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Differential Operators for Partial Differential Equations

INTRODUCTION

In [40] G. Darboux introduced differential operators in connection with the Euler equation. In recent years this method for the representation of solutions of partial differential equations has become the object of increasing interest. Particularly this is based on the fact that these representations permit a detailed investigation of the function theoretic properties of the solutions. Especially in case of the differential equation

$$(*) \qquad (1+\varepsilon z\bar{z})^2 w_{z\bar{z}} + \varepsilon n(n+1)w = 0, \qquad \varepsilon = \pm 1, \qquad n \in \mathbb{N},$$

it was possible to generalize a number of statements of the classical function theory. In the first place by reason of the results proved by St. Ruscheweyh a function theory associated with the differential equation (*) could be developed. These results are treated in the subsequent contribution of St. Ruscheweyh. On the other hand the assertions obtained by differential operators permit a number of applications. Moreover, in various papers certain connections between differential and integral operators were investigated. However, a general characterization of those partial differential equations which permit representations of solutions by differential operators could not be found up to now. So much the more in this stage of the investigations a survey of the known results is of particular interest.
In the first chapter in the case of the differential equation

$$w_{z\bar{z}} + A(z,\bar{z})w_{\bar{z}} + B(z,\bar{z})w = 0$$

general conditions are derived for the appearance of solutions which may be represented by differential operators of order n operating on holomorphic respectively antiholomorphic functions. By certain additional conditions concerning the form of these differential operators one is led to the known representations of solutions. Subsequently various methods are characterized which permit to get corresponding representations of solutions for certain classes of other partial differential equations. Here, apart from solutions of the equation $h_{z\bar{z}} = 0$, also solutions of other elliptic or parabolic differential equations are used as generators.
The second chapter deals with several applications of the representa-

tion of solutions derived here. First a new representation of the spherical surface harmonics is treated. Besides, a corresponding class of functions is considered which arise in connection with the wave equation and may be called hyperboloid functions. Moreover, a representation of the surface harmonics of degree n in p dimensions is treated. The real and imaginary parts of certain classes of pseudo-analytic functions satisfy elliptic differential equations of the type considered here. Thus, it is possible to derive simple representations of these pseudo-analytic functions in simply connected domains and in the neighbourhood of isolated singularities. Moreover, these results permit various applications, for example, for a differential equation in the theory of functions of several complex variables, for a class of pseudo-analytic functions with a "sharp" maximum principle and for the determination of Vekua resolvents. By use of the results proved in Chapter I,5 in connection with the generalized Darboux equation it is possible to obtain representations for further classes of pseudo-analytic functions. By means of the complex potentials corresponding to these functions one is led to elliptic partial differential equations for which a representation of the solutions was not known up to now. Finally the integro-differential-operators treated in Chapter I,4 may be used for the representation of pseudo-analytic functions.
In Chapter II,4 we deal with a class of generalized Tricomi equations which lead to differential equations of the form considered here in the elliptic half-plane, whereas we get an Euler-Poisson-Darboux equation in the hyperbolic half-plane. Thus, in either case the solutions can be represented by differential operators. Moreover, by means of these representations it is possible to determine integral-free fundamental solutions in the large. These results are of particular interest since generalized Tricomi equations arise, for instance, in connection with the theory of transonic flow.
The assertions proved in Chapter I,2a may be used also for the representation of solutions of generalized Stokes-Beltrami systems. First, a system is treated which is closely related to a Stokes-Beltrami system which was considered by A. Weinstein in connection with the development of the generalized axially symmetric potential theory. Moreover, we deal with several systems of first-order partial differential equations to which we are led by certain functional-differential-relations for solutions of the Euler equation. Finally, by using the results of Chapter I,2a representations of solutions of the iterated equation of generalized axially symmetric potential theory are derived which arises in a number of physical problems.

Within each chapter, the sections, theorems, and formulas are numbered consecutively. If quoted, or referred to within the same chapter, only their own number is mentioned. Otherwise the number of the chapter is added, for instance, Theorem II,4 or (I,17).

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August 1979 K. W. Bauer

CHAPTER I

Representation of solutions by differential operators

1) Polynomial operators for the differential equation $w_{z\bar{z}} + Aw_{\bar{z}} + Bw = 0$

a) Holomorphic generators

In the present paper we use the following notations. We set

$$z = x + iy$$

where x and y are real variables. i denotes the imaginary unit. Complex conjugates are denoted by bars, e.g.

$$\bar{z} = x - iy.$$

We use the formal differential operators

(1) $$r = \frac{\partial}{\partial z} = (\quad)_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and

(2) $$s = \frac{\partial}{\partial \bar{z}} = (\quad)_{\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) .$$

Apart from the usual rules in case of differentiable functions we have

(3) $$\bar{w}_z = \overline{(w_{\bar{z}})}, \qquad \bar{w}_{\bar{z}} = \overline{(w_z)} .$$

For a real-valued differentiable function, i.e. in case $w = \bar{w}$, we obtain

(4) $$w_{\bar{z}} = \overline{(w_z)} .$$

Furthermore, for a holomorphic function g(z) we have

(5)
$$\begin{cases} g_z = g', \quad g_{\bar z} = 0, \quad \bar g_{\bar z} = \overline{g'}, \\ (\mathrm{Re}\, g)_z = \frac{1}{2} g', \quad (\mathrm{Im}\, g)_z = -\frac{i}{2} g', \end{cases}$$

Re g denotes the real part of g, Im g denotes the imaginary part of g. Moreover, by (1) and (2) we get

(6)
$$4w_{z\bar z} = \Delta w,$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the Laplace operator.

We consider partial differential equations of the form

$$w_{z\bar z} + a(z,\bar z) w_z + b(z,\bar z) w_{\bar z} + c(z,\bar z) w = 0,$$

where a,b,c are given analytic functions in some domain. By a suitable transformation we can eliminate one of the two first derivatives. Therefore, we proceed from the normal form

(7)
$$w_{z\bar z} + A(z,\bar z) w_{\bar z} + B(z,\bar z) w = 0.$$

We denote by D a simply connected domain of the complex plane, and we suppose that $A(z,\bar z)$ and $B(z,\bar z)$ are analytic in D.

By a solution of (7) in D we mean a function defined in D which has continuous partial derivatives up to the order two and satisfies equation (7) in D. By a theorem of E. Picard (cf. e.g.[88], p.162) such a solution is analytic, in particular there exist derivatives of all order.

Now we consider polynomial operators of order n. We set

(8)
$$H(D) = \{ g(z) \mid g(z) \text{ holomorphic in } D \}$$

and get by

(9) $$P_n(r) = \sum_{k=0}^{n} a_k^*(z,\bar{z})r^k$$

the most general linear partial differential operator of order n on H(D). In view of future simple representations of solutions in place of (9) we use the form

(10) $$P_n(r) = \sum_{k=0}^{n} a_k(z,\bar{z})R_o^k ,$$

where the coefficients $a_k(z,\bar{z})$ are twice continuously differentiable functions in D and

(11) $$R_o = a(z)r ,$$

where a(z) is a holomorphic nonvanishing function in D.

Now we ask for all differential equations (7) which have solutions of the form

(12) $$w = P_n g, \quad g \in H(D).$$

If we substitute (12) into (7), first we find $a_n = a_n(z)$. On account of (11) we put $a_n \equiv 1$ and obtain

(13) $$\begin{cases} rsa_k + A\, sa_k + Ba_k + \dfrac{sa_{k-1}}{a} = 0 \\ \text{for } k = 0,1,\ \ldots,\ n-1 \text{ with } a_{-1} \equiv 0 \\ sa_{n-1} = -aB\ . \end{cases}$$

In characterizing the coefficients A and B in (7), in general, we are led to non-linear partial differential equations. In the sequel we consider some examples.

For n = 1 the system (13) reduces to

(14) $$\begin{cases} rsa_o + A\, sa_o + Ba_o = 0 \\ sa_o = -aB\ . \end{cases}$$

If $B \neq 0$ in D, we get

$$(\log B)_{z\bar{z}} + B + A_{\bar{z}} = 0. \tag{15}$$

Without loss of generality we may use $a \equiv 1$ and obtain the solution

$$w = g' + a_o g, \tag{16}$$

where $a_o = A + (\log B)_z$. If w is a solution of (7) which may be represented in the form (16), we have

$$g = -\frac{w_{\bar{z}}}{B},$$

i.e. for a given solution w of this kind the generator g(z) is uniquely determined.
If the coefficient A in (7) satisfies certain conditions, we can derive further assertions. If, for example,

$$A_{\bar{z}} = B[\alpha(z)\overline{\beta(z)} - 1],$$

where $\alpha(z)$ and $\beta(z)$ are holomorphic nonvanishing functions in D, it follows

$$(\log B)_{z\bar{z}} + \alpha\bar{\beta}B = 0,$$

and by

$$G = \alpha\bar{\beta}B$$

we get the differential equation

$$(\log G)_{z\bar{z}} + G = 0. \tag{17}$$

Setting here

$$\log G = 2W,$$

we obtain the Liouville equation

$$2W_{z\bar{z}} = -e^{2W}. \tag{18}$$

This is the special case of a differential equation which was investigated by G. Warnecke [104]. We quote some results of that paper as far as they are of interest for the following research.

Theorem 1

a) Let D^* be a simply connected domain of the complex plane. Let G be a solution of the differential equation (17)

$$(\log G)_{z\bar{z}} + G = 0$$

in D^*. Let D be a simply connected domain compact in D^*. Then, we can represent G in D by

$$G = \frac{-2\varphi'(z)\overline{\psi'(z)}}{(\varphi(z)+\overline{\psi(z)})^2} \quad , \tag{19}$$

where $\varphi(z)$ and $\psi(z)$ are suitable holomorphic or meromorphic functions which satisfy the following conditions:

(20)
- (i) $\varphi(z)$ and $\psi(z)$ have only a finite number of poles in D of at most first order.
- (ii) $\varphi(z)$ and $\psi(z)$ have no common poles in D.
- (iii) $(\varphi+\bar{\psi})\varphi'\psi' \neq 0$ in D.

b) Conversely, (19) represents a solution of (17) in D for each pair of holomorphic or meromorphic functions $\varphi(z)$ and $\psi(z)$ which satisfy the conditions (20).

c) Every real-valued solution of (17) defined in D may be represented by

$$G = \frac{2\varepsilon f'(z)\overline{f'(z)}}{[1+\varepsilon f(z)\overline{f(z)}]^2} \quad , \qquad \varepsilon = \pm 1 \quad , \tag{21}$$

where f(z) is a suitable holomorphic or meromorphic function in D which satisfies the conditions:

(22)
- (i) f(z) has only a finite number of poles in D of at most first order.
- (ii) $(1+\varepsilon f\bar{f})f' \neq 0$ in D.

d) Conversely, (21) represents a real-valued solution of (17) in D for each holomorphic or meromorphic function f(z) which satisfies the conditions (22).

On the supposition for D we get the following

Theorem 2

a) The differential equation (7)

$$w_{z\bar{z}} + Aw_{\bar{z}} + Bw = 0, \qquad B \neq 0 \text{ in } D,$$

has a solution of the form

(23) $$w = g' + a_o g, \qquad g(z) \in H(D),$$

if, and only of, the coefficients A and B satisfy the relation

(24) $$(\log B)_{z\bar{z}} + B + A_{\bar{z}} = 0.$$

Then, the coefficient a_o in (23) follows by

$$a_o = A + (\log B)_z.$$

b) For every given solution w of (7) in D which may be represented by (23) the generator g(z) is uniquely determined by

$$g(z) = -\frac{w_{\bar{z}}}{B}.$$

c) If $A_{\bar{z}} = B[\alpha(z)\overline{\beta(z)}-1]$, where $\alpha(z)$ and $\beta(z)$ are holomorphic nonvanishing functions in D, we obtain by (24) with $G = \alpha\bar{\beta}B$

$$(\log G)_{z\bar{z}} + G = 0.$$

If the domain D satisfies the supposition in Theorem 1, the coefficient B may be represented in the form

$$B = \frac{-2\varphi'(z)\overline{\varphi'(z)}}{\alpha(z)\overline{\beta(z)}[\varphi(z)+\overline{\varphi(z)}]^2},$$

where $\varphi(z)$ and $\psi(z)$ satisfy the conditions (20).

If we impose on the coefficients $a_k(z,\bar{z})$ in (10) certain conditions, we may expect that the system (13) can be solved for arbitrary $n \in \mathbb{N}$.[1] We suppose $B \neq 0$ in D and set

(25) $$a_k = c_k \eta^{n-k} , \quad n \geq 2,$$

where

$$c_k \in \mathbb{C}, \quad c_o \neq 0, \quad c_n = 1, \quad \eta = \eta(z,\bar{z}) \neq 0 \text{ in } D.$$

Then, (13) takes the form

(26) $$c_k\{(n-k)\eta^{n-k-1}\eta_{z\bar{z}} + (n-k)(n-k-1)\eta^{n-k-2}\eta_z\eta_{\bar{z}} + A(n-k)\eta^{n-k-1}\eta_{\bar{z}} + B\eta^{n-k}\} + \frac{c_{k-1}}{a}(n+1-k)\eta^{n-k}\eta_{\bar{z}} = 0$$

for $k = 0,1, \ldots, n-1$ with $c_{n-1} = 0$,

(27) $$c_{n-1}\eta_{\bar{z}} = -aB .$$

First we get $c_1, c_2, \ldots, c_{n-1} \neq 0$. Now we consider (26) with $k = 0$ and $k = 1$ and obtain by (27)

(28) $$-\frac{\eta_z}{\eta^2} = \frac{c}{a} ,$$

where

$$c = \frac{1}{n-1}\left[\frac{c_{n-1}}{n} - \frac{nc_o}{c_1}\right] .$$

In the case $\eta_z \equiv 0$, i.e.

$$\eta = \overline{\gamma(z)}, \quad \gamma(z) \in H(D),$$

[1] We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ the set of natural, integer, real, and complex numbers respectively. Moreover, we use $\mathbb{N}_o = \mathbb{N} \cup \{0\}$.