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Paul-Hermann Zieschang

An Algebraic Approach to Association Schemes

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Author

Paul-Hermann Zieschang
Christian-Albrechts-Universität zu Kiel
Mathematisches Seminar
Ludewig-Meyn-Straße 4
24098 Kiel, Germany

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Introduction

Let X be a set.

We define

$$1_X := \{(x, x) \mid x \in X\}.$$

Let $r \subseteq X \times X$ be given. We set

$$r^* := \{(y, z) \mid (z, y) \in r\},$$

and, for each $x \in X$, we define

$$xr := \{y \in X \mid (x, y) \in r\}.$$

Let G be a partition of $X \times X$ such that $\emptyset \notin G \ni 1_X$, and assume that, for each $g \in G$, $g^* \in G$. Then the pair (X, G) will be called an *association scheme* if, for all d, e, $f \in G$, there exists a cardinal number a_{def} such that, for all y, $z \in X$,

$$(y, z) \in f \Rightarrow |yd \cap ze^*| = a_{def}.$$ [1]

In these notes, we shall always say *scheme* instead of association scheme. The pair (X, G) will always denote a scheme. We shall always write 1 instead of 1_X. The elements of $\{a_{def} \mid d, e, f \in G\}$ will be called the *structure constants* of (X, G).[2] Occasionally we shall use the expression *regularity condition* in order to denote the condition which guarantees the existence of the structure constants.

The present text provides an algebraic approach to schemes. Similar to the theory of groups, the theory of schemes will be viewed as an elementary

[1] This definition of association schemes is slightly more general than the usual one. Usually one requires additionally at least that $|X|$ be finite. In this case, the term "homogeneous coherent configuration" is common, too; see [14]. (Also in the present text the finiteness of $|X|$ plays an important role.) It is also often required that, for all d, e, $f \in G$, $a_{def} = a_{edf}$; see, e.g., [1].

The term "association scheme" was introduced in [2]. There it is even required that, for each $g \in G$, $g^* = g$, and this additional condition (which implies that, for all d, e, $f \in G$, $a_{def} = a_{edf}$) is often included in the definition of association schemes.

In order to emphasize the algebraic treatment, association schemes (in the present sense) were called "generalized groups" in [34].

[2] In the literature, the structure constants are also called "intersection numbers".

algebraic theory which is naturally connected to certain geometric structures. In fact, the theory of schemes generalizes naturally the theory of groups.

Let us first see to what extent the class of groups may be viewed as a distinguished class of schemes.

For each $g \in G$, we abbreviate

$$n_g := a_{gg^*1}.$$

A non-empty subset F of G will be called *thin* if $\{1\} = \{n_f \mid f \in F\}$. (Note that we always have $n_1 = 1$.) The pair (X, G) will be called *thin* if G is thin.

Let E, $F \subseteq G$ be given. We define

$$EF := \{g \in G \mid \sum_{e \in E} \sum_{f \in F} a_{efg} \neq 0\}$$

and call it the *complex product* of E and F.

It follows readily from the definition of the complex product that, if (X, G) is thin,

$$\mathcal{G}(X, G) := \{\{g\} \mid g \in G\}$$

is a group with respect to the *complex multiplication*, with $\{1\}$ as identity element.

Conversely, let Θ be a group. For each $\theta \in \Theta$, we define $\tilde{\theta} := \{(\zeta, \eta) \in \Theta \times \Theta \mid \zeta\theta = \eta\}$, and we set $\tilde{\Theta} := \{\tilde{\theta} \mid \theta \in \Theta\}$. Then

$$\mathcal{T}(\Theta) := (\Theta, \tilde{\Theta})$$

is a thin scheme.[3]

Now it is readily verified that, if (X, G) is thin,

$$\mathcal{T}(\mathcal{G}(X, G)) \cong (X, G).^{4}$$

Moreover, for each group Θ,

$$\mathcal{G}(\mathcal{T}(\Theta)) \cong \Theta.$$

The elementary proofs of these two facts will be given as Theorem A(iii), (iv) in the appendix of these notes. What is important here for us is that these two facts allow us to identify each group with its corresponding thin scheme. Therefore, we may view the class of groups as a distinguished class of schemes, namely as the class of thin schemes.

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[3] This is easy to see. First of all, it is clear that $\tilde{1} = 1_\Theta$ and that, for each $\theta \in \Theta$, $\tilde{\theta}^* = \widetilde{\theta^{-1}}$. But also the regularity condition is easily verified for the pair $(\Theta, \tilde{\Theta})$. Let β, γ, ϵ, ζ, $\eta \in \Theta$ be given, and assume that $(\beta, \gamma) \in \tilde{\eta}$. Then, $\beta\tilde{\epsilon} \cap \gamma\tilde{\zeta}^* \neq \emptyset$ if and only if $\beta\epsilon = \gamma\zeta^{-1}$ if and only if $\beta\epsilon\zeta = \gamma$ if and only if $\epsilon\zeta = \eta$. Thus, $|\beta\tilde{\epsilon} \cap \gamma\tilde{\zeta}^*| = \delta_{\epsilon\zeta,\eta}$, where δ is the Kronecker delta.

[4] In Section 1.7, we shall say what it means for two schemes to be isomorphic.

There is still another class of important mathematical objects which can be viewed as a distinguished class of schemes, namely the buildings. Buildings were introduced by J. TITS in [27; (3.1)] as a particular class of chamber complexes.[5] Later, in [28; Theorem 2], Tits characterized the buildings as a particular class of "chamber systems". This characterization indicated already a strong relationship between buildings and schemes. In fact, due to this characterization, it is only a small step to see that, like groups, buildings can be viewed naturally as a distinguished class of schemes. Moreover, the embedding of the buildings into the class of schemes is similar to the one of groups. In other words, there exists a class of schemes which for the buildings plays exactly that role which is played by the class of thin schemes for the groups.

Let us give here a rough idea of the definition of this class of schemes.

In Section 1.4, we shall define what it means for an element of G to be a "(generalized) involution". The set of all involutions of G will be denoted by $\mathrm{Inv}(G)$. For each $L \subseteq \mathrm{Inv}(G)$, we shall define in Section 5.1 what it means for (X, G) to be a "Coxeter scheme with respect to L". Now the pair (X, G) will be called a *Coxeter scheme* if there exists $L \subseteq \mathrm{Inv}(G)$ such that (X, G) is a Coxeter scheme with respect to L.

The Coxeter schemes form the class of those schemes which represent the buildings within the class of all schemes. More precisely, the following three statements hold.

First, if (X, G) is a Coxeter scheme, there exists a natural way to construct a building

$$\mathcal{B}(X, G)$$

from (X, G).

Secondly and conversely, to each building $\mathfrak{g}$, say, there is associated naturally a Coxeter scheme which we shall denote by

$$\mathcal{A}(\mathfrak{g}).$$

Finally, if (X, G) is a Coxeter scheme,

$$\mathcal{A}(\mathcal{B}(X, G)) = (X, G).$$

Moreover, for each building $\mathfrak{g}$,

$$\mathcal{B}(\mathcal{A}(\mathfrak{g})) = \mathfrak{g}.$$

[5] In the literature, the definition of buildings changes occasionally. In these notes, buildings are always understood to be regular. Viewing buildings as chamber systems (and we shall do that always in this text) we say that a building in the sense of [28] is *regular* if any two members of one of the defining partitions have the same cardinality. It is an easy exercise in the theory of buildings to prove that buildings in the sense of [27] are regular. It is also obvious that thin buildings in the sense of [28] are regular. In particular, the class of buildings in the sense of the present text contains strictly the class of buildings in the sense of [27].

Occasionally we shall speak of the $(\mathcal{B}, \mathcal{A})$-*correspondence* in order to denote the above-mentioned correspondence between buildings and Coxeter schemes. Similarly, the above-mentioned correspondence between groups and thin schemes will be called the $(\mathcal{G}, \mathcal{T})$-*correspondence*.

The $(\mathcal{B}, \mathcal{A})$-correspondence is not as easy to describe as the $(\mathcal{G}, \mathcal{T})$-correspondence. Therefore we shall not give the details here. They will be given as Theorem E in the appendix of these notes.

There is a simple and natural way to identify Coxeter groups and thin buildings; see [28; (2.3.1)]. Modulo this identification, the $(\mathcal{B}, \mathcal{A})$-correspondence and the $(\mathcal{G}, \mathcal{T})$-correspondence coincide on the class of the thin Coxeter schemes. More precisely, the following two statements are true. If (X, G) is a thin Coxeter scheme, $\mathcal{G}(X, G) = \mathcal{B}(X, G)$. For each Coxeter group Θ, $\mathcal{T}(\Theta) = \mathcal{A}(\Theta)$. As a consequence, thin Coxeter schemes correspond to Coxeter groups.[6]

Viewing groups and buildings as the cornerstones of the class of schemes it seems to be promising to develop a structure theory of schemes between these cornerstones. The object of the present lecture notes is in the first place to develop a treatment of schemes analogous to that which has been so successful in the theory of finite groups.

As indicated in the first footnote, the condition $|X| \in \mathbb{N}$ plays an important role in these notes. Let us say that (X, G) is *finite* if $|X| \in \mathbb{N}$.

The starting point of our approach to schemes is the definition of the complex product. The complex product allows us to treat schemes as algebraic objects. The first chapter is devoted to elementary consequences of the definition of the complex product. Substructures of schemes as well as quotient structures of finite schemes are defined naturally, and we end this introductory chapter with the generalization of the isomorphism theorems [22; §2] for finite groups due to E. NOETHER.

The second chapter begins with a generalization of the fundamental group-theoretical theorem [16; §10] due to C. JORDAN and O. HÖLDER. As in the theory of finite groups, this theorem allows us to speak of "composition factors". In Sections 2.2, 2.3, and 2.4, we focus our attention on schemes which have thin composition factors. After that, other decompositions of schemes, such as "direct", "quasi-direct", and "semidirect products", are introduced. We include the theorem [9; Theorem 3.17] of P. A. FERGUSON and A. TURULL on "indecomposable" schemes.

In the third chapter, we collect various algebraic results which are needed for the representation theory (Chapter 4) and for the theory of generators of schemes (Chapter 5).

The fourth chapter gives a general introduction into the representation theory of finite schemes. We start with a generalization of the fundamental group-theoretical theorem [21] of H. MASCHKE on the semisimplicity of group algebras. After that, our approach is similar to that one given by D. G.

[6] ...as is to be expected...

HIGMAN in [14]. In this chapter, we restrict ourselves to general structural results ignoring the huge amount of literature which exists in this area.

For each $L \subseteq \mathrm{Inv}(G)$, we shall define in Section 5.1 what it means for (X, G) to be "L-constrained". The pair (X, G) will be called *constrained* if there exists $L \subseteq \mathrm{Inv}(G)$ such that (X, G) is L-constrained.

In the fifth chapter, we first investigate constrained schemes. The constrained schemes form a class of schemes which is slightly larger than the above-mentioned class of Coxeter schemes. Their definition as well as their treatment seems to be particularly natural. The constrained schemes provide an appropriate framework for showing how smoothly Coxeter schemes are embedded into the class of schemes. From a general algebraic point of view, we consider Section 5.1 as the heart of these notes.

Let us call the pair (X, G) *thick* if $\{1\} = \{g \in G \mid n_g = 1\}$. Then it is particularly easy to exhibit the significance of Theorem 5.1.8(ii). This theorem says in particular that thick constrained schemes and thick Coxeter schemes are the same thing. Therefore, if one is willing to view buildings generally as thick, in other words, if one assumes the definition of [27; (3.1)] for buildings, Theorem 5.1.8(ii) implies that, via the $(\mathcal{G}, \mathcal{T})$-correspondence, buildings and thick constrained schemes are the same thing. Since the definition of constrained schemes is particularly simple and natural, Theorem 5.1.8(ii) provides us with probably the most succinct definition of buildings.

In the fifth chapter, we include a complete proof of the famous Theorem [8; Theorem 1] of W. FEIT and G. HIGMAN on finite generalized polygons. (We shall present the proof which was given by R. KILMOYER and L. SOLOMON in [20].) Moreover, we give a conceptually alternate and simultaneous approach to the theorems [24; Theorem 2] of S. PAYNE and [23; Satz 1] of U. OTT on polarities of finite generalized quadrangles and finite generalized hexagons via semidirect products. The main focus of the remainder of the fifth chapter is then on an appropriate generalization of Coxeter schemes of "rank" 2. The chapter ends with a generalization of the (algebraic) characterization [34; Theorem B] of finite generalized polygons and Moore geometries; see Theorem 5.8.4.

The appendix of these notes is devoted to the embedding of the class of groups as well as the class of buildings into the class of schemes. In other words, we establish explicitly the $(\mathcal{G}, \mathcal{T})$-correspondence and the $(\mathcal{B}, \mathcal{A})$-correspondence.

Let us mention here that, apart from the class of groups and the class of buildings, other classes of mathematical objects as well can be viewed as specific classes of schemes. For the class of distance-regular graphs the embedding was given implicitly by P. DELSARTE in [6; Theorem 5.6]. As a consequence of this, one obtains the (elementary) fact that the class of strongly regular graphs forms a distinguished class of schemes. The fact that Moore geometries can be viewed as a specific class of schemes was partially shown in [34; (2.4)]. In particular, the class of the 2-designs with $\lambda = 1$ forms a distinguished class of schemes. As a consequence, a lot of problems and results in graph theory or design theory can be viewed naturally or formulated easily as problems or results on schemes; see, e.g., [32], [33], or [35].

It seems to be a thought-provoking question to ask for other algebraic or geometric objects which can be viewed as classes of schemes.

Apart from technical advantages there is at least one other reason for us to view mathematical objects such as groups, buildings, distance-regular graphs, or Moore geometries as schemes. Namely, the language of schemes provides a natural conceptual framework in which the above-mentioned objects may be characterized naturally. Theorem 5.8.4 and Corollary 5.8.5 are characterizations of that type for finite generalized polygons and Moore geometries. A similar characterization of the class of all finite buildings, i.e., a characterization without restriction of the rank, would be a challenging goal.

Let us conclude this introduction with three general remarks.

First of all, it might be helpful to mention here that the present text is thought to be an introductory monograph. No attempt has been made to give a complete account of the results available on schemes. In particular, the well-worked-out connection between schemes and graphs has been omitted completely. (This connection is discussed extensively in [3] and in [30].) On the other hand, apart from a few elementary facts about vector spaces and groups, the present notes are self-contained. They can be considered as an introduction to the structure theory of schemes.

Secondly, the meaning of the symbols $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{P}$ is fixed for the whole text. We shall denote by $\mathbb{Z}$ the set of rational integers. We set

$$\mathbb{N} := \{z \in \mathbb{Z} \mid 0 \leq z\},$$

and the set of prime numbers will be denoted by $\mathbb{P}$.

Finally, this is probably a good place for me to express my thanks to Professor J. Tits, who was the first to encourage me to seek a general structure theory of finite schemes. I am also grateful to Professor E. Bannai and to Professor T. Ito, who invited me twice to visit Kyushu University, Fukuoka, and to whom I owe my interest in schemes. Finally, I am grateful to M. Rassy, who carefully read large parts of the manuscript, and to whom I owe numerous simplifications of the text.

Kiel, September 30, 1995 — Paul-Hermann Zieschang

Table of Contents

1. Basic Results

In this introductory chapter, we develop some basic terminology which will be used throughout the remainder of the text.

1.1 Structure Constants

Our first lemma is an immediate consequence of the definition of the structure constants of (X, G). (By δ we mean the Kronecker delta.)

Lemma 1.1.1 *Let e, $f \in G$ be given. Then*
(i) $a_{1ef} = \delta_{ef} = a_{f1e}$.
(ii) $a_{e^*f1} = \delta_{ef} n_{e^*}$.

Lemma 1.1.2 *Let $g \in G$ be given. Then*
(i) *For each $x \in X$, $|xg| = n_g$.*
(ii) $|g| = n_g|X|$.
(iii) *If $|X| \in \mathbb{N}$, $n_{g^*} = n_g$.*

Proof. (i) The definition of g^* yields $g^{**} = g$. Therefore, we have $|xg| = a_{gg^*1} = n_g$.

(ii) follows from (i).

(iii) Assume that $|X| \in \mathbb{N}$. Then, as $g \subseteq X \times X$, $|g| \in \mathbb{N}$. But from (ii) we also obtain that $n_{g^*}|X| = |g^*| = |g| = n_g|X|$. Therefore, $n_{g^*} = n_g$. □

Lemma 1.1.3 *Let d, e, $f \in G$ be given. Then*
(i) *For each $g \in G$, $\sum_{b \in G} a_{deb} a_{bfg} = \sum_{c \in G} a_{dcg} a_{efc}$.*
(ii) $a_{def} = a_{e^*d^*f^*}$.
(iii) $a_{dfe} n_e = a_{ef^*d} n_d$.

Proof. (i) Let y, $z \in X$ be such that $(y, z) \in g$. We count in two different ways the elements of $e \cap (yd \times zf^*)$. This proves (i).

(ii) Let y, $z \in X$ be such that $(y, z) \in f$. Then, by definition,

$$a_{def} = |yd \cap ze^*| = |ze^* \cap yd^{**}| = a_{e^*d^*f^*}.$$

(iii) Apply (i) to $(f, e^*, 1)$ in the role of (e, f, g), and use (ii). □

Lemma 1.1.4 *Let e, $f \in G$ be given. Then*
(i) $\sum_{g \in G} a_{gef} = n_{e^*}$.
(ii) $\sum_{g \in G} a_{fge} = n_f$.
(iii) $\sum_{g \in G} a_{e^*fg} n_g = n_{e^*} n_f$.

Proof. (i) Let y, $z \in X$ be such that $(y, z) \in f$. We count in two different ways the elements of $\{(x, g) \in ze^* \times G \mid (y, x) \in g\}$. This proves (i).

(ii) For each $g \in G$, we apply Lemma 1.1.3(ii) to (f, g, e) in the role of (d, e, f). Then

$$\sum_{g \in G} a_{fge} = \sum_{g \in G} a_{g^*f^*e^*} = n_f;$$

use (i).

(iii) For each $g \in G$, we apply Lemma 1.1.3(iii) to (e^*, g) in the role of (d, e). Then

$$\sum_{g \in G} a_{e^*fg} n_g = \sum_{g \in G} a_{gf^*e^*} n_{e^*} = n_{e^*} n_f;$$

use (i). □

Let $g \in G$, let $n \in \mathbb{N} \setminus \{0, 1, 2\}$, and let $f_1, \ldots, f_n \in G$ be given. Then $a_{f_1 \cdots f_n g}$ is defined recursively by

$$a_{f_1 \cdots f_n g} := \sum_{e \in G} a_{f_1 \cdots f_{n-1} e} a_{e f_n g}.$$

The following lemma generalizes Lemma 1.1.3(i), (ii) and Lemma 1.1.4(iii).

Lemma 1.1.5 *Let $n \in \mathbb{N} \setminus \{0, 1\}$, and let $f_1, \ldots, f_n \in G$ be given.*
(i) *Assume that $3 \leq n$, and let $g \in G$ be given. Then we have $a_{f_1 \cdots f_n g} = \sum_{e \in G} a_{f_1 e g} a_{f_2 \cdots f_n e}$.*
(ii) *For each $g \in G$, $a_{f_1 \cdots f_n g} = a_{f_n^* \cdots f_1^* g^*}$.*
(iii) $\sum_{g \in G} a_{f_1 \cdots f_n g} n_g = n_{f_1} \cdots n_{f_n}$.

Proof. (i) If $n = 3$, the claim is just a restatement of Lemma 1.1.3(i) (with (e, f_1, f_2, f_3) in the role of (c, d, e, f)). Therefore, we assume that $4 \leq n$. Assuming that the claim holds for $n-1$, we obtain that

$$a_{f_1 \ldots f_n g} = \sum_{c \in G} a_{f_1 \ldots f_{n-1} c} a_{c f_n g} = \sum_{c \in G} (\sum_{d \in G} a_{f_1 d c} a_{f_2 \ldots f_{n-1} d}) a_{c f_n g} =$$

$$\sum_{d \in G} a_{f_2 \ldots f_{n-1} d} \sum_{c \in G} a_{f_1 d c} a_{c f_n g} = \sum_{d \in G} a_{f_2 \ldots f_{n-1} d} \sum_{e \in G} a_{f_1 e g} a_{d f_n e} =$$

$$\sum_{e \in G} a_{f_1 e g} (\sum_{d \in G} a_{f_2 \ldots f_{n-1} d} a_{d f_n e}) = \sum_{e \in G} a_{f_1 e g} a_{f_2 \ldots f_n e};$$

use Lemma 1.1.3(i).

(ii) If $n = 2$, the claim is just a restatement of Lemma 1.1.3(ii) (with (f_1, f_2, g) in the role of (d, e, f)). Therefore, we assume that $3 \leq n$. Assuming that the claim holds for $n-1$, we obtain that

$$a_{f_1 \ldots f_n g} = \sum_{e \in G} a_{f_1 \ldots f_{n-1} e} a_{e f_n g} = \sum_{e \in G} a_{f_n^* e^* g^*} a_{f_{n-1}^* \ldots f_1^* e^*} = a_{f_n^* \ldots f_1^* g^*};$$

use Lemma 1.1.3(ii) and (i).

(iii) If $n = 2$, the claim is just a restatement of Lemma 1.1.4(iii) (with (f_1, f_2) in the role of (e^*, f)). Therefore, we assume that $3 \leq n$. Assuming that the claim holds for $n-1$, we obtain that

$$\sum_{g \in G} a_{f_1 \ldots f_n g} n_g = \sum_{g \in G} (\sum_{e \in G} a_{f_1 \ldots f_{n-1} e} a_{e f_n g}) n_g = \sum_{e \in G} a_{f_1 \ldots f_{n-1} e} \sum_{g \in G} a_{e f_n g} n_g =$$

$$\sum_{e \in G} a_{f_1 \ldots f_{n-1} e} (n_e n_{f_n}) = (\sum_{e \in G} a_{f_1 \ldots f_{n-1} e} n_e) n_{f_n} = n_{f_1} \cdots n_{f_n};$$

use Lemma 1.1.4(iii). □

1.2 The Complex Product

Recall that, for all $E, F \subseteq G$,

$$EF := \{g \in G \mid \sum_{e \in E} \sum_{f \in F} a_{efg} \neq 0\}.$$

Lemma 1.2.1 *Let D, E, $F \subseteq G$ be given. Then*
(i) *If $E \subseteq F$, $DE \subseteq DF$ and $ED \subseteq FD$.*
(ii) $(DE)F = D(EF)$.

Proof. (i) follows immediately from the definition of the complex product.

(ii) Let $g \in (DE)F$ be given. Then, by definition, there exist $b \in DE$ and $f \in F$ such that $a_{bfg} \neq 0$. Since $b \in DE$, we find $d \in D$ and $e \in E$ such that $a_{deb} \neq 0$. It follows that $a_{deb}a_{bfg} \neq 0$.

Since $a_{deb}a_{bfg} \neq 0$, there exists $c \in G$ such that $a_{dcg}a_{efc} \neq 0$; see Lemma 1.1.3(i). From this we conclude that $a_{dcg} \neq 0$ and that $a_{efc} \neq 0$. From $a_{efc} \neq 0$ we obtain that $c \in EF$. Thus, as $a_{dcg} \neq 0$, $g \in D(EF)$.

Since $g \in (DE)F$ has been chosen arbitrarily, we have shown that $(DE)F \subseteq D(EF)$. Similarly, one obtains that $D(EF) \subseteq (DE)F$. □

From the definition of the complex product we obtain immediately that, for all E, $F \subseteq G$,

$$EF = \emptyset \;\Leftrightarrow\; \emptyset \in \{E, F\}.$$

Moreover, from Lemma 1.1.1(i) we deduce easily that, for each $F \subseteq G$,

$$F\{1\} = F = \{1\}F.$$

Therefore, Lemma 1.2.1(ii) says that the set of all non-empty subsets of G is a monoid with respect to the complex multiplication and with $\{1\}$ as identity element. This monoid will play an important role in the fifth chapter of these notes.

For each $F \subseteq G$, we define

$$F^* := \{f^* \mid f \in F\}.$$

It is obvious that $F^{**} = F$.

Lemma 1.2.2 *Let E, $F \subseteq G$ be given. Then $(EF)^* = F^*E^*$.*

Proof. For each $g \in G$, we have

$$g \in (EF)^* \;\Leftrightarrow\; g^* \in EF \;\Leftrightarrow\; \sum_{e\in E}\sum_{f\in F} a_{efg^*} \neq 0 \;\Leftrightarrow$$

$$\sum_{e\in E}\sum_{f\in F} a_{f^*e^*g} \neq 0 \;\Leftrightarrow\; \sum_{f\in F^*}\sum_{e\in E^*} a_{feg} \neq 0 \;\Leftrightarrow\; g \in F^*E^*;$$

use Lemma 1.1.3(ii). □

Let $n \in \mathbb{N} \setminus \{0, 1, 2\}$, and let $F_1, \ldots, F_n \subseteq G$ be given. Then $F_1 \cdots F_n$ is defined recursively by

$$F_1 \cdots F_n = (F_1 \cdots F_{n-1})F_n.$$

The following lemma generalizes Lemma 1.2.1(ii) and Lemma 1.2.2.

Lemma 1.2.3 *Let $n \in \mathbb{N} \setminus \{0,1\}$, and let $F_1, \ldots, F_n \subseteq G$ be given. Then we have*

(i) *If $3 \leq n$, $F_1 \cdots F_n = F_1(F_2 \cdots F_n)$.*
(ii) $(F_1 \cdots F_n)^* = F_n^* \cdots F_1^*$.

Proof. (i) If $n = 3$, the claim is just a restatement of Lemma 1.2.1(ii). Therefore, we assume that $4 \leq n$. Assuming that the claim holds for $n - 1$, we obtain that

$$F_1 \cdots F_n = (F_1 \cdots F_{n-1})F_n = (F_1(F_2 \cdots F_{n-1}))F_n =$$
$$F_1((F_2 \cdots F_{n-1})F_n) = F_1(F_2 \cdots F_n);$$

use Lemma 1.2.1(ii).

(ii) If $n = 2$, the claim is just a restatement of Lemma 1.2.2. Therefore, we assume that $3 \leq n$. Assuming that the claim holds for $n - 1$, we obtain that

$$(F_1 \cdots F_n)^* = (F_1(F_2 \cdots F_n))^* = (F_2 \cdots F_n)^* F_1^* =$$
$$(F_n^* \cdots F_2^*)F_1^* = F_n^* \cdots F_1^*;$$

use (i) and Lemma 1.2.2. □

Let $F \subseteq G$ be given. For each $x \in X$, we define

$$xF := \bigcup_{f \in F} xf.$$

For each $g \in G$, we set

$$gF := \{g\}F$$

and $Fg := F\{g\}$.

Lemma 1.2.4 *Let $n \in \mathbb{N} \setminus \{0,1\}$, and let $F_1, \ldots, F_n \subseteq G$ be given. Then, for each $g \in G$, the following conditions are equivalent.*

(a) $g \in F_1 \cdots F_n$.

(b) *Let $y, z \in X$ be such that $(y,z) \in g$. Then there exist $x_0, \ldots, x_n \in X$ such that $x_0 = y$, $x_n = z$, and, for each $i \in \{1, \ldots, n\}$, $x_i \in x_{i-1}F_i$.*

(c) *There exist $x_0, \ldots, x_n \in X$ such that $(x_0, x_n) \in g$ and, for each $i \in \{1, \ldots, n\}$, $x_i \in x_{i-1}F_i$.*

(d) *There exist $g_0, \ldots, g_n \in G$ such that $g_0 = 1$, $g_n = g$, and, for each $i \in \{1, \ldots, n\}$, $g_i \in g_{i-1}F_i$.*

(e) $\sum_{(f_1,\ldots,f_n) \in F_1 \times \ldots \times F_n} a_{f_1 \ldots f_n g} \neq 0$.

Proof. (a) ⇒ (b) The claim is obvious for $n = 2$. Therefore, we assume that $3 \leq n$.

Since $g \in (F_1 \cdots F_{n-1})F_n$, there exist $x \in X$ and $f \in F_1 \cdots F_{n-1}$ such that $(y, x) \in f$ and $z \in xF_n$. Now, by induction, there exist $x_0, \ldots, x_{n-1} \in X$ such that $x_0 = y$, $x_{n-1} = x$, and, for each $i \in \{1, \ldots, n-1\}$, $x_i \in x_{i-1}F_i$.

(b) ⇒ (c) This follows from the fact that $g \neq \emptyset$.

(c) ⇒ (d) For each $i \in \{0, \ldots, n\}$, let $g_i \in G$ be such that $(x_0, x_i) \in g_i$. Then $g_0 = 1$, $g_n = g$, and, for each $i \in \{1, \ldots, n\}$,

$$\sum_{f \in F_i} a_{g_{i-1} f g_i} \neq 0.$$

It follows that, for each $i \in \{1, \ldots, n\}$, $g_i \in g_{i-1}F_i$.

(d) ⇒ (e) The claim is obvious for $n = 2$. Therefore, we assume that $3 \leq n$.

By induction, we have

$$\sum_{(f_1, \ldots, f_{n-1}) \in F_1 \times \ldots \times F_{n-1}} a_{f_1 \cdots f_{n-1} g_{n-1}} \neq 0.$$

Since $g \in g_{n-1}F_n$, there exists $f_n \in F_n$ such that $a_{g_{n-1} f_n g} \neq 0$. It follows that

$$\sum_{(f_1, \ldots, f_n) \in F_1 \times \ldots \times F_n} a_{f_1 \cdots f_{n-1} g_{n-1}} a_{g_{n-1} f_n g} \neq 0.$$

Thus, we have

$$\sum_{(f_1, \ldots, f_n) \in F_1 \times \ldots \times F_n} a_{f_1 \cdots f_n g} = \sum_{(f_1, \ldots, f_n) \in F_1 \times \ldots \times F_n} \sum_{e \in G} a_{f_1 \cdots f_{n-1} e} a_{e f_n g} \neq 0.$$

(e) ⇒ (a) Again, the claim is obvious for $n = 2$. Therefore, we assume that $3 \leq n$.

Let $(f_1, \ldots, f_n) \in F_1 \times \ldots \times F_n$ be such that $a_{f_1 \cdots f_n g} \neq 0$. Then there exists $e \in G$ such that $a_{f_1 \cdots f_{n-1} e} \neq 0$ and $a_{e f_n g} \neq 0$. From $a_{f_1 \cdots f_{n-1} e} \neq 0$ we obtain by induction that $e \in F_1 \cdots F_{n-1}$. On the other hand, $a_{e f_n g} \neq 0$ implies that $g \in eF_n$. Thus, by Lemma 1.2.1(i), $g \in F_1 \cdots F_n$. □

For all e, $f \in G$, we abbreviate

$$ef := e\{f\}.$$

Note that, for all E, $F \subseteq G$,

$$EF = \bigcup_{e \in E} eF = \bigcup_{f \in F} Ef = \bigcup_{e \in E} \bigcup_{f \in F} ef.$$