PROBLEM BOOK
IN THE
THEORY OF FUNCTIONS

Volume I-2

BY KONRAD KNOPP

PROBLEM BOOK IN THE THEORY OF FUNCTIONS

VOLUME I

Problems in the Elementary Theory of Functions

PROBLEM BOOK IN THE THEORY OF FUNCTIONS

VOLUME I

Problems in the Elementary Theory of Functions

By DR. KONRAD KNOPP

Professor of Mathematics at the University of Tübingen

Translated by Dr. LIPMAN BERS

Associate Professor of Mathematics at Syracuse University

DOVER PUBLICATIONS, INC.

NEW YORK

THE DOVER SERIES IN MATHEMATICS AND PHYSICS

First American Edition

PRINTED IN THE UNITED STATES OF AMERICA

CONTENTS

FOREWORD

This translation follows the second edition of K. Knopp's *Aufgabensammlung zur Funktionentheorie*, I. Teil, except for a few minor changes.

The problems of the first five chapters concern material treated in the first volume of the same author's *Theory of Functions* (referred to as KI; the references are to the translation by Bagemihl, Dover Publications, Inc., 1945.) Familiarity with the first two chapters of C. Carathéodory's *Conformal Representation* (Cambridge University Press, 1932, referred to as C) is sufficient for the understanding of the problems in Chapter VI.

The solution of a problem often depends on that of a previous one. Some problems are more difficult than others; these are marked by asterisks. On many occasions, the reader will find carefully executed sketches helpful in the solution of problems. This applies particularly to the problems in Chapter VI.

The notations in this volume are as follows: Complex numbers (and points) are denoted by z_0, z_1, $\cdots$, w_0, w_1, $\cdots$ a, b $\cdots$, complex variables by z, ζ $\cdots$ w, w $\cdots$. (In §13, however, z_1, z_2 $\cdots$ denote variables.) Numbers conjugate to z, a, $\cdots$ are denoted by $\bar{z}$, $\bar{a}$ $\cdots$.

Real constants are denoted by x_0, x_1, $\cdots$, y_0, y_1, $\cdots$, u_0, $\cdots$, v_0, α, β, $\cdots$, λ, μ $\cdots$. We write $z = x + iy = r(\cos\varphi + i\cdot\sin\varphi)$, $x = \Re(z)$, $y = \Im(z)$, $r = |z|$, $\varphi = \operatorname{am} z$.

Positive constants are denoted by $r, \rho, \delta, \epsilon \cdots$, positive integers by $m, n, p \cdots$.

Regions are denoted by capital German letters: $\mathfrak{G}$, $\mathfrak{M}$, $\cdots$, paths and curves by l. c. German and capital roman letters: $\mathfrak{s}$, $\mathfrak{p}$, $\cdots$, C. L, $\cdots$.

Part I—PROBLEMS

CHAPTER I

FUNDAMENTAL CONCEPTS

§1. Numbers and Points

(KI, 1–2)

1. Given a complex number $z_0 \neq 0$, find its reflection with respect to a) the origin, b) the real axis, c) the imaginary axis, d) the line $x - y = 0$, e) the line $x + y = 0$.

2. Show that $(1/2^{\frac{1}{2}})(|x| + |y|) \leq |z| \leq |x| + |y|$.

3. Find the loci of points z satisfying the following relations:

a) $|z| \leq 2$; b) $|z| > 2$; c) $\Re(z) \geq \frac{1}{2}$;

d) $0 \leq \Re(iz) < 2\pi$; e) $\Re(z^2) = \alpha\left(\gtreqless 0\right)$;

f) $\Im(z^2) = \alpha\left(\gtreqless 0\right)$; g) $|z^2 - z| \leq 1$;

h) $|z^2 - 1| = \alpha > 0$; i) $\left|\frac{1}{z}\right| < \delta,\ \delta > 0$;

j) $\left|\frac{z-1}{z+1}\right| \leq 1$; k) $\left|\frac{z-1}{z+1}\right| \geq 2$;

l) $\left|\frac{z}{z+1}\right| = \alpha > 0$; m) $\left|\frac{z-z_1}{z-z_2}\right| = 1$.

4. When are z_1, z_2, z_3 collinear? (Consider the difference quotient $(z_1 - z_3)/z_2 - z_3)$.)

5. When do z_1, z_2, z_3, z_4 lie on a circle or on a straight line? (Consider the cross-ratio $(z_1 - z_3)/(z_2 - z_3) \div (z_1 - z_4)/(z_2 - z_4)$.)

6. What is the geometrical meaning of the identity $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$?

7. Find the point z dividing the segment $z_1 \cdots z_2$ in the ratio $\lambda_1 \div \lambda_2$ $(\lambda_1 + \lambda_2 \neq 0)$.

8. Find the mass center of the triangle z_1, z_2, z_3 when a) each vertex z_i carries the same mass λ, b) the vertices carry the masses $\lambda_1, \lambda_2, \lambda_3$. c) Show that the mass center found in b) lies within the triangle if all three masses are positive.

9. The masses $\lambda_1, \lambda_2, \cdots, \lambda_k$ are situated at $z_1, z_2, \cdots, z_k$. Show that the mass center of this system is $z = (\lambda_1 z_1 + \lambda_2 z_2 + \cdots + \lambda_k z_k)/(\lambda_1 + \lambda_2 + \cdots + \lambda_k)$.

10. Given that $z_1 + z_2 + z_3 = 0$, $|z_1| = |z_2| = |z_3| = 1$, show that z_1, z_2, z_3 are the vertices of an equilateral triangle inscribed into the unit circle.

11. Given that $z_1 + z_2 + z_3 + z_4 = 0$, $|z_1| = |z_2| = |z_3| = |z_4| = 1$, show that z_1, z_2, z_3, z_4 are the vertices of a rectangle inscribed into the unit circle.

12. When are two triangles, z_1, z_2, z_3 and z_1', z_2', z_3' similar and similarly situated? (Cf. problem 4.)

*13. a) Given two points z_1, z_2, $|z_1| < 1$, $|z_2| < 1$, show that for every point $z \neq 1$ belonging to the triangle z_1, z_2, 1

$$\frac{|1 - z|}{1 - |z|} \leq K$$

where $K = K(z_1, z_2)$ is a constant depending only on z_1 and z_2.

b) Determine the smallest value of K for $z_1 = (1 + i)/2$, $z_2 = (1 - i)/2$.

§2. Point Sets. Paths. Regions

(KI, 3–4)

1. Show that the set of roots of algebraic equations of the form

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0,$$

the a_ν's being Gaussian integers, is countable. (A set is countable if its elements can be arranged in a sequence. A complex number z is called a Gaussian integer if $\Re(z)$ and $\Im(z)$ are real integers.)

2. Show that the set of all numbers $z = x + iy$, x and y being rational numbers, is countable.

3. Order the set of all numbers $z = 1/m + i/n$ (m, n positive integers) into a sequence.

4. Find the greatest lower bound α, the least upper bound β, the lower limit λ, and the upper limit μ of the following real sets. (Indicate whether or not α, β, λ, μ belong to the set considered).

a) The set of rational numbers p/q with even q and $p^2/q^2 \leq 10$.
b) The set of numbers of the form $(1 \pm 1/n)^n$.
c) The set of numbers of the form $(1 \pm 1/n^2)^n$.
d) The set of numbers of the form $n \pm 1/n$.
e) The set of numbers of the form $n \pm 1/3$.
f) The set of numbers of the form $1/m + 1/n$.
g) The set of numbers of the form $(1/m + 1/n)^{m+n}$.
h) The set of numbers of the form $\pm 1/m \pm 1/n$.
i) The set of numbers of the form $1 + (-1)^n + (-1)^n/n$.
k) The set of all numbers which may be written as infinite decimal fractions of the form $.\alpha_1\alpha_2\alpha_3 \cdots$, α_i odd.

(In b) to i) n and m denote arbitrary positive integers.)

*5. Show that each point of the set defined in 4k) is a limit point of the set.

6. Show that $\alpha = \lambda$ whenever α does not belong to the set and $\beta = \mu$ whenever β does not belong to the set.

7. Is the set defined by the relation $|z| + \Re(z) \leq 1$ bounded? What domain does it occupy?

8. Find all limit points of the following sets:
a) $1/m + i/n$, m and n positive integers,
b) $|z| < 1$,
c) $|z| > 1$,
d) the set defined in problem 2,
e) the set of all non-real z in the domain interior to the unit circle.

*9. Is the set defined in problem 4k) closed?

10. Is a limit point of a point set which does not belong to the set a boundary point of the set?

11. Show that a boundary point of a point set M which belongs to M is a limit point of the complementary set M'. (M' consists of all points which do not belong to M.)

12. Show that the set of all boundary points of a point set M is closed.

*13. Given two disjunct *closed* point sets M' and M'', one of which, say M', is bounded, show that there exists a positive number d such that $| z' - z'' | \geq d$ whenever z' belongs to M' and z'' to M''. Show that among all such numbers d there exists a largest number d_0.

14. Show that an arc of the continuous curve

$$y = \begin{cases} x \sin (\pi/x), & x \neq 0 \\ 0 \text{ for } x = 0 \end{cases}$$

containing the origin is not rectifiable.

15. Let $\mathfrak{M}$ consist of all points of the upper half-plane $[\mathfrak{I}(z) > 0]$ except those lying on the segments $z = it, z = \pm 1/n + it, n = 1, 2, 3, \cdots, 0 < t \leq 1$. Is $\mathfrak{M}$ a region? Find the boundary points of $\mathfrak{M}$. Is $i/2$ a boundary point? Does there exist a path leading from $z_0 = 2 + i$ to $i/2$ and situated (except for the end-point $i/2$) within $\mathfrak{M}$?

16. Consider the spiral S defined by

$$z = z(t) = \begin{cases} e^{(-1+i)/t}, & 0 < t \leqq 1, \\ 0 \text{ for } t = 0. \end{cases}$$

Is S a path leading from $z_1 = z(1)$ to $z_0 = 0$?

*17. Let $\mathfrak{G}$ be a plane region, $\mathfrak{G}_1$ its image under stereographic projection, M the set of boundary points of $\mathfrak{G}_1$. Show that $\mathfrak{G}$ is simply connected if and only if M is connected. (A closed set is called connected if it can not be divided into two closed sub-sets without a common element.)

18. Is the region defined in problem 15 simply connected?

19. Show that a simply connected region $\mathfrak{M}$ on the surface of a sphere which does not contain two points of the sphere does not contain infinitely many points of the sphere.

CHAPTER II

INFINITE SEQUENCES AND SERIES

§3. Limits of Sequences. Infinite Series with Constant Terms

(KI, 2–3)

1. Let ζ be a limit point of the sequence z_1, z_2, $\cdots$, z_n, $\cdots$. Show that the sequence contains a subsequence z'_1, z'_2, $\cdots$ which converges to ζ.

2. If $z_n \to \zeta$, then

$$z'_n = \frac{z_1 + z_2 + \cdots + z_n}{n} \to \zeta.$$

Is this true if $\zeta = \infty$?

3. If $z_n \to \zeta$, then

$$z'_n = \frac{p_1 z_1 + p_2 z_2 + \cdots + p_n z_n}{p_1 + p_2 + \cdots + p_n}$$

$$= \frac{P_1 z_1 + (P_2 - P_1) z_2 + \cdots + (P_n - P_{n-1}) z_n}{P_n} \to \zeta$$

where p_1, p_2, $\cdots$ is any sequence of positive numbers such that $P_n = (p_1 + p_2 + \cdots + p_n) \to +\infty$.

4. If $z_n \to \zeta$, then

$$z'_n = \frac{b_1 z_1 + b_2 z_2 + \cdots + b_n z_n}{b_1 + b_2 + \cdots + b_n}$$

$$= \frac{B_1 z_1 + (B_2 - B_1) z_2 + \cdots + (B_n - B_{n-1}) z_n}{B_n} \to \zeta$$

if $b_1, b_2, \cdots$ are complex numbers such that for all n the numbers $\beta_n = (|b_1 + b_2 + \cdots + b_n|)/(|b_1| + |b_2| + \cdots + |b_n|)$ exceed some fixed positive number β, and such that $(|b_1| + |b_2| + \cdots + |b_n|) \to +\infty$.

5. Let there be given infinitely many numbers $a_{\kappa\lambda}$ arranged in the form

$$\begin{array}{cccc} a_{11} & & & \\ a_{21} & a_{22} & & \\ a_{31} & a_{32} & a_{33} & \\ a_{41} & a_{42} & a_{43} & a_{44} \\ \cdots & \cdots & \cdots & \cdots \end{array}$$

and satisfying the conditions: 1) for every fixed p, $a_{np} \to 0$, 2) there exists a positive constant M such that $|a_{n1}| + |a_{n2}| + \cdots + |a_{nn}| \leq M$ for all n. Show that if $z_n \to 0$, then

$$z'_n = a_{n1}z_1 + a_{n2}z_2 + \cdots + a_{nn}z_n \to 0.$$

6. Assume that in addition to conditions 1) and 2) of the preceding problem the numbers $a_{\kappa\lambda}$ also satisfy the condition 3) $A_n = a_{n1} + a_{n2} + \cdots + a_{nn} \to 1$. Show that if $z_n \to \zeta$, then

$$z'_n = a_{n1}z_n + a_{n2}z_n + \cdots + a_{nn}z_n \to \zeta.$$

Show that this theorem contains as special cases the theorems stated in problems 2, 3, 4.

7. a) If $z'_n \to \zeta'$ and $z''_n \to \zeta''$, then

$$z_n = \frac{z'_1z''_1 + z'_2z''_2 + \cdots + z'_nz''_n}{n} \to \zeta'\zeta'',$$