

Chromatographic Methods

Abdel Salam Said

Theory and Mathematics of Chromatography

Hüthig

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by Abdel Salam Said

with 44 Figures

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Preface

Chromatography has been a subject where experiment is ahead of theory, and even though the literature on the theory is quite extensive yet a great part of it is either repetitive or unnecessary.

Until now chromatography has been paying off generously with little help from theory, but as time goes by and as evident applications are exhausted workers in the field will have to rely more and more on guidance from theory.

This book represents an attempt to collect the fundamental principles of the subject in a concise volume and to give students of chromatography enough background in the mathematics needed for them to pursue their research activities in the various branches of this important field.

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A. S. Said

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I Introduction

Until now, theory has played a secondary role in chromatography. There have been so many obvious applications and efforts have so far been directed mainly towards getting the most out of these applications. Scientists thus had no time to concentrate on theory. All they wanted from theory was just a rough guide as to where they could start experimenting, but they were willing to use trial and error experimentation to achieve their goal

A scientist would develop his own technique and his own theory and publish them both. It is not a general rule that the best experimentalist is always the best theoretician or mathematician, and a good experiment may carry so much unneeded, possibly dubious theoretical ballast that the literature on the theory ultimately becomes swollen with repetitive and unnecessary material.

This has been the general trend as far as theory goes in chromatography, and it seems that this trend will continue for some time to come. It is a free-for-all that many workers in this field seem to enjoy and we will have to live with it whether we like it or not. After all, chromatography is a rich and flourishing field and can afford some irregularities of this kind, at least for the time being.

In a situation like this it would appear essential that chemists be well equipped mathematically so that they may retain for themselves the original ideas in the literature and recognize repetitive or unnecessary data as such. An adequate training in mathematics could have saved much of the effort wasted on mathematical problems of practical value like the problem of analytical resolution of overlapping peaks. The number of articles published on this subject is certainly not in proportion to the mathematics involved.

It is quite an important practical problem and should indeed be settled satisfactorily, but a fair knowledge of the mathematics involved would serve as a reliable guide as to which approach is likely to lead to useful results.

The main aim of this boook, therefore, is to present the fundamentals of the theory of chromatography and to provide the reader with the mathematics necessary for understanding this part. It also provides the basis for more advanced topics of the theory and the corresponding higher level mathematics, like partial differential equations and statistical mathematics.

With chromatography at this high level of practical importance it should also assume the same level in theory and should be acknowledged in colleges as a subject in its own right, alongside others like spectrophotometry and electrochemical analysis. There should be standard courses including the basic fundamentals and it is hoped that this book can contribute in this direction.

Since this book was intended more as a text book for college students, the main aim was to present the material according to its logical development rather than its historical development. On the subject of resolution, for example, it was difficult to start with a presentation of *Purnell's* treatment which dealt with a special case ($w_1 \cong w_2$). The logical procedure was to present the general case first and deduce from it other formulae as special cases.

There are several treatments of the plate theory; most important among them is the original algebraic treatment by *Martin and Synge*. There is also the *Glueckauf* treatment which is quite complex and is incorrect in parts, but authors keep reproducing it in books and monographs on the subject, probably because *Glueckauf* was able to extract from it quite useful results. However, this is a price many students will not be willing to pay.

There is also the stochastic random walk model treatment of the theory by *Giddings*, which leads to the final results in a simple and elegant way. Unfortunately, however, many students of chemistry and chemical engineering for whom this book is intended still

find the concepts of probability and statistics quite perplexing.

The calculus treatment as presented in this book seems to be the logical approach for a student who has some previous knowledge of elementary differential equations. As can be seen, this treatment may be extended to many cases of practical interest like a zone of finite size, the cases of deposition and washing, and even to gradient elution leading to simple analytic expressions. Furthermore, it can also be used in the development of difference equations utilized in the numerical solution for the case of the non-linear adsorption isotherm.

No references are included in the body of the text but a short bibliography is given at the end. The references were chosen mostly from work with which the author is most familiar including his own publications which represent the main source of the material presented in this book.

In doing this the author was well aware of the confusion that may arise if the student gets involved in treatments which might look different but really amount to the same thing. Sometimes one may find an algebraic treatment of an original idea, and a subsequent treatment of the very same idea in calculus. The latter represents an improvement, and the calculus treatment should replace the algebraic one. On the other hand, the literature on the theory is full of examples where an algebraic development follows an elegant and much simpler calculus treatment. In this case, the second treatment is unnecessary and trivial, and can help only in adding even more confusion to the already confused literature on the theory of chromatography.

In the mathematics part, no references to text books of mathematics are given. Almost any mathematics book on the respective subject may be consulted. Much of the material presented was taken from the author's personal notes

A short list of unsolved problems is also given at the end of the book. They are meant for students wishing to familiarize themselves with some of the equations derived. Solutions to these problems may be obtained from the author on request.

Limitations of space precluded an exhaustive treatment of the entire subject, and the material included was selected according to the dual criteria of importance and simplicity. A future monograph will be devoted to more advanced topics and their applications. A detailed account of the rate theory together with the non-linear isotherm were considered as advanced topics together with the mathematics that goes with them. Emphasis is placed on the plate theory of chromatography throughout Part III.

There have been many unkind statements and remarks made about the plate theory; probably the most serious among these statements are the following:

- 1) It is naive to talk about plates in chromatography. There are no such things as plates in a chromatographic column.
- 2) The plate model is not a realistic model because it assumes equilibrium between the solute in the mobile phase and the solute in the stationary phase, and such equilibrium does not exist.
- 3) The number of theoretical plates does not mean anything. A column having a large number of theoretical plates can give less resolution than one having a smaller number of plates.

Of course all these statements are misleading and incorrect.

In answer to the first statement, one can say that the plate theory actually states that the plate model is only a mathematical model or that the chromatographic column is mathematically equivalent to a plate column where equilibrium is established for the solute between the mobile and stationary phases on each plate and that the greater the number of these plates or equilibrium stages, the better is the column and the more resolution we get from it.

Similarly, in answer to the second statement one can say that, in effect, the plate theory states that no equilibrium exists at any point between the mobile phase and the stationary phase and that the mobile phase needs to move a certain distance H up the column before the solute concentration in it is in equilibrium with the average solute concentration in the station-

ary phase in this distance H . This distance is called the HETP or the height equivalent to a theoretical plate. It is clear also that the larger the solute diffusivity in the liquid phase, the smaller the diffusivity in the gas phase, and the greater the interfacial area, the smaller is the value of the HETP and the greater is the number of theoretical plates N .

And finally, the answer to the third statement is quite straightforward. It is simply that resolution is a function of both the number of theoretical plates N and the void peak retention volume and not a function of N only.

It is not the aim of the plate theory to find the relation between H and column parameters like velocity, polarity, and flow pattern of mobile phase, and diffusivities in both phases, and thickness and polarity of the stationary phase. This is rather the function of the rate theory. Once H is known either from the rate theory or experimentally from the peak width and retention volume according to equations derived from the plate theory itself, the value H can be fed into equations deduced from the plate model to obtain useful results as resolution, peak capacity, and peak shape. The theoretical plate concept and the alternative concept of HTU (the height of a transfer unit) are well established and useful concepts in many chemical engineering unit operations as in distillation packed columns, in absorption and adsorption columns, and other unit operations so why not in chromatography as well.

It is a good sign, however, that criticism of the plate theory is waning and that more and more scientists share this author's views about it. It is particularly useful in an introductory course for chromatography students because it demonstrates in a simple and elegant way both qualitatively and quantitatively the mechanism of zone separation on a column, and an elementary or intermediate book on the theory cannot afford to ignore the plate theory of chromatography. This author is quite convinced that the plate theory is here to stay.

The role of the rate theory is not only to get values of H to be fed into plate theory equations but also to calculate the parameters which will lead to the optimum selection of mobile and stationary phases, mobile phase velocity, etc. in order to get the smallest value

of H or the best possible resolution. In fact the two theories complement one another. They are not two rivals as many think. Let us hope that both will win.

Finally, it is sincerely hoped that this book will stimulate the interest of chemistry and chemical engineering students in this highly important and useful field of chromatography. There is much to be done if we want this technique to attain a high level of perfection and if we want finally to have efficient preparative and continuous chromatography. I trust that this book is a step in the right direction towards achieving this goal.

II Mathematics for Chromatography

Mathematics has been used extensively throughout the development of the theory of chromatography. Because authors found it necessary to elaborate on pure mathematical derivations, the theory of chromatography has become unduly lengthy and repetitive. In many instances, exactly the same result is obtained by using different mathematical approaches. One of these might be precise and elegant and should consequently be retained whereas the others may serve only as exercises but not as alternative derivations. Of course, the author who first reaches a certain scientific result will receive credit for it whether he used the best approach or not, but the reader would certainly prefer an easier or a more elegant derivation if there is one.

The purpose of this part of the present book is to provide the reader with enough mathematical background to enable him to choose for himself the best mathematical approach to a certain result and to sort out trivial, repetitious, or even wrong equations which are not uncommon in the literature pertaining to chromatographic theory. Another purpose of this section is to collect most of the mathematics encountered in the theory in one place so that it might serve as a reference source throughout the next section dealing with the theory of chromatography. No pure mathematical details will be included in the theory. This will considerably reduce its extent and make it concise and more to the point.

Since this book is intended as a textbook for undergraduate as well as graduate students it was found necessary to exclude some parts of the theory and the mathematics associated with it. This is not because they are unimportant, but because they were classified as more advanced subjects. The rate theory

of chromatography is not discussed in detail and the partial differential equations needed for its development are not included. Such subjects and other topics of a more advanced nature should constitute the subject matter of an advanced textbook on the theory and mathematics of chromatography.

Some of the material included in this section will probably appear quite simple and elementary to some readers. It was found necessary to include this elementary material for the sake of continuity and also because it will be referred to quite often in the theory of chromatography. Thus we shall start with simple mathematics, namely logarithms and solution of algebraic equations, and then proceed gradually to more advanced subjects. Worked examples are included in several chapters.

1 Solution of Algebraic Equations

1.1 Simultaneous First Order Linear Equations

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = a_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = a_2$$

$$\vdots$$

$$a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n = a_n$$

x_1, x_2, \dots, x_n are the unknowns and a are constant coefficients.

The solution of this system of n first order linear algebraic equations is given by

$$x_r = \frac{D_r}{D}. \quad (1)$$

The symbol D denotes a determinant

$$D = \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{vmatrix}.$$

D_r is the same determinant as D except that the column corresponding to r is replaced by the a -column on the right hand side.

Example 1

Solve the following two simultaneous equations

$$2x - y = 5$$

$$3x + 2y = 11.$$

The solution is easily accomplished by multiplying the top equation by 2 and adding the two equations together, to give

$$7x = 21 \quad \text{or} \quad x = 3.$$

Substituting this value in the first equation gives $y = 1$.

Alternatively, using determinants gives

$$x = \frac{D_1}{D} = \frac{\begin{vmatrix} 5 & -1 \\ 11 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}} = \frac{5 \times 2 - (-1 \times 11)}{2 \times 2 - (-1 \times 3)} = 3$$

$$y = \frac{D_2}{D} = \frac{\begin{vmatrix} 2 & 5 \\ 3 & 11 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}} = \frac{2 \times 11 - 3 \times 5}{7} = 1.$$

Example 2

The following is an example of how to calculate the value of a 3×3 determinant:

$$\begin{vmatrix} 2 & -3 & 5 \\ -1 & 2 & -4 \\ 4 & 1 & 3 \end{vmatrix} = 2[2 \times 3 - (-4 \times 1)] - (-3)[(-1 \times 3) - (-4 \times 4)] + 5[(-1 \times 1) - (2 \times 4)] = 14.$$

When the number of simultaneous equations n is more than 3, the first method in example 1 becomes complex and the use of determinants is necessary. For large values of n , computers are used; sub-routines are available for any n .

1.2 Second Order Algebraic Equations

$$ax^2 + bx + c = 0 \quad (2)$$

x is given by the well known formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (3)$$

1.3 Third Order and Other Algebraic Equations

$$x^3 + bx^2 + cx + d = 0 \quad (4)$$

There is an analytical solution for this equation but it is not simple enough. There is also an analytical solution, though quite complex, for the fourth order algebraic equation. For higher order algebraic equations, analytical solutions are impossible except in a few special cases. For such equations and for all