Lecture Notes in Mathematics

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Jay Jorgenson
Serge Lang

$Pos_n(R)$ and Eisenstein Series

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Lecture Notes in Mathematics

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Preface

We are engaged in developing a systematic theory of theta and zeta functions, to be applied simultaneously to geometric and number theoretic situations in a more extensive setting than has been done up to now. To carry out our program, we had to learn some classical material in several areas, and it wasn't clear to us what would simultaneously provide enough generality to show the effectiveness of some new methods (involving the heat kernel, among other things), while at the same time keeping knowledge of some background (e.g. Lie theory) to a minimum. Thus we experimented with the quadratic model of G/K in the simplest case $G = \mathrm{GL}_n(\mathbf{R})$. Ultimately, we gave up on the quadratic model, and reverted to the G/K framework used systematically by the Lie industry. However, the quadratic model still serves occasionally to verify some things explicitly and concretely for instance in elementary differential geometry.

The quadratic forms people see the situation on $K\backslash G$, with right G-action. We retabulated all the formulas with left G-action. Just this may be useful for readers since the shift from right to left is ongoing, but not yet universal.

Some other people have found our notes useful. For instance, we include some reduction theory and Siegel's formula (after Hlawka's work). We carry out with some variations material in Maass [Maa 71], dealing with $\mathrm{GL}_n(\mathbf{R})$, but also include more material than Maass. We have done some things hinted at in Terras [Ter 88]. Her inclusion of proofs is very sporadic, and she leaves too many "exercises" for the reader. Our exposition is self-contained and can be used as a naive introduction to Fourier analysis and special functions on spaces of type G/K, making it easier to get into more sophisticated treatments.

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Contents

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$\mathrm{GL}_n(\mathbf{R})$ Action on $\mathrm{Pos}_n(\mathbf{R})$

Let $G = \mathrm{GL}_n(\mathbf{R})$ or $\mathrm{SL}_n(\mathbf{R})$ and $\Gamma_n = \mathrm{GL}_n(\mathbf{Z})$. Let $\mathrm{Pos}_n(\mathbf{R})$ be the space of positive symmetric real $n \times n$ matrices. Recall that symmetric real $n \times n$ matrices Z have an ordering, defined by $Z \geqq 0$ if and only if $\langle Zx, x\rangle \geqq 0$ for all $x \in \mathbf{R}^n$. We write $Z_1 \geqq Z_2$ if and only if $Z_1 - Z_2 \geqq 0$. If $Z \geqq 0$ and Z is non-singular, then $Z > 0$, and in fact $Z \geqq \lambda I$ if λ is the smallest, necessarily positive, eigenvalue.

The group G acts on $\mathrm{Pos}_n(\mathbf{R})$ by associating with each $g \in G$ the automorphism (for the C^∞ or real analytic structure) of Pos_n given by

$$[g]Z = gZ^t gs \ .$$

We are interested in $\Gamma_n \backslash \mathrm{Pos}_n(\mathbf{R})$, and we are especially interested in its topological structure, coordinate representations, and compactifications which then allow effective computations of volumes, spectral analysis, differential geometric invariants such as curvature, and heat kernels, and whatever else comes up.

The present chapter deals with finding inductively a nice fundamental domain and establishing coordinates which are immediately applied to describe Grenier's compactification, following Satake.

Quite generally, let X be a locally compact topological space, and let Γ be a discrete group acting on X. Let Γ_0 be the kernel of the representation $\Gamma \to \mathrm{Aut}(X)$. A **strict fundamental domain** F for Γ is a Borel measurable subset of X such that X is the disjoint union of the translates γF for $\gamma \in \Gamma/\Gamma_0$. In most practices, X is also a C^∞ manifold of finite dimension. We define a **fundamental domain** F to be a measurable subset of X such that X is the union of the translates γF, and if $\gamma x \in F$ for some $\gamma \in \Gamma$, and γ does not act trivially on X, then x and γx are on the boundary of F. In practice, this boundary will be reasonable, and in particular, in the cases we look at, this boundary will consist of a finite union of hypersurfaces. By resolution of singularities, the boundary can then be parametrized by C^∞ maps defined on cubes of Euclidean space of dimension $\leqq \dim X - 1$. Thus the boundary has n-dimensional measure 0.

In this chapter, we have essentially reproduced aspects of Grenier's papers [Gre 88] and [Gre 93]. He carried out on $\mathrm{GL}_n(\mathbf{R})$ and $\mathrm{SL}_n(\mathbf{R})$ Satake's compactification of the Siegel upper half space [Sat 56], [Sat 58], see also Satake's general results [Sat 60]. It was useful to have Grenier's special case worked out in the literature, especially Grenier's direct inductive method.

Note that to a large extent, this chapter proves results compiled in Borel [Bor 69], with a variation of language and proofs. These are used systematically in treatments of Eisenstein series, partly later in this book, and previously for instance in Harish-Chandra [Har 68].

1 Iwasawa-Jacobi Decomposition

Let:
$G = G_n = \mathrm{GL}_n(\mathbf{R})$
$\mathrm{Pos}_n = \mathrm{Pos}_n(\mathbf{R})$ = space of symmetric positive real matrices
$K = O(n) = \mathrm{Uni}_n(\mathbf{R})$ = group of real unitary $n \times n$ matrices
U = group of real unipotent upper triangular matrices, i.e. of the form

$$u(X) = u = \begin{pmatrix} 1 & & & x_{ij} \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & 1 \end{pmatrix} \text{ so } u(X) = I + X,$$

with

$$X = (x_{ij}), 1 < i < j \leqq n\ .$$

A = group of diagonal matrices with positive components,

$$a = \begin{pmatrix} a_1 & & & \\ & a_2 & & 0 \\ & & \ddots & \\ 0 & & & \\ & & & a_n \end{pmatrix} \quad a_i > 0 \text{ all } i\ .$$

Theorem 1.1. *The product mapping*

$$U \times A \times K \to UAK = G$$

is a differential isomorphism. Actually, the map

$$U \times A \to \mathrm{Pos}_n(\mathbf{R}) \text{ given by } (u, a) \mapsto ua\,{}^t u$$

is a differential isomorphism.

Proof. Let $\{e_1, \ldots, e_n\}$ be the standard unit vectors of $\mathbf{R}^n$, and let $x \in \mathrm{L}_n(\mathbf{R})$. Let $v_i = xe_i$. We orthogonalize $\{v_1, \ldots, v_n\}$ by the standard Gram-Schmidt process, so we let

$$\begin{aligned} w_1 &= v_1, \\ w_2 &= v_2 - c_{21}w_1\, w_1, \\ w_3 &= v_3 - c_{32}w_2 - c_{31}w_2 \perp w_1 \text{ and } w_2 \end{aligned}$$

and so on. Then $e_i' = w_i/\|w_i\|$ is a unit vector, and the matrix a having $\|w_i\|^{-1}$ for its diagonal elements is in A. Let $k = aux$ so $x = u^{-1}a^{-1}k$. Then k is unitary, which proves that $G = UAK$. To show uniqueness, suppose that

$$u_1 a\, {}^t u_1 = u_2 b\, {}^t u_2 \text{ with } u_1, u_2 \in U \text{ and } a, b \in A,$$

then putting $u = u_2^{-1}u_1$ we find

$$ua = b\, {}^t u\ .$$

Since u and ${}^t u$ are triangular in opposite direction, they must be diagonal, and finally $a = b$. That the decomposition is differentially a product is proved by computing the Jacobian of the product map, done in Chap. 2.

The group K is the subset of elements of G fixed under the involution

$$g \mapsto {}^t g^{-1}\ .$$

We write the transpose on the left to balance the inverse on the right. We have a surjective mapping

$$G \to \mathrm{Pos}_n \text{ given by } g \mapsto g\, {}^t g\ .$$

This mapping gives a bijection of the coset space

$$\varphi : G/K \to \mathrm{Pos}_n\ ,$$

and this bijection is a real analytic isomorphism. Furthermore, the group G acts on Pos_n by a homomorphism $g \mapsto [g] \in \mathrm{Aut}(\mathrm{Pos}_n)$, where $[g]$ is given by the formula

$$[g]p = gp\, {}^t g\ .$$

This action is on the left, contrary to right wing action by some people. On the other hand, there is an action of G on the coset space G/K by translation

$$\tau : G \to \mathrm{Aut}(G/K) \text{ such that } \tau(g)g_1K = gg_1K\ .$$

Under the bijection φ, a translation $\tau(g)$ corresponds precisely to the action $[g]$.

Next we tabulate some results on partial (inductive) Iwasawa decompositions. These results are purely algebraic, and do not depend on real matrices or positivity. They depend only on routine matrix computations, and it will prove useful to have gotten them out of the way systematically.

Let $G = \mathrm{GL}_n$ denote the general linear group, wherever it has its components. Vectors are column vectors. An element $g \in \mathrm{GL}_n$ can be written

$$g = \begin{pmatrix} A & b \\ {}^tc & d \end{pmatrix} = \begin{pmatrix} & & b_1 \\ & \boxed{\mathrm{A}} & \vdots \\ & & b_{n-1} \\ c_1 & \cdots c_{n-1} & d \end{pmatrix}$$

where b and c are $(n-1)$-vectors, so ${}^tc$ is a row vector of dimension $n-1$, A is an $(n-1) \times (n-1)$ matrix, and d is a scalar. We write

$$d = \mathbf{d}_n(g)$$

for this lower right corner of g.

For induction purposes, we do not deal with fully diagonal matrices but with an inductive decomposition

$$\begin{pmatrix} W & 0^{(n-1)} \\ {}^t0^{(n-1)} & v \end{pmatrix} = \begin{pmatrix} W & 0 \\ 0 & v \end{pmatrix} \quad \text{with } W \in \mathrm{GL}_{n-1} \text{ and } v \text{ scalar } \neq 0\,.$$

We have the left action of GL on Mat_n given on a matrix M by

$$[g]M = gM{}^tg,$$

so $g \mapsto [g]$ is a representation. For an $(n-1)$-vector x, we denote

$$u(x) = \begin{pmatrix} I_{n-1} & x \\ 0 & 1 \end{pmatrix}.$$

We write ${}^tx = (x_1, \ldots, x_{n-1})$. Then $x \mapsto u(x)$ is an injective homomorphism. In particular,

$$u(x)^{-1} = u(-x)\,.$$

The usual matrix multiplication works to yield

$$gu(x) = \begin{pmatrix} A & Ax+b \\ {}^tc & {}^tcx+d \end{pmatrix} \tag{1}$$

An expression

$$Z = [u(x)]\begin{pmatrix} W & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} W+[x]v & xv \\ v{}^tx & v \end{pmatrix}$$

for a matrix Z will be called a **first order Iwasawa decomposition** of Z. We note that with such a decomposition, we have

$$\mathbf{d}_n(Z) = v\,. \tag{2}$$

Straightforward matrix multiplication yields the expression:

$$(3)\ [g]Z = [g][u(x)]\begin{pmatrix} W & 0 \\ 0 & v \end{pmatrix}$$

$$= \begin{pmatrix} [A]W + [Ax+b]v & AW_c + (Ax+b)v({}^t xc + d) \\ {}^t cW{}^t A + ({}^t cx + d)v^t(Ax+b) & [{}^t c]W + [{}^t cx + d]v \end{pmatrix}.$$

In particular,

$$\mathbf{d}_n([g]Z) = [{}^t c]W + [{}^t cx + d]v = [{}^t c]W + ({}^t cx + d)^2 v\,. \tag{4}$$

Indeed, ${}^t cx$ is a scalar, so is ${}^t cx + d$, so $[{}^t cx + d]v = ({}^t x + d)^2 v$. Note that directly from matrix multiplication, one has also

$$\mathbf{d}_n([g]Z) = [{}^t c, d]Z\,. \tag{5}$$

For later purposes, we record the action of a semidiagonalized matrix:

$$\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} Z = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} I_{n-1} & x \\ 0 & 1 \end{bmatrix}\begin{pmatrix} W & 0 \\ 0 & v \end{pmatrix} \tag{6}$$

$$= \begin{bmatrix} I_{n-1} & Ax \\ 0 & 1 \end{bmatrix}\begin{pmatrix} [A]W & 0 \\ 0 & v \end{pmatrix}.$$

One way to see this is to multiply both sides of (6) by

$$[u(-Ax)] = [u(Ax)]^{-1},$$

and to verify directly the identity

$$\begin{pmatrix} I_{n-1} & -Ax \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} I_{n-1} & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}. \tag{7}$$

We have a trivial action

$$\begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix}\begin{pmatrix} W & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} W & 0 \\ 0 & v \end{pmatrix}. \tag{8}$$

In other words, $\begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix}$ acts as the identity on a semidiagonalized matrix $\begin{pmatrix} W & 0 \\ 0 & v \end{pmatrix}$. On the other hand, on $u(x)$ we can effect a change of sign by the transformation

$$\begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} I_{n-1} & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I_{n-1} & -x \\ 0 & 1 \end{pmatrix} . \tag{9}$$

We then derive the identity

$$\begin{aligned} &\begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} I_{n-1} & x \\ 0 & 1 \end{bmatrix} \begin{pmatrix} W & 0 \\ 0 & v \end{pmatrix} \\ &= \begin{bmatrix} I_{n-1} & -x \\ 0 & 1 \end{bmatrix} \begin{pmatrix} W & 0 \\ 0 & v \end{pmatrix} . \end{aligned} \tag{10}$$

Indeed, in the left side of (10) we insert

$$\begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 1 \end{bmatrix}$$

just before $\begin{pmatrix} W & 0 \\ 0 & v \end{pmatrix}$, and use (8), (9) to obtain the right side, and thus prove (10).

2 Inductive Construction of the Grenier Fundamental Domain

This section is taken from [Gre 88].

Throughout the section, we let:

$G = \mathrm{GL}_n(\mathbf{R})$

$\Gamma = \mathrm{GL}_n(\mathbf{Z})$

$\mathrm{Pos}_n = \mathrm{Pos}_n(\mathbf{R}) =$ space of symmetric positive $n \times n$ real matrices.

We write $Z > 0$ for positivity. We use the action of G on Pos_n given by

$$g \mapsto [g], \text{ where } [g]Z = gZ{}^tg .$$

Thus $g \mapsto [g]$ is now viewed as a representation of G in $\mathrm{Aut}(\mathrm{Pos}_n)$. We note that the kernel of this representation in Γ_n is $\pm I_n$, in other words, if $g \in \Gamma_n$ and $[g] = id$ then $g = \pm I_n$.

We use the notation of Sect. 1. An element $Z \in \mathrm{Pos}_n$ has a **first order Iwasawa decomposition**

$$Z = \begin{bmatrix} I_{n-1} & x \\ 0 & 1 \end{bmatrix} \begin{pmatrix} W & 0 \\ 0 & v \end{pmatrix}$$

with $W \in \mathrm{Pos}_{n-1}$ and $v \in \mathbf{R}^{+}$.

Since we shall deal with the discrete group Γ_n, the following fact from algebra is useful to remember.

Let R be a principal ideal ring. A vector in ${}^tR^n$ is primitive, i.e. has relatively prime components, if and only if this vector can be completed as the first (or any) row of a matrix in $GL_n(R)$.

This fact is immediately proved by induction. In dealing with $\mathbf{d}_n([g]Z)$ and $g \in \Gamma_n$, we note that this lower right component depends only on the integral row vector $({}^tc, d) \in {}^t\mathbf{Z}^n$. Here we continue to use the notation of Sect. 1, that is

$$g = \begin{pmatrix} A & b \\ {}^tc & d \end{pmatrix} .$$

Note that we have an obvious lower bound for v. If λ is the smallest eigenvalue of Z (necessarily > 0), then using the n-th unit vector e_n and the inequality $[e_n]Z \geqq \lambda \|e_n\|^2$ we find

(1) $$v \geqq \lambda .$$

For $n \geqq 2$, we define the set F_n to consist of those $Z \in \mathrm{Pos}_n$ such that:

Fun 1. $\mathbf{d}_n(Z) \leqq \mathbf{d}_n([g]Z)$ for all $g \in \Gamma_n$, or in terms of coordinates,

$$v \leqq [{}^tc]W + ({}^tcx + d)^2 v$$

for all $({}^tc, d)$ primitive in $\mathbf{Z}^n$.

Fun 2. $W \in F_{n-1}$

Fun 3. $0 \leqq x_1 \leqq \frac{1}{2}$ and $|x_j| \leqq \frac{1}{2}$ for $j = 2, \ldots, n-1$.

Minkowski had defined a **fundamental domain** Min_n by the following conditions on matrices $Z = (z_{ij}) \in \mathrm{Pos}_n$:

Min 1. For all $a \in \mathbf{Z}^n$ with $(a_i, \ldots, a_n) = 1$ we have $[{}^ta]Z \geqq z_{ii}$.

Min 2. $z_{i,i+1} \geqq 0$ for $i = 1, \ldots, n-1$.

Minkowski's method is essentially that followed by Siegel in numerous works, for instance [Sie 40], [Sie 55/56]. Grenier's induction (following Satake) is simpler in several respects, and we shall not use Minkowski's in general. Grenier followed a recursive idea of Hermite. However, we shall now see that for $n = 2$, F_2 is the same as Minkowski's Min_2.

The case $n = 2$.

We tabulate the conditions in this case. The positivity conditions imply at once that $v, w > 0$, so **Fun 2** doesn't amount to anything more. We have with $x \in \mathbf{R}$:

$$Z = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{pmatrix} w & 0 \\ 0 & v \end{pmatrix} .$$

The remaining **Fun** conditions read:

Fun 1. $v \leqq c^2 w + (cx + d)^2 v$ for all primitive vectors (c, d).
Fun 3. $0 \leqq x \leqq \frac{1}{2}$.

Proposition 2.1. *For $v, w > 0$ the above conditions are equivalent to the conditions*

$$v \leqq w + x^2 v \text{ and } 0 \leqq x \leqq \frac{1}{2} \text{ for all } x \in \mathbf{R} .$$

Under these conditions, we have

$$w \geqq \frac{3}{4} v .$$

Proof. The inequality $v \leqq w + x^2 v$ comes by taking $c = 1, d = 0$. Then

$$w \geqq v(1 - x^2),$$

and since $0 \leqq x \leqq \frac{1}{2}$, the inequality $w \geqq 3v/4$ follows. Then it also follows that for all primitive pairs (c, d) of integers, we have **Fun 1** (immediate verification), thus proving the proposition.

Write Z in terms of its coordinates:

$$Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix} = \begin{pmatrix} w + x^2 v & xv \\ vx & v \end{pmatrix} .$$

Proposition 2.2. *The inequalities* **Fun 1** *and* **Fun 3** *are equivalent to:*

$$0 \leqq 2z_{12} \leqq z_{22} \leqq z_{11} .$$

Thus F_2 is the same as the Minkowski fundamental domain Min_2. *If $z_{12} = 0$, then the inequalities are equivalent to $0 < z_{22} \leqq z_{11}$.*

Proof. The equivalence is immediate in light of the explicit determination of z_{ij} in terms of v, w and x, coming from the equality of matrices above.

After tabulating the case $n = 2$, we return to the general case. We shall prove by induction:

Theorem 2.3. *The set F_n is a fundamental domain for* $\mathrm{GL}_n(\mathbf{Z})$ *on* $\mathrm{Pos}_n(\mathbf{R})$.

Proof. The case $n = 2$ follows from the special tabulation in Proposition 2.2. So let $n \geqq 3$ and assume F_{n-1} is a fundamental domain. Let $Z \in \mathrm{Pos}_n$ and let $g \in \mathrm{GL}_n(\mathbf{Z})$ have the matrix expression of Sect. 1, so A, b, c, d are integral matrices. We begin by showing: