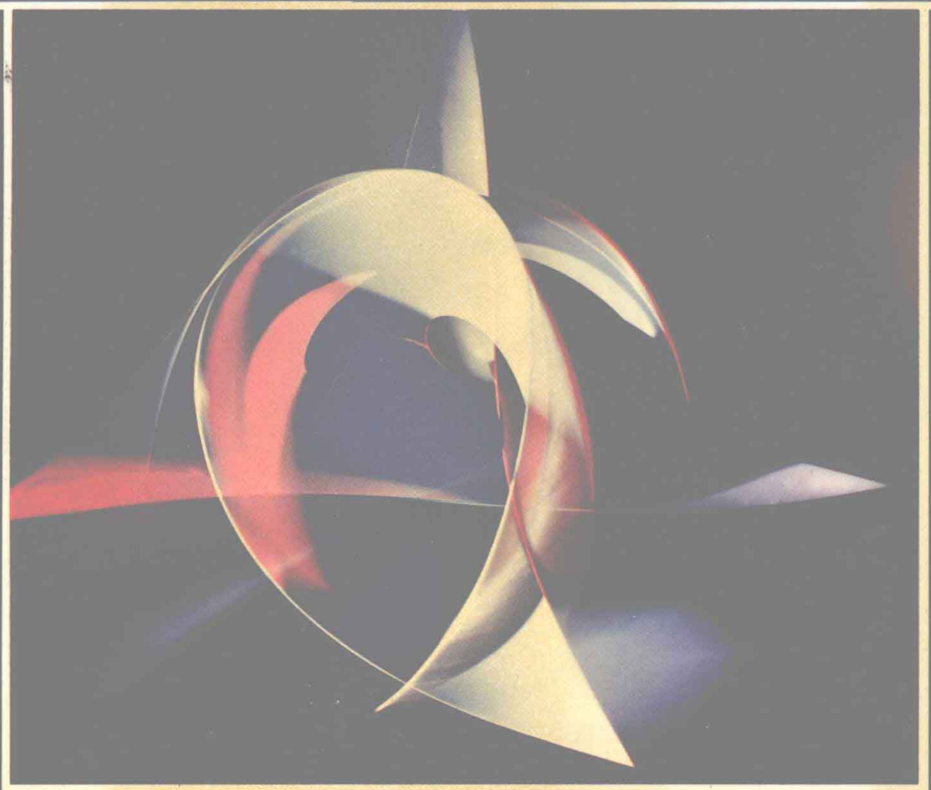
MODERN INTRODUCTORY ANALYSIS



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COVER

The design on the cover is a motion-study photograph of the painted aluminum sculpture of José de Rivera entitled "Construction in Blue and Black," 1951. It is used by permission of the Whitney Museum of American Art, New York. Here the photographer has been interested in describing photographically the path of the sculpture through space over a brief interval of time. In a similar manner, mathematicians are interested in describing with equations the change in position of objects as a function of time.

MODERN INTRODUCTORY ANALYSIS

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Photograph Credits

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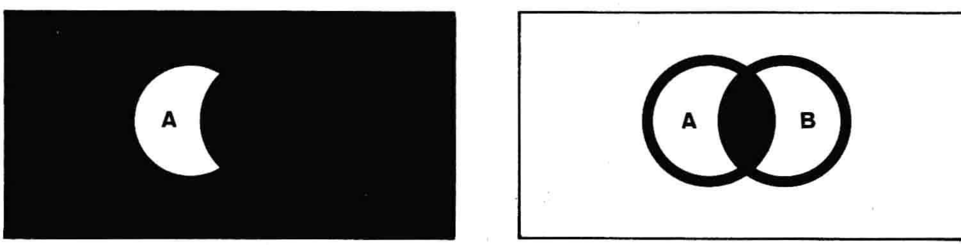
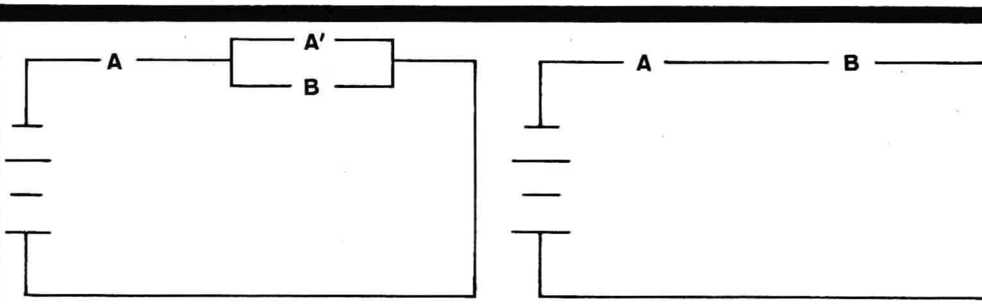
1

p	q	p'	$q \vee p'$	$p \wedge (q \vee p')$	$p \wedge q$
T	T	F	T	T	T
T	F	F	F	F	F
F	T	T	T	F	F
F	F	T	T	F	F

$$p \wedge (q \vee p') = p \wedge q$$



$$A \cap (B \cup A') = A \cap B$$



$$a \cdot (b + a') = a \cdot b$$

Above, the same theorem is stated in terms of logic, set theory, and an abstract mathematical system called Boolean algebra. In logic the theorem is proved by means of a truth table. In set theory it is illustrated by Venn diagrams and in Boolean algebra by circuit diagrams.

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Statements and Sets in Mathematics

One way to view mathematics is to think of it as a study of patterns. In this book, you will learn to work with a wide variety of patterns, some familiar to you, such as $x^2 - y^2 = (x + y)(x - y)$, and others quite new. In order to learn the mathematics you will encounter, you will need to understand the logical machinery used to discuss mathematical statements; this is what you will study in the present chapter.

SIMPLE STATEMENTS AND SETS

1–1 Logical Statements; Sets

In studying algebra and geometry you have discovered many logical connections between mathematical statements. But do you know what constitutes a statement in logic?

A **statement** or **proposition** is a set of symbols (and *words* are symbols, of course) which forms a meaningful assertion that is either true or false, but not both true and false.

The following assertions are examples of statements:

True Statements	*False Statements*
(1) $2 + 3 = 5$.	(1) $0 > 1$.
(2) Shakespeare wrote *Hamlet*.	(2) Plato was a Chinese philosopher.
(3) Ottawa is not a province of Canada.	(3) $\sqrt[3]{125} \neq 5$.

On the other hand, none of the following sentences is a statement because none can be said to be definitely true or definitely false.

(1) Give me liberty or give me death.

(2) $x^2 + 1 = 5$.

(3) This sentence is false.

Logical statements in mathematics frequently involve the use of symbols of equality and inequality. You should recall that the symbol "$=$" represents the word "equals" or the phrase "is equal to," and is used between expressions to show that they represent the same object. Similarly, the inequality symbols "$>$," "$<$," "$\geq$," and "$\leq$" represent "is greater than," "is less than," "is greater than or equal to," and "is less than or equal to," respectively. These symbols are used in sentences and statements comparing numbers. A bar, / or |, is used in conjunction with certain symbols to denote negation. For example, the symbol "$\neq$" placed between two expressions asserts that the expressions do not represent the same object, and the symbol $\nless$ stands for "is not less than."

Some mathematical statements concern *membership* in a *set*. You should recall from your earlier study of mathematics that the objects in a set are called **members** or **elements** of the set, and are said to *belong to* or to be *contained in* the set. The symbol $\in$ is used to denote "is an element of," while $\notin$ means "is not an element of." For example, if J is the set of integers, then $3 \in J$ is a true statement, whereas $4 \notin J$ is a false statement.

A statement that two sets are equal is an assertion that they contain the same elements. Thus, the statement $A = B$ means that A and B name the same set. This definition of the equality of sets implies that to *specify* a set, you must be able to identify its elements. You can sometimes specify a set by listing the names of its elements within braces, $\{\ \}$. For example, $\{-1, 0, 1\}$, read "the set whose members are -1, 0, and 1," is a *roster* (list) of the numbers -1, 0, and 1. Quite frequently, however, a set has so many elements that it is inconvenient or impossible to list them all. Thus, we may write $\{1, 2, 3, \ldots, 50\}$ and expect the reader to understand that the set specified contains the integers from 1 to 50, inclusive. Similarly, $\{1, 2, 3, \ldots\}$ specifies the set of all positive integers.

The members of a set can, themselves, be sets. $\{\{1, 3\}, \{5, 7\}, \{9, 11\}\}$, for example, is the set whose members are $\{1, 3\}$, $\{5, 7\}$, and $\{9, 11\}$. Observe that the set $\{\{a, b\}\}$ is not the same as the set $\{a, b\}$. The first contains just one element, $\{a, b\}$, while the second contains two elements, a and b.

Another way to specify a set consists in giving a rule or condition that enables you to decide whether or not any given object belongs to the set. For example, {the teachers of mathematics}, read "the set of the teachers of mathematics," specifies a set whose roster contains many thousands of names. Of course, you certainly recognize that every mathematics teacher you know belongs to

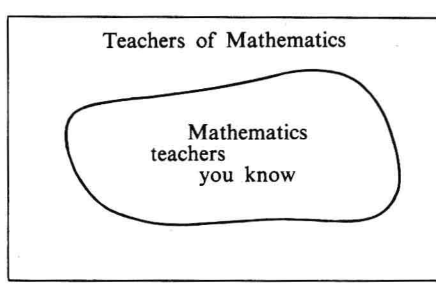

Figure 1–1

{the teachers of mathematics}. Whenever each element of a set R is also an element of a set S, we say that R is a **subset** of S, in symbols "$R \subset S$." Thus, {mathematics teachers you know} $\subset$ {teachers of mathematics}. Diagrams such as the one shown are used to illustrate a set and a subset; they are called **Venn diagrams.**

Since every positive integer is a member of the set of positive integers, it is certainly true that the set of positive integers is a subset of itself. Each set is said to be the **improper subset** of itself; every other subset is called a **proper subset** of the set.

Can you list the members of the set of integers between $\frac{1}{2}$ and $\frac{2}{3}$? This set contains no elements at all and is therefore called the **empty set** or **null set.** We use the symbol $\emptyset$, written without braces, to designate the empty set. Because of our agreement on the meaning of equality of sets, there is only one empty set, $\emptyset$. Furthermore, $\emptyset$ is taken to be a *proper* subset of every set except itself.

Exercises

Which of the sentences in Exercises 1–16 are statements? Of the statements, which are true? Give a reason for each answer.

A

1. $5 \cdot 4 = 20$
2. $5 - 4 = 2$
3. $3 + 7(2) = 2 + 15$
4. $7(4 + 2) = 7 \cdot 4 + 7 \cdot 2$
5. $\dfrac{6 + 2}{2} = 6 + 1$
6. $(-2)^3 \geq (-2)^4$
7. $1 + 3 \neq 1 + 6$
8. $2 + 3 > 0$
9. Man the lifeboats!
10. Help!

In Exercises 11–16, let J = {integers} = $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

11. $\{1, 2, 7\} \subset J$
12. $0 \in J$
13. $\{0\} \in J$
14. $\emptyset$ is not a subset of J.
15. $-1 \subset J$
16. $\{-1, 0, 2\} \notin J$

Copy each sentence, making it a true statement by replacing each question mark with a numeral or with one of the symbols $=$, $\neq$, $\in$, $\notin$, $\subset$. (Note: There may be more than one correct answer.)

17. $-3 \times ? = -3$
18. $? \times 7 = 0$
19. $5(? - 3) = 20$
20. $\dfrac{24 - 6}{6}\ ?\ 4 - 6$
21. $\dfrac{10 \times 4}{2}\ ?\ 5 \times 2$
22. $\dfrac{21 + ?}{3} \neq 7 + 1$
23. $\{0 \cdot 2\} \subset \{1, 3, ?\}$
24. $\{1, \frac{6}{2}, -1\} = \{-2 \div 2, 3, ? \times 4\}$

B

25. $\{0, 2, 4\}$? {the even integers}

26. $5 + ? \notin$ {the positive and negative numbers}

27. $\dfrac{8 \times ?}{4} \in \{8\}$

28. $\emptyset ? \{7 - 7\}$

29. $0 ? \{0\}$

30. $\{\emptyset\} ? \emptyset$

31. $\{1, \{2\}\} ? \{\{2\}, 1\}$

32. $\{1, \{2\}\} ? \{\{1\}, 2\}$

Let $U = \{-4, 0, 8\}$. List all the subsets of U that

33. Have exactly one element.

34. Have no elements.

35. Have at least two elements.

36. Have no more than 2 elements.

Which of the following rosters or rules (Exercises 37–40) specify sets? Justify your answer by explaining whether or not it can be decided that an arbitrary object is or is not a member.

37. {the authors of this textbook}

38. {bell, book, candle}

39. {the digits appearing in the decimal numeral for $\frac{1}{7}$}

40. $\{3, 1, 4, 5, 9, 2, 6, \ldots\}$

C

41. Make a list of all subsets of $\{\emptyset, \{\emptyset\}\}$.

42. A set contains $n > 0$ elements. How many subsets does it have?

43. A set containing $k + 1$ elements has 8 more subsets than a set containing k elements. Find k.

44. Argue that if $A \subset B$ and $B \subset A$, then $A = B$.

45. Argue that if $A \subset B$ and $B \subset C$, then $A \subset C$.

46. Can an element of a set be a subset of the set? Justify your answer.

1–2 Variables and Quantifiers

In logic and mathematics you encounter sentences, such as "He is an elected official" or "$x + 1 > 0$," which cannot be described as true or as false. To work with such sentences, you must understand the role of the pronoun *He* and the letter x. Each is a symbol, called a **variable**, and is used to represent any element of a specified set. The set whose elements may serve as replacements for the variable is called the **domain**, or **replacement set**, or **universe** of the variable. The members of the domain are called the **values** of the variable. A variable with just one value is called a **constant**. If the domain of the pronoun *He* in the sentence "He is an elected official" is {the public officeholders in New York City}, you obtain a true statement when you replace *He* by the name of the mayor, because in New York City the mayor is elected. On the other hand, when you write the name of the superintendent of schools in place of *He*, a false statement results, because