicable. However, as will be shown in another paper, the ideas of this paper can be extended to yield such weak ergodic theorems.

There is an enormous literature concerning linear and nonlinear maps of cones, and there is a large subliterature in which the so-called Hilbert's projective metric or its variants (see [6, 78]) have been used in the study of linear and nonlinear cone maps. We refer, for example, to the work of Bushell [18, 21], Potter [67, 68] and Krause [46] on nonlinear maps and to an extensive linear theory beginning with G. Birkhoff [9, 10] and E. Hopf [42]. Bushell's expository article [18] still provides an excellent introduction. Existing nonlinear theory is inadequate for our applications, the technical reason being that while our maps are frequently nonexpansive with respect to Hilbert's projective metric d they are often not strict contractions with respect to d.

Since this paper is long, it may be worthwhile to summarize its contents. In the first half of Section 1 we attempt to make the paper as self-contained as possible by summarizing relevant facts and definitions about cones, cone mappings, Hilbert's projective metric d and a variant of Hilbert's projective metric due to A.C. Thompson [78]. This material can be safely omitted by the expert. The second half of Section 1 contains some geometric results about Hilbert's projective metric d and Thompson's variant $\bar{d}$. In particular a number of propositions concerning minimal geodesics with respect to d or $\bar{d}$ are proved, and the reader may find it interesting that Proposition 1.10 has connections to linear operator theory. These results are elementary but appear to be new and are needed later.

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The first few pages of Section 2 summarize known results from the theory of linear, order–preserving maps and can be omitted by experts. Theorem 2.5 provides an answer to Question 2. If K is a finite dimensional cone and $f:\overset{\circ}{K} \to \overset{\circ}{K}$ is order–preserving, and homogeneous of degree one and f is C^1 on $\overset{\circ}{K}$ with $f'(x)$ irreducible for all $x \in \overset{\circ}{K}$, then Theorem 2.5 reduces to the assertion that f has at most one (to within scalar multiples) eigenvector in $\overset{\circ}{K}$. Theorem 2.7 provides an answer to Question 3. In the special case just described, Theorem 2.7 implies that if f has an eigenvector $u \in \overset{\circ}{K}$ with $\|u\| = 1$, $f'(u)$ is primitive and f is C^1 near u, then for all $x \in \overset{\circ}{K}$ one has

$$u = \lim_{k \to \infty} f^k(x)\|f^k(x)\|^{-1}.$$

A crucial role in the theorems of Section 2 is played by purely linear results which may have some independent interest. Lemmas 2.6 and 2.7 provide information about the spectrum of Λ, where

$$\Lambda(x) = L(x) - \psi(Lx)u$$

and L is a given positive linear map, ψ is a positive linear functional and $u \in \overset{\circ}{K}$ is such that $\psi(u) = 1$ and $Lu \leq u$. Essentially best possible results are given which insure that the spectral radius of Λ is strictly less than one. The reader should also note Corollary 2.4, which describes a situation under which $f^k(x)$ approaches a point in ∂K for all $x \in \overset{\circ}{K}$.

Section 3 would perhaps better be titled "means and their iterates", after the excellent survey article [3]. If $f:\overset{\circ}{K} \to \overset{\circ}{K}$ is homogeneous of degree one and order–preserving, Theorem 3.2 provides a very general positive answer to Question 6. However, there are also important examples in which the function is not order–preserving and possibly not homogeneous of degree one but for which Question 6 has a positive answer. Propositions 3.1 – 3.4 treat such situations. In generalizing the arithmetic–geometric mean and its extensions to bounded linear operators, the author and Joel Cohen [63] have encountered situations in which the map f is not order–preserving and has a fixed point set of dimension greater than one.

Section 4 treats Question 4 and Question 5. The basic idea is that while Hilbert's projective metric d is not a norm, it enjoys enough convexity properties that certain arguments from the theory of nonexpansive maps in Banach spaces can be modified to our situation. A difficulty is that in the most important examples, d is analogous to a norm which is not strictly convex. Theorem 4.2 provides an answer to Question 5 and Theorems 4.5 – 4.7 give answers to Question 4.

We should note, finally, that many of the results of this paper were summarized in [62].

I. BASIC PROPERTIES OF HILBERT'S PROJECTIVE METRIC

In this section we shall recall some definitions and notations and establish some geometrical results which will be useful later.

If K is a closed convex subset of a Banach space X, we shall say that K is a cone (with vertex at 0) if (a) $x \in K$ and $x \neq 0$ implies that $-x \notin K$ and (b) $x \in K$ and λ a non-negative real implies that $\lambda x \in K$. Some authors do not demand that cones be closed or that condition (a) hold. A cone K induces a partial ordering on X by

$$x \leq y \quad \text{if and only if} \quad y-x \in K. \tag{1.1}$$

A cone K is called "normal" if there exists a constant A such that for all x and y in K with $0 \leq x \leq y$ one has

$$\|x\| \leq A \ \|y\|, \tag{1.2}$$

where $\|\cdot\|$ is the norm on X. It is known (see [71], Chapter 5, Section 3, p. 215) that if K is a normal cone in a Banach space X with norm $\|\cdot\|$, then there exists an equivalent norm $|\cdot|$ such that

$$0 \leq x \leq y \quad \text{implies} \quad |x| \leq |y|. \tag{1.3}$$

A norm which satisfies equation (1.3) is called "monotonic". If S is a compact Hausdorff space, $X = C(S)$, the space of continuous real-valued functions with

$$\|x\| = \sup_{s \in S} |x(s)|$$

and K is the cone of non–negative functions, K is a normal cone. We note for future reference that we shall always think of R^n as $C(S)$, where $S = \{1,2,...,n\}$ and S is given the discrete topology. If (Σ,μ) is a measure space, $X = L^p(\Sigma,\mu)$, $1 \leq p \leq \infty$, and the standard norm on X is used, then the set of non–negative functions in X provides another example of a normal cone.

If K is a cone in a Banach space X and if there exists a finite dimensional Banach space Y such that $K \subset Y$, then K will be called "finite dimensional". The next proposition is well–known and can be proved by a simple compactness argument which is left to the reader.

Proposition 1.1: Any finite dimensional cone K is normal.

A cone K in a Banach space X is called "total" if X is the closure of $K-K$, where

$$K-K = \{x-y:x,y \in K\}.$$

The cone is "reproducing" if $X = K-K$. If K is a cone in a Banach space X, X^* will always denote the Banach space of continuous linear functionals on X and K^* will be defined by

$$K^* = \{\psi \in X^*;\psi(x) \geq 0 \text{ for all } x \in K\}.$$

It will sometimes be convenient to use "pairing notation", i.e., $\langle\psi,x\rangle = \psi(x)$ for $\psi \in X^*$ and $x \in X$. It is easy to prove that K^* is a cone if K is total, but even if K^* is not a cone (condition (a) in the definition of cones may fail) the above definitions of "total" and "reproducing" still make sense. Classical theorems (see [71]) assert that a cone K in a Banach space X is normal if and only if K^* is reproducing, and K^* is normal if and only if K is reproducing.

We shall be interested later in the cone of positive semi-definite operators in a Hilbert space. Thus let H be a real Hilbert space with inner product $\langle\ ,\ \rangle$ and let X be the Banach space of bounded linear operators $L:H \to H$ with

$$\|L\| = \sup\{\|Lx\|:x \in H \text{ and } \|x\| = 1\}.$$

Let K be the set of positive semidefinite, self-adjoint operators in X, so

$$\begin{aligned} K = \{A:A \in X,\ A \text{ is self-adjoint and} \\ \langle Ax,x\rangle \geq 0 \text{ for all } x \in H\}. \end{aligned} \tag{1.4}$$

Recall that if A is bounded and self-adjoint and if $\sigma(A)$ denotes the spectrum of the complexification of A, then $\sigma(A) = \{\lambda \in \mathbb{R}:\lambda-A$ is not one-one and onto $H\}$, and one has

$$\begin{aligned} \lambda_1 &= \inf\{\langle Ax,x\rangle:\|x\| = 1\} \in \sigma(A), \\ \lambda_2 &= \sup\{\langle Ax,x\rangle:\|x\| = 1\} \in \sigma(A), \end{aligned} \tag{1.5}$$

and

$$\|A\| = \sup\{|\langle Ax,x\rangle|:\|x\| = 1\}. \tag{1.6}$$

<u>Proposition 1.2</u>: The set K of non-negative definite self-adjoint operators defined in equation (1.4) is a normal cone.

<u>Proof</u>. K is clearly closed and convex and if $A \in K$, then $tA \in K$ for all $t \geq 0$. If $A \in K$ and $-A \in K$, then $\langle Ax,x\rangle = 0$ for all x, and equation (1.6) implies that $\|A\| = 0$. Thus K is a cone. If $0 \leq A \leq B$, then

$$0 \leq \langle Ax,x\rangle \leq \langle Bx,x\rangle$$

for all x, so

$$\|A\| = \sup\{\langle Ax,x\rangle : x \in H \text{ and } \|x\| = 1\}$$
$$\leq \sup\{\langle Bx,x\rangle : x \in H \text{ and } \|x\| = 1\} = \|B\|.$$

The latter inequality proves K is normal. ∎

If K is now a general cone in a Banach space X and x and y are elements of $K-\{0\}$, say that x and y are "comparable" if there exist real numbers $\alpha > 0$ and $\beta > 0$ such that

$$\alpha x \leq y \leq \beta x. \tag{1.7}$$

We say that two elements x and y of $K-\{0\}$ are "equivalent" if they are comparable, this defines an equivalence relation on $K-\{0\}$ and divides $K-\{0\}$ into disjoint subsets which we shall call "components of K". Two elements of $K-\{0\}$ are equivalent if and only if they lie in the same component. If $u \in K-\{0\}$ and

$$C_u = \{x \in K-\{0\} : x \text{ is comparable to } u\}, \tag{1.8}$$

then C_u is the component containing u; and one easily checks that $C_u \cup \{0\}$ satisfies all conditions to be a cone except that $C_u \cup \{0\}$ need not be closed. If $\overset{\circ}{K}$, the interior of K, is nonempty and $u \in \overset{\circ}{K}$, then $C_u = \overset{\circ}{K}$.

If x and y are comparable, we define (following notation in [18]) numbers $m(y/x)$ and $M(y/x)$ by

$$m(y/x) = \sup\{\alpha > 0 : \alpha x \leq y\} \text{ and} \tag{1.9}$$

$$M(y/x) = \inf\{\beta > 0 : y \leq \beta x\}. \tag{1.10}$$

If x and y are comparable and $\alpha = m(y/x)$ and $\beta = M(y/x)$, define $d(x,y)$, Hilbert's projective metric, by

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