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John W. Morgan   Kieran G. O'Grady

Differential Topology of Complex Surfaces

Elliptic Surfaces with $p_g=1$: Smooth Classification

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With the collaboration of Millie Niss

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Authors

John W. Morgan
Department of Mathematics
Columbia University
New York, NY 10027, USA

Kieran G. O'Grady
Institute for Advanced Study
Olden Lane
Princeton NJ 08540, USA

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Contents

1 Introduction

In [K] Kodaira introduced the operation of logarithmic transform on elliptic surfaces. He noticed that applying certain types of log transforms to an elliptic $K3$ surface produced new surfaces which were not $K3$ surfaces but were homotopy equivalent to $K3$ surfaces. He then asked about their diffeomorphism[1] classification. In this paper we shall completely answer this question – in fact, two such surfaces are diffeomorphic if and only if they are deformation equivalent which means that their multiple fibers have the same multiplicity.

This is part of a more general question about the relationship between the complex geometry and the differential topology of algebraic surfaces. For example, one can ask (cf. [D1, FM2]), "In general for complex surfaces, what is the relationship of deformation type and diffeomorphism type?" Similarly, "For minimal non-rational surfaces is the canonical class invariant up to sign under orientation-preserving diffeomorphisms?" What we do here completely answers these two questions for regular elliptic surfaces of geometric genus one.

The base of the elliptic fibration of any regular elliptic surface is the projective line. In the case when there are more than two multiple fibers the fundamental group of the elliptic surface determines the number of multiple fibers as well as the multiplicities of these multiple fibers. That is to say the fundamental group determines the deformation type, see [FM1], for example. Thus, for the rest of this paper we shall assume that the elliptic fibration has at most two multiple fibers. The fundamental group of such a surface is a cyclic group of order equal to the gcd of the multiplicities of the multiple fibers. (This means that the group is trivial if there is at most one multiple fiber.) The main body of the paper is concerned with the case of simply connected elliptic surfaces. In the appendix we extend the argument to the non-simply connected case.

Let us restrict for a moment to minimal regular elliptic surfaces of geometric genus one. Given any pair of positive integers, there is such an elliptic surface with base the projective line with multiplicities given by this pair. (By convention we set one or both of the multiplicities equal to one if there are respectively one or no multiple fibers). As is well-known, e.g. [FM1], the equivalence classes of these elliptic surfaces up to deformation type (as defined in [FM1]) are indexed by the multiplicities. Our main result is that the multiplicities are diffeomorphism invariants. Thus, we have as an immediate corollary that for these elliptic surfaces deformation type agrees with diffeomorphism type. Our second main result is that a diffeomorphism between minimal regular elliptic surfaces of geometric genus one preserves the canonical class up to sign modulo torsion. From this fact one can deduce, by the arguments given in [FM1], the analogues of these results hold for non minimal simply connected ellitpic surfaces with $p_g = 1$ as well. By way of contrast, notice that by Freedman's Theorem [Fre] there are only two homeomorphism types of such minimal elliptic surfaces – those that are spin and those that are not.

[1] He actually asked about the topological classification, but in those days the distinction between the two notions was not as clear as it is today.

1.1 Statement of the main results

If S is any algebraic surface, we let $k_S \in H^2(S;\mathbf{Q}) = c_1(K_S)$ where K_S is the canonical class of S. Let $\mathcal{E}$ be the collection of minimal regular elliptic surfaces of geometric genus one with at most two multiple fibers. To each $S \in \mathcal{E}$ we associate a pair of positive integers $(m_1(S), m_2(S)) = (m_1, m_2)$, called the *multiplicities* of S, as follows. If S has two multiple fibers F_1 and F_2, then m_1 and m_2 are their multiplicities with the proviso that if one of the multiplicities is even then we take m_1 to be this multiplicity. If S has only one multiple fiber, we set m_1 equal to the multiplicity of this fiber and m_2 equal to 1. If S has no multiple fibers (so that it is a $K3$-surface), then we set both the m_i equal to 1.

Let S be a surface in $\mathcal{E}$. Let

$$\kappa_S = c_1(F)/m_1m_2 \in H^2(S;\mathbf{Q})$$

where F is a fiber of the elliptic fibration. Notice that $gcd(m_1, m_2)\kappa_S$ is the image of an indivisible integral cohomology class. According to [FM1] each Donaldson polynomial $\gamma_c(S)$ of S is a polynomial in κ_S and the quadratic intersection form q_S of the surface. It follows from the diffeomorphism invariance of Donaldson polynomials up to sign that the diffeomorphism type determines the coefficients up to sign. We shall compute two of the coefficients in this expansion for the invariant $\gamma_3(S)$ as functions of the multiplicities. An simple analysis of these functions shows that they determine the multiplicities.

Our main result is the following partial evaluation of the Donaldson polynomial.

Theorem 1.1.1 *Let S be a minimal elliptic surface over $\mathbf{P}^1$ with at most two multiple fibers and with $p_g(S) = 1$. Let (m_1, m_2) be its multiplicities. Let $\gamma_3(S)$ be the unstable Donaldson polynomial as defined in Chapter 3 of [FM1]. It is a polynomial function of degree 6 on $H_2(S;\mathbf{Q})$. According to [FM1] there is an expansion*

$$\gamma_3(S) = \sum_{i=0}^{3} c_i(m_1, m_2) q_S^{3-i} \kappa_S^{2i}$$

where the $c_i(m_1, m_2)$ are rational numbers depending only on (m_1, m_2). In fact,

$$c_0(m_1, m_2) = 15m_1m_2 \tag{1}$$

and

$$c_1(m_1, m_2) = 15(m_1m_2)\left(2(m_1m_2)^2 - (m_1^2 + m_2^2)\right). \tag{2}$$

Suppose that $\varphi: S \to S'$ is a diffeomorphism between simply connected surfaces in $\mathcal{E}$. As is shown in [FM1] (Chapter 4) it follows that all the coefficients in the expansions for $\gamma_3(S)$ and $\gamma_3(S')$ are equal up to sign, i.e., $c_i(m_1, m_2) = \pm c_i(m_1', m_2')$ for all i. Furthermore, if $c_i(m_1, m_2) \neq 0$ for some $i > 0$, then $\varphi^* k_{S'} = \lambda_\varphi k_S$ for some $\lambda_\varphi \in \mathbf{Q}$.

It is obvious that the unordered pair (m_1, m_2) is determined by Formulas (1) and (2). Thus, we come to our main result about elliptic surfaces of geometric genus one.

Theorem 1.1.2 *Let S, S' be minimal elliptic surfaces over $\mathbf{P}^1$ with at most two multiple fibers and with geometric genera equal to one. Then S and S' are diffeomorphic if and only if they are deformation equivalent in the sense of [FM1], i.e., they are diffeomorphic if and only if the multiplicities of S equal those of S'. Furthermore, if $\varphi: S \to S'$ is a diffeomorphism then*

$$\varphi^* k_{S'} = \pm k_S \ . \tag{3}$$

Bauer, in [B1] also establishes this result in the simply connected case by proving a theorem similar to Theorem 1.1.1.

As proved in [FM1] Theorem 1.1.2 implies a similar result for nonminimal elliptic surfaces.

Theorem 1.1.3 *Let S, S' be possibly non-minimal regular elliptic surfaces with geometric genus equal to one. Let $p: S \to S_{\min}$ and $p': S' \to S'_{\min}$ be the maps to the minimal models. Then S and S' are diffeomorphic if and only if they are deformation equivalent, i.e., if and only if they have the same Euler characteristic (i.e. they are blown up the same number of times) and the multiplicities of the multiple fibers of $S_{\min}$ and $S'_{\min}$ are equal. If $\varphi: S \to S'$ is a diffeomorphism then $\varphi^*(p')^* k_{S'_{\min}} = \pm p^* k_{S_{\min}}$. Also, φ^* maps $(p')^* H^2(S'_{\min})$ isomorphically to $p^* H^2(S_{\min})$.*

This paper is concerned with establishing Theorem 1.1.1. The main body of the paper (Sections 1 through 6) establishes the necessary algebro-geometric description of the relevant moduli spaces. Section 7 deals with combinatorics of the simply connected case, i.e., the case when the multiplicities are relatively prime. In the appendix, written jointly with Millie Niss, we treat the combinatorics in the case when the multiplicities are not relatively prime.

1.2 Background

In the 1980's Donaldson introduced a family of invariants for smooth four-manifolds, see [D3]. These invariants are the only known invariants for four-manifolds which go beyond homotopy or homeomorphism invariants. They are defined using a riemannian metric and a moduli space of solutions to a non-linear elliptic PDE associated to the metric (the anti-self-dual equations for connections on principal bundles). They are homogeneous polynomials on the second homology of the manifold and are defined by integrating cup products of 2-dimensional cohomology classes over the fundamental class of a compactification of this moduli space.

Let us give more details on the definition of the invariants. General references for the material below are [FU], [DK], [D3], and [FM1]. Fix a closed oriented riemannian 4-manifold (X, g). Assume for simplicity that $b_1(X) = 0$. For each integer $c > 0$, let $P_c \to X$ be the unique (up to isomorphism) principal $SU(2)$-bundle over X with $c_2 = c$. We denote by $\mathcal{M}(P_c, g)$ the moduli space of gauge equivalence classes of g-ASD connections on P_c. If g is generic and $b_2^+(X) > 0$, then $\mathcal{M}(P_c, g)$ is a smooth manifold of dimension $2d(c) = 8c - 3(b_2^+(X) + 1)$. There is a natural compactification, the so-called Uhlenbeck compactification, $\overline{\mathcal{M}}(P_c, g)$ of this moduli space. The points of $\overline{\mathcal{M}}(P_c, g)$ parametrize generalized g-ASD connections, generalized in the sense that they can have points of concentrated curvature. For g generic and c sufficiently large

$\overline{\mathcal{M}}(P_c, g)$ has a fundamental class $[\overline{\mathcal{M}}(P_c, g)]$. The condition on c that there be a fundamental class is that the codimension of every stratum added at infinity be at least two. For g generic all the strata have codimension at least 4, except possibly the one parametrizing generalized connections with flat background connection (i.e., generalized connections all of whose curvature is concentrated at points). This stratum has dimension $4c$. Thus, we need $8c - 3(b_2^+(M) + 1) \geq 4c + 2$ or

$$c \geq (3b_2^+(M) + 5)/4 \ .$$

The c satisfying this condition are said to be in the *stable range* for X. All others are said to be in the *unstable range* for X.

There is the Donaldson μ-map

$$\mu: H_2(X) \to H^2(\overline{\mathcal{M}}(P_c, g))$$

extending the slant product with minus one-quarter the first Pontrjagin class of the universal $SO(3)$-bundle over $X \times \mathcal{M}(P_c, g)$. The Donaldson polynomial invariant is defined by

$$\gamma_c(X, g)(\alpha_1, \ldots, \alpha_{d(c)}) = \langle \mu(\alpha_1) \cup \cdots \cup \mu(\alpha_{d(c)}), [\overline{\mathcal{M}}(P_c, g)] \rangle$$

for any classes $\alpha_i \in H_2(X)$. Clearly, this is a multilinear function on $H_2(X)$ or equivalently, a homogeneous polynomial in $H^2(X)$ of degree $d(c)$. Provided that $b_2^+(X) > 1$ this polynomial does not depend on the metric g and therefore is an invariant of the underlying smooth oriented manifold. We denote it by $\gamma_c(X)$. (Actually, here we have not discussed orientations for the moduli space but there is a consistent way to orient these involving one choice of sign. Hence, we have invariants of X defined up to sign.) The polynomials $\gamma_c(X)$ are called the Donaldson invariants. So far, we have indicated how they are defined for c in the stable range.

We can extend the discussion to the unstable range by using a blow up formula. Let $X' = X \# \overline{\mathbf{P}}^2_{\mathbf{C}}$, and suppose that c is in the stable range for X. (Notice that the stable range for a manifold depends on b_2^+ of the manifold, so that X and X' have the same stable range.) Then we have the gauge-theoretic blow up formula [FM1]

$$\frac{-1}{2}\gamma_{c+1}(X')(e, e, e, e, \alpha_1, \ldots, \alpha_{d(c)}) = \gamma_c(X)(\alpha_1, \ldots, \alpha_{d(c)})$$

where $e \in H^2(\overline{\mathbf{P}}^2_{\mathbf{C}})$ is a generator and $\alpha_i \in H_2(X)$. Now suppose that $c+1$ is in the stable range for X and X' but c is not. Then we can use the expression on the left-hand-side of this equation to define $\gamma_c(X)$ (provided that we are willing to invert 2). Blowing up more than one point and repeatedly using this device allows one to define $\gamma_c(X)$ for any $c > 0$, enjoying the same formal properties as the stable invariants, [FM1].

The Donaldson invariants are hard to compute directly from the definition. Most computations to date have been done for algebraic surfaces using a correspondence established by Donaldson in [D2] between anti-self-dual (ASD) connections and stable holomorphic bundles in the case that the metric is Kähler. (The $(0,1)$-part of an anti-self-dual connection is an integrable complex structure on the bundle and defines a stable holomorphic structure. This gives the correspondence.) Let S be a smooth projective surface and H a polarization of S. Let g_H be the Kähler metric associated to the polarization H. Let $\mathcal{M}_c(S,H)$ be the moduli space of rank-two H-slope stable holomorphic vector bundles over S with trivial determinant and with $c_2 = c$. Let P_c be the $SU(2)$-bundle on S with $c_2(P_c) = c$. The Donaldson correspondence [D2] proves that moduli space $\mathcal{M}(P_c, g_H)$ is naturally homeomorphic to $\mathcal{M}_c(S,H)$.

As a consequence of these ideas Donaldson showed that these invariants are non-zero for algebraic surfaces, [D3]. On the other hand, gauge-theoretic arguments [D3] show that the invariants are zero for manifolds which are connected sums of manifolds with 2-dimensional classes with positive self-intersection. A striking consequence of these two results is that no algebraic surface can be decomposed as such a connected sum. (This is not directly related to what we shall in this paper, but it does give an indication of the power of the methods.)

Let us consider other results proved by of these techniques which are more directly related to this paper. The first is Donaldson's result [D4] that there are two homeomorphic but non-diffeomrophic simply connected elliptic surfaces with $p_g = 0$, thus showing that the smooth h-cobordism theorem does not extend to dimension four. This result was generalized by [FM3] and [OV] to show that in fact there are infinitely many homeomorphic but pairwise non-diffeomorphic simply connected elliptic surfaces with $p_g = 0$. Making more explicit computations along the same lines, Bauer [B2] showed the following. Suppose that $T(m_1, m_2)$ is a simply connected elliptic surface with multiple fibers of multiplicity m_1, m_2, then the product $(m_1^2 - 1)(m_2^2 - 1)$ is a diffeomorphism invariant of $T(m_1, m_2)$. By the Castelnuovo criterion, for any $m_1 \geq 1$ the elliptic surface $T(m_1, 1)$ is a rational surface and hence diffeomorphic to the connected sum of $\mathbf{P}^2$ with a number of copies of $\overline{\mathbf{P}}^2$. It is natural to conjecture that for all pairs (m_1, m_2) with $m_1, m_2 > 1$ that the unordered pair (m_1, m_2) is a diffeomorphism invariant of $T(m_1, m_2)$. All of these results come by studying the moduli space of stable bundles with $c_2 = 1$.

Let us turn now to minimal regular elliptic surfaces of higher geometric genus with at most two multiple fibers. In [FM1] it is proved that the product of the multiplicities of the multiple fibers is a diffeomorphism invariant for such surfaces. In proving this result it is first established that the Donaldson invariants of such a surface are polynomial expressions in the cohomology class κ_S and the intersection form q_S, as defined above. In fact if S is a minimal regular elliptic surface with $p_g = p_g(S) \geq 1$ and with multiple fibers of multiplicities m_1, m_2 it is shown that the highest order term in q_S in the expression for $\gamma_c(S)$ is

$$\frac{d!}{2^a a!}(m_1 m_2)^{p_g} \kappa_S^{p_g - 1} q_S^a$$

where $d = 4c - 3(p_g + 1)$ and $a = 2c - 2p_g - 1$. The result on the diffeomorhism invariance of $m_1 m_2$ follows immediately. It is natural to conjecture that in this context the unordered pair (m_1, m_2) is a diffeomorphism invariant. It would follow from this conjecture that the diffeomorphism classification of such surfaces coincides with the classification up to deformation equivalence of complex analytic structure.

It also follows from these computations that for all $p_g > 1$ the line spanned by the canonical class k_S is left invariant by the group of self-diffeomorphisms. In fact, this result is also established for complete intersections with p_g even. For both these results see [FM1].

We mentioned above the fact that the moduli space of ASD connections over an algebraic surface is identified with the moduli space of stable holomorphic bundles on the surface. Taking the point of view of algebraic geometry, one is lead to consider algebro-geometric analogues of the Donaldson invariants defined as follows cf, [O1, O2]. We denote by $\mathcal{M}_c^G(S,H)$ the moduli space of H Gieseker semi-stable rank-two torsion-free sheaves on S with trivial determinant and with $c_2 = c$, see [G]. According to [G], this is a projective variety containing $\mathcal{M}_c(S,H)$ as a Zariski open subset. We denote by $\overline{\mathcal{M}}_c(S,H)$ the closure of $\mathcal{M}_c(S,H)$ in $\mathcal{M}_c^G(S,H)$. When c is odd there is a universal sheaf over $S \times \mathcal{M}_c^G(S,H)$. We denote by $\mathcal{F}$ the restriction of this sheaf to $S \times \overline{\mathcal{M}}_c(S,H)$. Slanting with $c_2(\mathcal{F})$ defines $\nu\colon H_2(S) \to H^2(\overline{\mathcal{M}}_c(S,H))$, which is an algebro-geometric analogue of Donaldson's μ-map. By a theorem of Donaldson, provided that c is sufficiently large, $\overline{\mathcal{M}}_c(S,H)$ will be generically smooth and reduced of the expected (complex) dimension $d = 4c - 3\chi(\mathcal{O}_S)$. For any c satisfying these conditions we define $\delta_c^{st}(S,H)$ by setting

$$\delta_c^{st}(S,H)(\alpha_1,\ldots,\alpha_d) = \langle \nu(\alpha_1 \cup \cdots \cup \alpha_d), [\overline{\mathcal{M}}_c(S,H)]\rangle \ .$$

(The use of 'st' in the notation for the δ-invariant refers to the fact that it is defined using the moduli space of stable bundles. This stability has nothing to do with c being in the stable range.) One expects that when c is in the stable range and $\delta_c^{st}(S,H)$ is defined, up to correction terms coming from moduli spaces with lower c_2, it agrees with the Donaldson invariant $\gamma_c(S)$. One expects that will be no such correction terms when c is sufficiently large. If c is not in the stable range and $\delta_c^{st}(S,H)$ is defined, then we not only expect correction terms as above, but also correction terms coming from 'moduli spaces' of properly slope semi-stable bundles with $c_2 \leq c$.

For the surfaces that we shall consider, it turns out that γ_3 is equal to $\delta_3^{st} + (1/2)\delta_3^{ss}$, where δ_3^{ss} is defined using the moduli space of properly semi-stable bundles with $c_2 = 3$.

1.3 Outline of the paper

Let S be a regular elliptic surfaces with $p_g = 1$. The stable range for such a surface is $c \geq 4$. But we shall compute the invariant $\gamma_3(S)$ which is unstable. As we indicated above, we shall do this by computing $\delta_3^{st}(S,H)$, by computing $\delta_3^{ss}(S,H)$ which is the contribution of the moduli space of properly slope semi-stable bundles with $c_2 = 3$, and by showing that the moduli spaces for bundles with lower c_2 do not contribute to $\gamma_3(S)$. At this point the reader might ask, Why not compute a stable Donaldson polynomial? The reason is that by the Riemann-Roch Theorem it is easier to describe the **reduced** moduli spaces $\mathcal{M}_c(S,H)$ for $c \leq 3$ than it is for $c \geq 4$.

We now give an exposition of the contents of the various sections. In Section 2 we will define the unstable algebro-geometric polynomials $\delta_{c,\mathbf{r}}(S,H)$ for any polarized surface (S,H). These are defined in terms of moduli spaces $\overline{\mathcal{M}}_{c+k}(\tilde{S},\tilde{H})$, where $\pi\colon \tilde{S} \to S$ is the blow-up of S at k distinct points (the subscript $\mathbf{r}$ in $\delta_{c,\mathbf{r}}(S,H)$ has to do with the choice of the polarization $\tilde{H}$). For the definition to make sense we must show that for k big enough the moduli space $\mathcal{M}_{c+k}(\tilde{S},\tilde{H})$ is generically smooth of the expected dimension. This is proved by defining a scheme-theoretic stratification of the moduli space. The strata are in one-to-one correspondence with a subset of the connected components of the "moduli space" of rank-two bundles on the completion $\hat{E}$ of $\tilde{S}$ along the exceptional set of π. Of course this correspondence is given by restricting vector bundles to $\hat{E}$. Then we describe each stratum in terms of moduli spaces of rank-two vector bundles on S. This is done by associating to a point $[V]$ of a stratum the isomorphism class of $(\pi_* V)^{**}$. A dimension count will allow us, thanks to a Theorem of Donaldson ([D3, Fri1]), to show that for k big the moduli space $\mathcal{M}_{c+k}(\tilde{S}, \tilde{H})$ is generically reduced of the expected dimension. Finally we will prove that the unstable polynomials are independent of the points we choose to blow up, if these points are generic. To prove it we need to consider relative moduli spaces over families of surfaces and compare the algebro-geometric polynomials for different surfaces in the family.

In Section 3 we prove that for every pair of integers (m_1, m_2) there exists a regular, minimal elliptic surface S with $p_g = 1$ with with multiplicities m_1, m_2 and a polarization H of S such that $\delta_{3,\mathbf{r}}(S, H) = \gamma_3(S)$. In fact S and H can be thought of as generic. The proof is based on the surjectivity of the period map for $K3$ surfaces and monodromy arguments. For this reason it does not generalize to cover other cases (e.g. elliptic surfaces with geometric genus bigger than one). Nonetheless, it is to be expected that the algebro-geometric polynomials are equal to Donaldson's polynomials in general (with some genericity assumption on the polarization H).

In Section 4 we analyze the moduli spaces $\mathcal{M}_c(S, H)$ for $1 \leq c \leq 3$, and parameter spaces $P_c(S, H)$ for H properly slope semistable rank-two vector bundles with $c_1 = 0$, $c_2 = c$. The moduli space $\mathcal{M}_3(S, H)$ turns out to be of the expected dimension. It has many irreducible components, which are in general non-reduced. We determine the multiplicity of those components which contribute to the computation of $c_0(m_1, m_2)$ and $c_1(m_1, m_2)$.

In Section 5 we prepare the way for the proof of the blow-up formula and the computations of $\delta_{3,\mathbf{r}}(S, H)$ by giving representatives for $\nu(\alpha)$ in two cases. The first is when α is the homology class of a curve C of genus g on S. The choice of a line bundle $L \in \mathrm{Pic}^{g-1}(C)$ determines (under some assumptions) an effective Cartier divisor $\Delta(C, L)$ on $\overline{\mathcal{M}}_c(S, H)$ representing $\nu([C])$. These are the algebro-geometric analogues of the real codimension two subsets of $\mathcal{M}_c(S, g_H)$ employed by Donaldson ([D3]). The second case is when α is Poincaré dual to a holomorphic two-form $\omega \in H^0(\Omega_S^2)$: we reprsent $\nu(\alpha)$ by a Kähler two-form on $\overline{\mathcal{M}}_c(S, H)$ which was introduced by Mukai and Tyurin ([Mu1, T]).

Section 6 is devoted to the proof of the blow-up formula i.e., Theorem 6.0.1. This formula says that for regular minimal ellipic surfaces S with suitable polarizations H we have

$$\gamma_3(S) = \delta_3^{\mathrm{st}}(S, H) + \frac{1}{2}\delta_3^{\mathrm{ss}}(S, H) + 30\epsilon_2(S, H)q_S$$

at least as multilinear functions on $H_2^+(S; \mathbf{Q})$. (Of course, it follows from the general form of $\gamma_3(S)$, that this invariant is determined by its restriction to $H_2^+(S; \mathbf{Q})$.) Here $\delta_3^{st}(S, H)$ is an invariant of degree 6 defined in the usual way using the components of $\mathcal{M}_3(S, H)$; δ_3^{ss} an invariant of degree 6 defined using the components of $P_3(S, H)$; and $\epsilon_2(S, H)$ is an invariant of degree 4 defined using the four-dimensional components of $\mathcal{M}_2(S, H)$. The proof of this result is quite involved and relies heavily on the results of Sections 2, 4, and 5.

In Section 7 we compute $c_0(m_1, m_2)$ and $c_1(m_1, m_2)$. By the the blow up formula and the computation of the contribution of the moduli space of properly semi-stable bundles this is equivalent to computing $\delta_3^{st}(S,H)(\Gamma+\overline{\Gamma})$ and

$$\delta_3^{st}(S,H)([H],[H],\underbrace{\Gamma+\overline{\Gamma},\ldots,\Gamma+\overline{\Gamma}}_{4})$$

where $\Gamma \in H_2(S)$ is the Poincaré dual of a holomorphic two-form. These are the computations that we actually do.

In the appendix we turn to the non-simply connected case and discuss those parts of the argument which must be modified when m_1 and m_2 are not relatively prime.

1.4 Conventions and notation

General conventions on schemes. By a scheme we mean a scheme of finite type over $\mathbf{C}$. The topology on a scheme is the Zariski topology unless we explicitely state the contrary. A stratification of a scheme X is a finite collection $\{X_i\}$ of pairwise disjoint subschemes of X such that as a set X is the union of the X_i. If X is a scheme then $H_*(X;R)$ ($H^*(X;R)$) denote the homology (cohomology) of the reduced scheme associated to X with the classical topology. If $X = Y \times Z$ is a product then p_Y, p_Z will be the projections of X to Y and Z, respectively. A polarization of a projective scheme X is an $H \in \mathrm{Pic}(S)\otimes\mathbf{Q}$ such that some positive multiple of H is a very ample line bundle. We use the notation (X,H) to denote a projective scheme together with a polarization of X; we then say that (X,H) is a polarized scheme. If D is a Cartier divisor on X (or its equivalence class modulo rational equivalence) we let $[D]$ be the corresponding line bundle on X. On the other hand if $X \subset Y$ is a closed compact (in the classical topology) subset we let $[X] \in H_*(Y;\mathbf{Z})$ be the homology class represented by X. If X is a scheme and U is an open subset of X then we implicitly give U the induced scheme structure. Let X be an irreducible scheme. We say that the *general* $x \in X$ has a certain property if the set of geometric points of X which do not have this property is contained in a proper closed subset of X. We say that the *generic* $x \in X$ has a certain property if the set of geometric points of X which do not have this property is contained in the union of a countable family of proper closed subsets of X.

Surfaces. By surface we mean a smooth projective irreducible surface. If S is a surface we let $q(S) = h^1(\mathcal{O}_S)$, $p_g(S) = h^2(\mathcal{O}_S)$. An elliptic surface is a surface S and a map $\varphi: S \to C$ to a curve such that the generic fiber is an elliptic curve. Assume that the elliptic surface S is regular. We let $F = F_S$ be any scheme-theoretic fiber of φ, or its class in $\mathrm{Pic}(S)$. We let $\kappa = \kappa_S \in \mathrm{Pic}(S)$ be the indivisible class some positive multiple of which is equal to $[F_S]$. We will often abuse notation and also denote by κ_S the first Chern class of κ_S.

A family of surfaces is a proper map $f: \mathcal{S} \to B$ of smooth varieties which is surjective and whose fibers are surfaces. If $b \in B$ we let $S_b = f^{-1}(b)$. A relative polarization of $\mathcal{S}$ is an $\mathcal{H} \in \mathrm{Pic}(\mathcal{S})\otimes\mathbf{Q}$ which restricts to a polarization on each S_b. If S is a surface we let $Hilb^n(S)$ be the Hilbert scheme of zero-dimensional subschemes of S of length equal to n. If Z is such a subscheme we let $[Z] \in Hilb^n(S)$ be the corresponding point.

Sheaves. By a sheaf on a scheme we mean a coherent sheaf. Let X be a smooth scheme. If F is a sheaf on X then the Chern class $c(F)$ will be viewed as an element of $H^*(X)$ (and not as an element of $A^*(X)$). A family of sheave on X parametrized by B with Chern class c is a B-flat sheaf $\mathcal{F}$ on $X \times B$ such that for all $b \in B$ one has $c(\mathcal{F}|_{X\times\{b\}}) = c$. If U is a subscheme of B we let $\mathcal{F}_U = \mathcal{F}|_{X\times U}$. If $\iota: X \hookrightarrow Y$ is an inclusion and F is a sheaf on X we abuse notation and denote by F the sheaf on Y given by $\iota_* F$.

Let (X, H) be a polarized smooth scheme. There are two notions of stability for sheaves on X. Let F be a torsion-free sheaf on X. We say that F is H slope semistable if, for every inclusion $G \hookrightarrow F$, we have that

$$\frac{1}{\mathrm{rk}(G)} c_1(G) \cdot H \leq \frac{1}{\mathrm{rk}(F)} c_1(F) \cdot H \ . \tag{4}$$

If in Inequality (4) there is strict inequality whenever $0 < \mathrm{rk}G < \mathrm{rk}F$ then F is H slope stable; otherwise, we say that F is properly H slope semistable. We say that F is H Gieseker semistable if, for every inclusion $G \hookrightarrow F$, we have that

$$\frac{1}{\mathrm{rk}(G)} \chi(G \otimes [nH]) \leq \frac{1}{\mathrm{rk}(F)} \chi(F \otimes [nH]) \tag{5}$$

for $n \gg 0$. If in Inequality (5) there is strict inequality for all proper subsheaves G (and for all $n >> 0$) then F is H Gieseker stable; if there exists a proper subsheaf G such that we have equality then F is properly H Gieseker semistable. Notice that if F is H slope stable then it is also H Gieseker stable and that if it is H Gieseker semistable then it is H slope semistable. We also notice that if two sheaves F, F' on X are isomorphic outside of a codimension two subset then F' is H slope stable (semistable) if and only if F is. A sheaf F is simple if $\mathrm{Hom}(F, F) \cong H^0(\mathcal{O}_X)$. Any Gieseker stable sheaf is simple.

Let F be a torsion-free sheaf on a scheme X. There is a canonical exact sequence

$$0 \to F \to F^{**} \to \mathcal{Q}(F) \to 0 \, .$$

The sheaf $\mathcal{Q} = \mathcal{Q}(F)$ is supported on the singularity set of F, which, since F is torsion-free, is of codimension at least two. Hence F is H slope stable (semistable) if and only if F^{**} is. We let $Z(\mathcal{Q})$ be the subscheme of S whose ideal sheaf is the annihilator of $\mathcal{Q}$.

Sheaves on surfaces. Let (S, H) be a polarized surface. Then by [G] there exists a moduli space for equivalence classes of H Gieseker semistable torsion-free sheaves on S with given rank and Chern class. The moduli space is a projective scheme. Two sheaves on S are equivalent if the Graded sheaves associated to their respective Jordan-Hölder filtrations are isomorphic. If a sheaf is Gieseker stable then its Jordan-Hölder filtration is trivial, hence two stable sheaves are equivalent if and only if they are isomorphic. We will denote by $\mathcal{M}_c^G(S, H)$ the moduli space of H Gieseker semistable rank-two sheaves F with $c_1(F) = 0$ and $c_2(F) = c$. We let $\mathcal{M}_c(S, H)$ be the open subscheme of $\mathcal{M}_c^G(S, H)$ parametrizing H slope stable locally free sheaves and we let $\overline{\mathcal{M}}_c(S, H)$ be its closure in $\mathcal{M}_c^G(S, H)$. Notice that $\overline{\mathcal{M}}_c(S, H)$ is a projective subset of $\mathcal{M}_c^G(S, H)$ but it does not inherit a natural scheme structure. If F is an H Gieseker semistable rank-two sheaf with $c_1(F) = 0$, $c_2(F) = c$ we let $[F]$ be the corresponding point of $\mathcal{M}_c^G(S, H)$. We say that $\mathcal{M}_c(S, H)$ is *good* if in every irreducible component of $\mathcal{M}_c(S, H)$ there is a point $[F]$ such that $h^2(\mathrm{ad}F) = 0$.

Claim 1.4.1 *The moduli space $\mathcal{M}_c(S, H)$ is good if and only if it is reduced and of pure dimension equal to the expected dimension $d(c) = 4c - 3\chi(\mathcal{O}_S) + q(S)$.*

Proof. Let $[F] \in \mathcal{M}_c(S, H)$. The germ of $\mathcal{M}_c(S, H)$ at $[F]$ is isomorphic to $\mathbf{Def}(F)$, the universal deformation space of F. There is an obstruction map $\Phi: H^1(\mathrm{End}F) \to H^2(\mathrm{ad}F)$ describing $\mathbf{Def}(F)$, in the sense that $\mathbf{Def}(F) = \Phi^{-1}(0)$. Furthermore the tangent space to $\mathbf{Def}(F)$ at at its closed point is canonically identified with $H^1(\mathrm{End}F)$. Let $\mathrm{ad}F$ be the sheaf of traceless endomorphisms of F. Since F is stable we have $h^0(\mathrm{ad}F) = 0$; by Riemann-Roch we conclude that

$$h^1(\mathrm{End}F) - h^2(\mathrm{ad}F) = 4c - 3\chi(\mathcal{O}_S) + q(S).$$

From these facts it follows that $\mathcal{M}_c(S, H)$ is good if and only if every irreducible component is of the expected dimension and generically reduced. What is left to prove is that if $\mathcal{M}_c(S, H)$ is good then it is everywhere reduced. By the above discussion it is a local complete intersection and hence Cohen-Macauley. Since it is generically reduced it follows from Theorem 17.3 in [M] that it is everywhere reduced. □

Elementary modifications. Let X be a scheme and let D be a Cartier divisor on X. Let F be a sheaf on X and let $\mathcal{O}_D(Q)$ be a sheaf on D. Let $\phi: F \to \mathcal{O}_D(Q)$ be a surjective map. Then we call the kernel G of ϕ the *elementary modification of F determined by ϕ*. By definition G fits into the exact sequence

$$0 \to G \to F \xrightarrow{\phi} \mathcal{O}_D(Q) \to 0 . \tag{6}$$

As is easily checked if F is locally free and $\mathcal{O}_D(Q)$ is locally free (as a $\mathcal{O}_D$ module) then G is locally free. Let $\mathcal{O}_D(K)$ be the kernel of the restriction of ϕ to D, so that we have

$$0 \to \mathcal{O}_D(K) \to F|_D \xrightarrow{\phi|_D} \mathcal{O}_D(Q) \to 0 .$$

From the Exact Sequence (6) it follows easily that we have an exact sequence

$$0 \to \mathcal{O}_D(Q) \otimes [-D] \to G|_D \xrightarrow{\psi} \mathcal{O}_D(K) \to 0 \tag{7}$$

provided that the map

$$F \otimes \mathcal{O}_X(-D) \to F \tag{8}$$

given by multiplication is injective. Under these circumstances we define the *inverse elementary modification* of the modification determined by ϕ to be the kernel, F', of the map ψ. Thus we have

$$0 \to F' \to G \xrightarrow{\psi} \mathcal{O}_D(K) \to 0 . \tag{9}$$

The reason for calling it the inverse modification is that, as is easily checked, F' is canonically isomorphic to $F \otimes [-D]$.

Now assume that F has rank two and $\mathcal{O}_D(Q)$ has rank one. Let $\iota: D \hookrightarrow X$ be the inclusion. Then as is easily checked we have

$$c_1(G) = c_1(F) - c_1(D) \tag{10}$$

$$c_2(G) = c_2(F) - \iota_* c_1(K) . \tag{11}$$

Polynomials and polarizations. Let V be a finite dimensional vector space over a field k of characteristic zero (for us k equals $\mathbf{Q}$, $\mathbf{R}$, or $\mathbf{C}$) and let $f: V \to k$ be a homogeneous polynomial function of degree d. Then we define the polarization of f to be the unique d-linear symmetric function

$$\tilde{f}: \underbrace{V \times \cdots V}_{d \text{ times}} \to k$$

such that $\tilde{f}(x, \ldots, x) = f(x)$ for all $x \in V$. Since f and $\tilde{f}$ determine each other, we shall abuse notation and denote them by the same symbol. We let $\mathrm{Sym}^d V^*$ be the vector space of degree d homogeneous polynomial functions on V.

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