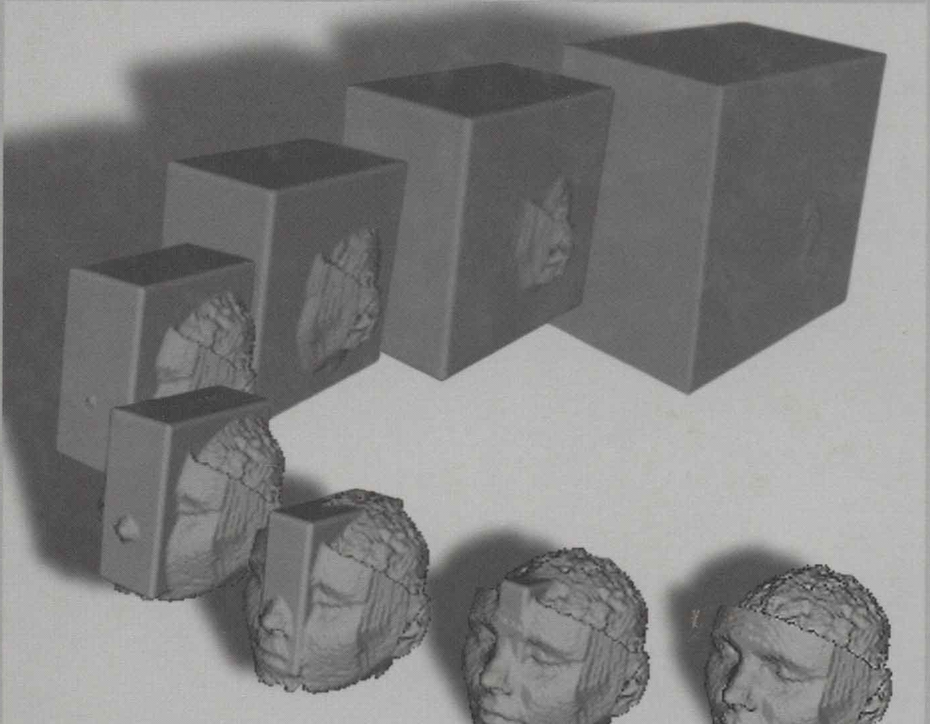
Geometric Partial Differential Equations and Image Analysis



Guillermo Sapiro

GEOMETRIC PARTIAL DIFFERENTIAL EQUATIONS AND IMAGE ANALYSIS

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To Eitan, Dalia, and our little one on the way...

They make the end of my working journey and return home something to look forward to from the moment the day starts....

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*These figures are available in color at http://www.ece.umn.edu/users/guille/book.html

Preface

This book is an introduction to the use of geometric partial differential equations (PDEs) in image processing and computer vision. This relatively new research area brings a number of new concepts into the field, providing, among other things, a very fundamental and formal approach to image processing. State-of-the-art practical results in problems such as image segmentation, stereo, image enhancement, distance computations, and object tracking have been obtained with algorithms based on PDE's formulations.

This book begins with an introduction to classical mathematical concepts needed for understanding both the subsequent chapters and the current published literature in the area. This introduction includes basic differential geometry, PDE theory, calculus of variations, and numerical analysis. Next we develop the PDE approach, starting with curves and surfaces deforming with intrinsic velocities, passing through surfaces moving with image-based velocities, and escalating all the way to the use of PDEs for families of images. A large number of applications are presented, including image segmentation, shape analysis, image enhancement, stereo, and tracking. The book also includes some basic connections among PDEs themselves as well as connections to other more classical approaches to image processing.

This book will be a useful resource for researchers and practitioners. It is intended to provide information for people investigating new solutions to image processing problems as well as for people searching for existent advanced solutions. One of the main goals of this book is to provide a resource for a graduate course in the topic of PDEs in image processing. Exercises are provided in each chapter to help with this.

Acknowledgments

Writing the acknowledgments part of this book is a great pleasure for me. Not only because it means that I am getting close to the end of the writing process (and although writing a book is a very enjoyable experience, finishing it is a great satisfaction!), but also because it gives me the opportunity to thank a number of people who have been very influential to my career. Although a book is always covered with subjective opinions, even a book in applied mathematics and engineering like this, in the acknowledgments part, the author is officially allowed to be completely subjective. This gives to the writing of this part of the book an additional very relaxed pleasure.

This books comes after approximately 10 years of research in image processing and computer vision. My interest in image processing and computer and biological vision began when I was an undergraduate assistant at the Signal and Image Processing Lab, Department of Electrical Engineering, Technion, Haifa, Israel. This lab, as well as the Electrical Engineering Department at the Technion, provides the perfect research environment, and without any doubt, this was extremely influential in my career. I always miss that place! I was deeply influenced as well by the basic undergraduate course given by Y. Y. Zeevi on computational vision; this opened my eyes (and brain!) to this exciting area. Later, I became a graduate student in the same department. I was very lucky to be accepted by David Malah as one of his M.Sc. students. David is a TEACHER and EDUCATOR (with capital letters). David's lessons go way beyond image processing; he basically teaches his students all the basics of research (Research 101 we could call it). His students are the ones that spend the most time working for their Master's degree, but after those difficult 2 years, you realize that all of it was worth the effort. I could not continue in a better direction, and I was very lucky again when Allen Tannenbaum accepted me as his Ph.D. student.

This gave me the chance to move more toward mathematical aspects, my all-time academic love, and to learn a whole new area of science. But this is not as important as the fact that being his student gave me the opportunity to get to know a great person. Allen and I became colleagues and great friends. He was my long-distance Ph.D. advisor, and between ski, planes, Rosh HaShana in Cambridge, steaks in Montreal, all-you-can-eat restaurants in Minneapolis, and zoos in San Diego, we managed to have a lot of fun and also to do some research. We continue to have a lot of fun and sometimes to do some research as well. Both David and Allen go way beyond the basic job description of being graduate advisors, and without their continuous help and teaching, I would not be writing this book and most probably I would be doing something completely different. I learned a great deal from you, and for this I thank you both.

After getting my education at the Technion, I spent one very enjoyable year at MIT. I thank Sanjoy Mitter for inviting me and the members of the Laboratory for Information and Decisions Systems for their hospitality. From MIT I moved to Hewlett-Packard Labs in Palo Alto, California, where some of the research reported in this book was performed. I thank members of the labs for the three great years I spent there.

Getting to know great people and researchers and having the chance to work with them is without any doubt the most satisfactory part of my job. During my still short academic life, I had the great pleasure to have long collaborations with and to learn from Freddy Bruckstein, Vicent Caselles, Peter Giblin, Ron Kimmel, Jean-Michael Morel, Peter Olver, Stan Osher, and Dario Ringach. We are not just colleagues, but friends, and this makes our work much more enjoyable. Thanks for all that I learned from you. I also enjoyed shorter, but very productive, collaborations with Michael Black, David Heeger, Patrick Teo, and Brian Wandell. Parts of this book are based on material developed with these collaborators (and with David and Allen), and a big thanks goes to them.

My graduate and post-doctoral students Alberto Bartesaghi, Santiago Betelu, Marcelo Bertalmio, Do Hyun Chung, Facundo Memoli, Hong Shan Neoh, and Bei Tang, "adopted" graduate students Andres Martinez Sole, Alvaro Pardo, and Luis Vazquez, as well as students from my courses in advanced image processing are, in part, the reason for this book. I wanted them and future students to have a reference book, and I hope this is it. My graduate students have also provided a number of the images used in this book, and our joint research is reported in this book as well.

Many other people provided figures for this book, and they are explicitly acknowledged when those figures appear. They helped me to make this book

more illustrative, and this is a necessary condition when writing a book in image analysis.

I started this book while enjoying a quarter off teaching at the University of Minnesota, my current institution. I thank Mos Kaveh and the people who took over my teaching responsibilities for this break and the Electrical and Computer Engineering department for supporting me in this task.

Stan Osher, Peter Olver, Ron Kimmel, Vicent Caselles, Marcelo Bertalmio, and Dario Ringach read parts of this book and provided a lot of corrections and ideas. I thank them enormously for this volunteer job.

While writing this book, I was supported by grants from the U.S. Office of Naval Research and the U.S. National Science Foundation. The support of these two agencies allowed me to spend hours, days, and months writing, and it is greatfully acknowledged. I especially thank my Program Managers John Cozzens (National Science Foundation), Dick Lau (U.S. Office of Naval Research), Wen Masters (U.S. Office of Naval Research), and Carey Schwartz (U.S. Office of Naval Research) for supporting and encouraging me from the early days of my career.

My editor, Dr. Alan Harvey, helped a lot with my first experience writing a book.

The long hours I spent writing this book I could have spent with my family. So, without their acceptance of this project I could not write the book. This is why the book is dedicated to them.

Collaborations

As mentioned above, portions of this book describe results obtained in collaboration with colleagues and friends, particularly the following:

- The work on affine curve evolution was done with Allen Tannenbaum, and Peter Olver joined us when it was generalized to other groups and dimensions.
- The work on mathematical morphology is the result of collaboration with Ronny Kimmel, Doron Shaked, and Alfred Bruckstein.
- The initial work on geodesic active contours is in collaboration with Vicent Caselles and Ronny Kimmel.
- The work on morphing contours and the work on tracking regions on level sets is in collaboration with Marcelo Bertalmio and Gregory Randall.

- The work on edge tracing is in collaboration with Luis Vazquez and Do Hyun Chung.
- The work on affine invariant detection is in collaboration with Allen Tannenbaum and Peter Olver.
- The work on robust diffusion is in collaboration with Michael Black, David Heeger, and Dave Marimont.
- The work on the introduction of prior knowledge into anisotropic diffusion started with Brian Wandell and Patrick Teo and was followed up with Steven Haker, Allen Tannenbaum, and Alvaro Pardo.
- The work on vector-valued diffusion is in part with Dario Ringach and in part with Vicent Caselles and Do Hyun Chung.
- The work on image inpainting is in collaboration with Marcelo Bertalmio, Vicent Caselles, and Coloma Ballester.
- The original work on diffusion on general manifolds by means of harmonic maps is in collaboration with Vicent Caselles and Bei Tang. It continued with Alvaro Pardo, and extensions follow with Stan Osher, Marcelo Bertalmio, Facundo Memoli, and Li-Tien Cheng.
- The work on contrast enhancement started with Vicent Caselles and then continued with Vicent Caselles, Jose Luis Lisani, and Jean Michael Morel.
- Many additional images were produced with the software package developed in collaboration with Do Hyun Chung.
- Images from their own work were provided by V. Caselles, T. Chan, R. Deriche, R. Kimmel, J. M. Morel, S. Osher, N. Paragios, L. Vese, C. K. Wong, and H. Zhao. Additional images were provided by my graduate students.

Introduction

The use of partial differential equations (PDEs) and curvature-driven flows in image analysis has become an interest-raising research topic in the past few years. The basic idea is to deform a given curve, surface, or image with a PDE, and obtain the desired result as the solution of this PDE. Sometimes, as in the case of color images, a system of coupled PDEs is used. The art behind this technique is in the design and analysis of these PDEs.

Partial differential equations can be obtained from variational problems. Assume a variational approach to an image processing problem formulated as

$$\arg\{\mathrm{Min}_I \mathcal{U}(u)\},$$

where $\mathcal{U}$ is a given energy computed over the image (or surface) I. Let $\mathcal{F}(\Phi)$ denote the Euler derivative (first variation) of $\mathcal{U}$. Because under general assumptions a necessary condition for I to be a minimizer of $\mathcal{U}$ is that $\mathcal{F}(I) = 0$, the (local) minima may be computed by means of the steady-state solution of the equation

$$\frac{\partial I}{\partial t} = \mathcal{F}(I),$$

where t is an artificial time-marching parameter. PDEs obtained in this way have already been used for quite some time in computer vision and image processing, and the literature is large. The most classical example is the Dirichlet integral,

$$\mathcal{U}(I) = \int |\nabla I|^2(x)\,\mathrm{d}x,$$

which is associated with the linear heat equation

$$\frac{\partial I}{\partial t}(t, x) = \Delta I(x).$$

More recently, extensive research is being done on the direct derivation of evolution equations that are not necessarily obtained from the energy approaches. Both types of PDEs are studied in this book.

Clearly, when introducing a new approach to a given research area, one must justify its possible advantages. Using partial differential equations and curve/surface flows in image analysis leads to modeling images in a continuous domain. This simplifies the formalism, which becomes grid-independent and isotropic. The understanding of discrete local nonlinear filters is facilitated when one lets the grid mesh tend to zero and, thanks to an asymptotic expansion, one rewrites the discrete filter as a partial differential operator.

Conversely, when the image is represented as a continuous signal, PDEs can be seen as the iteration of local filters with an infinitesimal neighborhood. This interpretation of PDEs allows one to unify and classify a number of the known iterated filters as well as to derive new ones. Actually, we can classify all the PDEs that satisfy several stability requirements for image processing such as locality and causality [5].

Another important advantage of the PDE approach is the possibility of achieving high speed, accuracy, and stability with the help of the extensive available research on numerical analysis. Of course, when considering PDEs for image processing and numerical implementations, we are dealing with derivatives of nonsmooth signals, and the right framework must be defined. The theory of *viscosity solutions* provides a framework for rigorously using a partial differential formalism, in spite of the fact that the image may be not smooth enough to give a classical sense to derivatives involved in the PDE. Last, but not least, this area has a unique level of formal analysis, giving the possibility of providing not only successful algorithms but also useful theoretical results such as existence and uniqueness of solutions.

Ideas on the use of PDEs in image processing go back at least to Gabor [146] and, a bit more recently, to Jain [196]. However, the field really took off thanks to the independent works of Koenderink [218] and Witkin [413]. These researchers rigorously introduced the notion of *scale space*, that is, the representation of images simultaneously at multiple scales. Their seminal contribution is, to a large extent, the basis of most of the research in PDEs for image processing. In their work, the multiscale image representation is obtained by Gaussian filtering. This is equivalent to deforming the original image by means of the classical heat equation, obtaining in this way an isotropic diffusion flow. In the late 1980s, Hummel [191] noted that the heat flow is not the only parabolic PDE that can be used to create a scale space, and indeed he argued that an evolution equation that satisfies the maximum principle will define a scale space as well (all these concepts will be described in this book). Maximum principle appears to be a natural mathematical translation of *causality*. Koenderink once again made a major contribution into the PDE arena (this time probably involuntarily, as the consequences were not clear at all in his original formulation), when he suggested adding a thresholding operation to the process of Gaussian filtering. As later suggested by Osher and his colleagues and as proved by a number of groups, this leads to a geometric PDE, actually, one of the most famous ones: curvature motion.

The work of Perona and Malik [310] on anisotropic diffusion has been one of the most influential papers in the area. They proposed replacing Gaussian smoothing, equivalent to isotropic diffusion by means of the heat flow, with a selective diffusion that preserves edges. Their work opened a number of theoretical and practical questions that continue to occupy the PDE image processing community, e.g., Refs. [6 and 324]. In the same framework, the seminal works of Osher and Rudin on shock filters [293] and Rudin et al. [331] on total variation decreasing methods explicitly stated the importance and the need for understanding PDEs for image processing applications. At approximately the same time, Price et al. published a very interesting paper on the use of Turing's reaction-diffusion theory for a number of image processing problems [319]. Reaction-diffusion equations were also suggested to create artificial textures [394, 414]. In Ref. [5] the authors showed that a number of basic axioms lead to basic and fundamental PDEs.

Many of the PDEs used in image processing and computer vision are based on moving curves and surfaces with curvature-based velocities. In this area, the level-set numerical method developed by Osher and Sethian [294] was very influential and crucial. Early developments on this idea were provided by Ohta et al. [274], and their equations were first suggested for shape analysis in computer vision in Ref. [204]. The basic idea is to represent the deforming curve, surface, or image as the level set of a higher dimensional hypersurface. This technique not only provides more accurate numerical implementations but also solves topological issues that were previously very difficult to treat. The representation of objects as level sets (zero sets) is of course not completely new to the computer vision and image processing communities, as it is one of the fundamental techniques in mathematical morphology [356]. Considering the image itself as a collection of its level sets and not just as the level set of a higher dimensional function is a key concept in the PDE community [5].

Other works, such as the segmentation approach of Mumford and Shah [265] and the snakes of Kass et al. [198] have been very influential in the PDE community as well.

It should be noted that a number of the above approaches rely quite heavily on a large number of mathematical advances in differential geometry for curve evolution [162] and in viscosity solution theory for curvature motion (see, e.g., Evans and Spruck [126].)

Of course, the frameworks of PDEs and geometry-driven diffusion have been applied to many problems in image processing and computer vision since the seminal works mentioned above. Examples include continuous mathematical morphology, invariant shape analysis, shape from shading, segmentation, tracking, object detection, optical flow, stereo, image denoising, image sharpening, contrast enhancement, and image quantization. Many of these contributions are discussed in this book.

This book provides the basic mathematical background necessary for understanding the literature in PDEs applied to image analysis. Fundamental topics such as differential geometry, PDEs, calculus of variations, and numerical analysis are covered. Then the basic concepts and applications of surface evolution theory and PDEs are presented.

It is technically impossible to cover in a single book all the great literature in the area, especially when the area is still very active. This book is based on the author's own experience and view of the field. I apologize in advance to those researchers whose outstanding contributions are not covered in this book (maybe there is a reason for a sequel!). I expect that the reader of this book will be prepared to continue reading the abundant literature related to PDEs. Important sources of literature are the excellent collection of papers in the book edited by Romeny [324], the book by Guichard and Morel [168], which contains an outstanding description of the topic from the point of view of iterated infinitesimal filters, Sethian's book on level sets [361], which are covered in a very readable and comprehensive form, Osher's long-expected book (hopefully ready soon; until then see the review paper in Ref. [290]), Lindeberg's book, a classic in scale-space theory [242], Weickert's book on anisotropic diffusion in image processing [404], Kimmel's lecture notes [207], Toga's book on brain warping [389], which includes a number of PDE-based algorithms for this, the special issue (March 1998) of the *IEEE Transactions on Image Processing* (March 1998), the special issues of the *Journal of Visual Communication and Image Representation* (April 2000