Dirk van Dalen

Logic and Structure

Second Edition

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Springer-Verlag
Berlin Heidelberg New York Tokyo 1983

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AMS Subject Classification (1980): 03-01

ISBN 3-540-12831-X Springer-Verlag Berlin Heidelberg New York Tokyo
ISBN 0-387-12831-X Springer-Verlag New York Heidelberg Berlin Tokyo

ISBN 3-540-09893-3 1. Auflage Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-09893-3 1st Edition Springer-Verlag New York Heidelberg Berlin

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© by Springer-Verlag Berlin Heidelberg 1980, 1983
Printed in Germany

Printing and bookbinding: Beltz, Offsetdruck, Hemsbach
2141/3140-543210

Preface

Logic appears in a 'sacred' and in a 'profane' form. The sacred form is dominant in proof theory, the profane form in model theory. The phenomenon is not unfamiliar, one observes this dichotomy also in other areas, e.g. set theory and recursion theory. For one reason or another, such as the discovery of the set theoretical paradoxes (Cantor, Russell), or the definability paradoxes (Richard, Berry), a subject is treated for some time with the utmost awe and diffidence. As a rule, however, sooner or later people start to treat the matter in a more free and easy way.

Being raised in the 'sacred' tradition, I was greatly surprised (and somewhat shocked) when I observed Hartley Rogers teaching recursion theory to mathematicians as if it were just an ordinary course in, say, linear algebra or algebraic topology. In the course of time I have come to accept his viewpoint as the didactically sound one: before going into esoteric niceties one should develop a certain feeling for the subject and obtain a reasonable amount of plain working knowledge.

For this reason I have adopted the profane attitude in this introductory text, reserving the more sacred approach for advanced courses. Readers who want to know more about the latter aspect of logic are referred to the immortal texts of Hilbert-Bernays or Kleene.

The present book has developed out of courses given at the University at Utrecht in the mathematics department to undergraduates. The experience drawn from these courses and the reactions of the participants was that one should try to incorporate bits of real mathematics as soon as possible. For that reason the well-known structures, such as groups, partially ordered sets, projective planes, are introduced at the earliest possible occasion.

By now, it is generally agreed that a mathematician should know how to formalize his language and his semantics. One of the traditional stumbling blocks in any logic course is the awkward business of actually proving theorems. As a rule, one gets over this part as quickly as possible and ignores formal proof. I have, in an effort to stimulate the students, introduced Gentzen's system of natural deduction, without going into the esoteric issues of proof theory. As an extra benefit, Gentzen's natural deduction represents intuitive reasoning quite faithfully. In this system students can fairly easily construct derivations for

themselves, and what is more: they generally like it. The technical details, required for the proof of the completeness theorem, are kept to a minimum in this approach. In the third chapter we initiate the study of models of various theories. The basic theorems, e.g. the compactness theorem, the Skolem-Löwenheim theorems, are applied to a number of 'real life' examples, such as arithmetic and analysis. It is shown how to obtain non-standard models, and how to exploit infinitely large numbers and infinitesimals. Skolem functions are also introduced, and the introduction of definable functions and predicates is studied. Finally, using the already gathered knowledge, a quick exposition of second-order logic is given; a topic that has regained some attention after being overshadowed for a time by set theory. We have refrained from treating the incompleteness theorems and recursion theory; however important those subjects are, they are not of central importance for mathematicians who do not intend to specialize in logic or computer science. Neither have we included intuitionistic logic. Again, the subject deserves a treatment, but not (yet?) in an introduction for general mathematicians.
Strictly speaking there are hardly any prerequisites for the reader; the first chapter can be understood by everybody. The second chapter uses a few elementary facts about algebra for the sake of illustrating the logical notions. In the last chapter some facts from algebra are used, but certainly not more than one would find in any introductory text.
As a matter of fact the reader does not have to be an accomplished algebraist, as long as he is to a certain degree familiar with the main notions and structures of elementary mathematics. One should recognize certain structures as old friends, and one should be prepared to make new friends.

In a text on logic one has the choice of slipping over tedious, technical details, or elaborating them ad nauseam. I have tried to avoid both extremes by covering some proofs in full detail, and leaving some routine matters to the reader. It should be stressed, however, that the majority of the proofs is trivial, but that the reader should not take my word for it and convince himself by devising proofs by himself, only consulting the book when stuck or when not certain of the correctness.

In particular the reader is urged to do the exercises, some of which are merely a training ground for definitions,etc. There are also a number of problems that constitute minor (or even major) theorems, but that can be handled strictly on the basis of the material in the text. Finally, some problems require a bit of knowledge of such subjects as set theory, algebra or analysis. Since the latter kind of problems present a glimpse of the use of logic in the real world, they should not be neglected.

Some of the material in this book is optional, e.g. one can safely skip the parts that deal with normal forms, duality, functional completeness, switching and (when in a hurry) the sections on the missing connectives (1.7 and 2.9).

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In section 3.1 it is advisable to skip at first reading the proof of the completeness theorem and to go straight for the applications in 3.2. After getting the right feeling for this kind of material it is wise to get a thorough grasp of the proof of the completeness theorem, as the technique is very basic indeed.

Various people have contributed to the shaping of the text at one time or another. I wish to thank *Henk Barendregt*, who also tested the material in class, *Dana Scott* and *Jane Bridge* for their advice and criticism, and in particular *Jeff Zucker* who offered criticism and advice in a most unselfish way. It was he who suggested to incorporate the conjunction in our treatment of natural deduction.

Finally I want to express my gratitude and appreciation to *Sophie van Sterkenburg* for her patient typing and retyping.

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Contents

0. INTRODUCTION 1

1. PROPOSITIONAL LOGIC
   - 1.1. *Propositions and connectives* 4
   - 1.2. *Semantics* 13
   - 1.3. *Some properties of propositional logic* 20
   - 1.4. *Logic and Switching* 29
   - 1.5. *Natural deduction* 31
   - 1.6. *Completeness* 42
   - 1.7. *The missing connectives* 52

2. PREDICATE LOGIC
   - 2.1. *Quantifiers* 58
   - 2.2. *Structures* 59
   - 2.3. *The language of a similarity type* 61
   - 2.4. *Semantics* 69
   - 2.5. *Simple properties of predicate logic* 74
   - 2.6. *Identity* 82
   - 2.7. *Examples* 84
   - 2.8. *Natural deduction* 94
   - 2.9. *Adding the existential quantifier* 99
   - 2.10. *Natural deduction and identity* 101

3. COMPLETENESS AND APPLICATIONS
   - 3.1. *The completeness theorem* 106
   - 3.2. *Compactness and Skolem-Löwenheim* 115
   - 3.3. *Some model theory* 124
   - 3.4. *Skolem functions, or how to enrich your language* 144

4. SECOND-ORDER LOGIC 154

5. INTUITIONISTIC LOGIC
   - 5.1. *Constructive reasoning* 165
   - 5.2. *Intuitionistic propositional and predicate logic* 167
   - 5.3. *Kripke's semantics* 174
   - 5.4. *Some model theory* 185

6. APPENDIX 200

7. BIBLIOGRAPHY 203

8. GOTHIC ALPHABET 204

9. INDEX 205

0. Introduction

Without adopting one of the various views advocated in the foundations of mathematics, we may agree that mathematicians need and use a language, if only for the communication of their results and their problems. Whereas mathematicians have been claiming the greatest possible exactness for their methods, they should also be sensitive as to their means of communication. It is well known that Leibniz proposed to put the practice of mathematical communication and mathematical reasoning on a firm base; it was, however, not before the nineteenth century that those enterprises were (more) successfully undertaken by G. Frege and G. Peano. No matter how successfully and rigorously Frege, Russell, Hilbert, Bernays and others developed mathematical logic, it was onl in the second half of this century that logic and its language showed any features of interest to the general mathematician. The sophisticated results of Gödel were valued of course, but any pratical use was not made of them. Even Tarski's result on the decidability of elementary algebra and geometry had to bide its time before any applications turned up.

Nowadays the application of logic to algebra, analysis, topology, etc. are numerous and well-recognized. It seems strange that quite a number of simple facts, within the grasp of any student, were overlooked for such long time. It is not possible to give proper credit to all those who opened up this new territory, any list would inevitably show the preferences of the author, and neglect some fields and persons.

Let us note that mathematics has a fairly regular, canonical way of formulating its material, partly by its nature, partly under the influence of strong schools, like the one of Bourbaki. Furthermore the crisis at the beginning of this century has forced mathematicians to pay attention to the finer details of their language and to their assumptions oncerning the nature and the extent of the mathematical universe. This attention started to pay off when it was discovered that there was in some cases a close connection between a class of mathematical structures and its syntactical description.

We will give some examples.

It is well known that a subset of a group G which is closed under multiplication and inverse, is a group; however, a subset of an algebraically closed field F which is closed under sum, product, minus, and inverse, is in general not an algebraically closed field. This phenomenon is an instance of something quite general: an axiomatizable class of structures is axiomatized by a set of universal sentences (of the form $\forall x_1,\ldots,x_n\varphi$, with φ quantifier free) iff it is closed under substructures. If we check the axioms of group theory we see that indeed all axioms are universal, while not all the axioms of the theory of algebraically closed fields are universal. The latter fact could of course be accidental, it could be the case that we were not clever enough to discover a universal axiomatization of the class of algebraically closed fields. The above theorem of Tarski and Łos tells us, however, that it is impossible to find such an axiomatization!

The point of interest is that we have a simple syntactic criterion for some property of a class of structures. We can, so to speak, read the behaviour of the real mathematical world (in some simple cases) off from its syntactic description.

There are numerous examples of the same kind, e.g. *Lyndon's Theorem*: if a class of structures is axiomatizable then it is closed under homomorphisms iff it can be axiomatized by a set of positive sentences (i.e. sentences which, in prenex normal form with the open part in disjunctive normal form, do not contain negations).

The most basic and at the same time monumental example of such a connection between syntactical notions and the mathematical universe is of course *Gödel's completeness theorem*, which tells us that provability in any of the many available formal systems is extensionally identical with *truth* in all structures. That is to say, although provability and truth are totally different notations, (the first is combinatorial in nature, the latter set theoretical), they determine the same class of sentences: φ is provable iff φ is true in all structures.

Given the fact the study of logic involves a great deal of syntactical toil, we will set out by presenting an efficient machinery for dealing with syntax. We use the technique of *inductive definitions* and as a consequence we are rather inclined to see trees wherever possible e.g. we prefer natural deduction in the tree form to the linear versions that are here and there in use.

One of the amazing phenomena in the development of the foundations of mathematics is the discovery that the language of mathematics itself can be studied by mathematical means. This is far from a futile play: Gödel's incompleteness theorems, for instance, lean heavily on a mathematical analysis of the language of arithmetic, and the work of Gödel and Cohen in the field of the independence proofs in set theory requires a thorough knowledge of the mathematics of mathematical language. These topics are not in the scope of the present book, so we can confine ourselves to the simpler parts of the syntax. Nonetheless we will aim at a thorough treatment, in the hope that the reader will realize that all these things which he suspects to be trivial, but cannot see why, are perfectly amenable to proof. It may help the reader to think of himself as a computer with great mechanical capabilities, but with no creative insight, in those cases where he is puzzled because 'why should we prove something so utterly evident!' On the other hand the reader should keep in mind that he is not a computer and that, certainly when he gets to chapter 3, certain details should be recognized as trivial.

For the actual practice of mathematics predicate logic is doubtlessly the perfect tool, since it allows us to handle individuals. All the same we start this book with an exposition of propositional logic. There are various reasons for this choice.

In the first place propositional logic offers in miniature the problems that we meet in predicate logic, but there the additional difficulties obscure some of the relevant features e.g. the completeness theorem for propositional logic already uses the concept of 'maximal consistent set', but without the complications of the Henkin axioms.

In the second place there are a number of truly propositional matters that would be difficult to treat in a chapter on predicate logic without creating an impression of discontinuity that borders on chaos. Finally it seems a matter of sound pedagogy to let propositional logic precede predicate logic. The beginner can in a simple context get used to the proof theoretical, algebraic and model theoretic skills that would be overbearing in a first encounter with predicate logic.

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1. Propositional Logic

1.1. PROPOSITIONS AND CONNECTIVES

Traditionally, logic is said to be the art (or study) of reasoning; so in order to describe logic in this tradition, we have to know what 'reasoning' is. According to some traditional views reasoning consists of the building of chains of linguistic entities by means of a certain relation '... follows from ....', a view which is good enough for our present purpose. The linguistic entities occurring in this kind of reasoning are taken to be *sentences*, i.e. entities that express a complete thought, or state of affairs. We call those sentences *declarative*. This means that, from the point of view of natural language, our class of acceptable linguistic objects is rather restricted.

Fortunately this class is wide enough when viewed from the mathematician's point of view. So far logic has been able to get along pretty well under this restriction. True, one cannot deal with questions, or performative statements, but the role of these entities is negligible in pure mathematics. I must make an exception for performative statements, which play an important role in programming; think of instructions as "goto, if ... then, else ...", etc. For reasons given below, we will, however, leave them out of consideration.

The sentences we have in mind are of the kind "25 is a square number", "every positive integer is the sum of four squares", "there is only one empty set". A common feature of all those declarative sentence is the possibility of assigning them a truth value, *true* or *false*. We do not require the actual determination of the truth value in concrete cases, such as for instance Fermat's last problem of Riemann's hypothesis. It suffices that we can "in principle" assign a truth value.

Our so-called *two-valued* logic is based on the assumption that every sentence is either true or false, it is the cornerstone of the practice of truth tables.

Some sentences are minimal in the sense that there is no proper part which is also a sentence, e.g. $5 \in \{0,1,2,5,7\}$, or $2 + 2 = 5$; others can be taken apart into smaller parts, e.g. 'ρ is rational or ρ is irrational' (where ρ is some constant). Conversely, we can build larger sentences from smaller ones by using *connectives*. We know many connectives in natural language; the following list is by no means meant to be exhaustive: *and*, *or*, *not*, *if ... then ...*, *but*, *since*, *as*, *for*, *although*, *neither ... nor ...* . In ordinary discourse, and also in informal mathematics, one uses these connectives incessantly; however, in formal mathematics we will economize somewhat on the connectives we admit. This is mainly for reasons of exactness. Compare, for example, the following two sentences: "π is irrational, but it is not algebraic", "Max is a Marxist, but he is not humorless". In the second statement we may discover a suggestion of some contrast, as if we should be surprised that Max is not humorless. In the first case such a surprise cannot be so easily imagined (unless, e.g. one has just read that almost all irrationals are algebraic); without changing the meaning one can transform this statement into "π is irrational and π is not algebraic".

So why use (in a formal text) a formulation that carries vague, emotional undertones? For these and other reasons (e.g. of economy) we stick in logic to a limited number of connectives, in particular those that have shown themselves to be useful in the daily routine of formulating and proving.

Note, however, that even here ambiguities loom. Each of the connectives has already one or more meanings in natural language. We will give some examples:

1. John drove on and hit a pedestrian.
2. John hit a pedestrian and drove on.
3. If I open the window then we'll have fresh air.
4. If I open the window then $1 + 3 = 4$.
5. If $1 + 2 = 4$, then we'll have fresh air.
6. John is working or he is at home.
7. Euclid was a Greek or a mathematician.

From 1 and 2 we conclude that 'and' may have an ordering function in time. Not so in mathematics; "π is irrational and 5 is positive" simply means that both parts are the case. Time just does not play a role in formal mathematics. We could not very well say "π was neither rational nor irrational before 1882". What we would want to say is "before 1882 it was unknown whether π was rational or irrational".

In the examples 3 - 5 we consider the implication. Example 3 will be generally accepted, it displays a feature that we have come to accept as inherent to implication: there is a relation between the premise and the conclusion. This feature is lacking in the examples 4 and 5. Nonetheless we will allow cases such as 4 and 5 in mathematics. There are various reasons to do so. One is the consideration that meaning should be left out of syntactical considerations. Otherwise syntax would become unwieldy and we would run into an esoteric practice of exceptional cases. This general implication, in use in mathematics, is called *material implication*. The other implications have been studied under the names of *strict* *implication*, *relevant implication*, etc.

Finally 6 and 7 demonstrate the use of 'or'. We tend to accept 6 and to reject 7. One mostly thinks of 'or' as something exclusive. In 6 we more or less expect John not to work at home, while 7 is unusual in the sense that we as a rule do not use 'or' when we could actually use 'and'. There is also a habit of not using a disjunction if we already know which of the two parts is the case e.g. "32 is a prime or 32 is not a prime" will be considered artificial (to say the least) by most of us, since we already know that 32 is not a prime. Yet mathematics freely uses such superfluous disjunctions, for example "2 $\geqslant$ 2" (which stands for "2 > 2 or 2 = 2").

In order to provide mathematics with a precise language we will create an artificial, formal language, which will lend itself to mathematical treatment. First we will define a language for propositional logic, i.e. the logic which deals only with *propositions* (sentences, statements). Later we will extend our treatment to a logic which also takes properties of individuals into account.

The process of *formalization* of propositional logic consists of two stages: (1) present a formal language, (2) specify a procedure for obtaining *valid* or *true* propositions.

We will first describe the language, using the technique of *inductive definitions*. The procedure is quite simple:

First give the smallest propositions, which are not decomposable into smaller propositions; next describe how composite propositions are constructed out of already given propositions.

1.1.1. Definition. The language of propositional logic has an alphabet consisting of (i) proposition symbols: $p_0, p_1, p_2, \ldots$,

(ii) connectives : $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\leftrightarrow$, $\perp$,

(iii) auxilliary symbols : (,).

The connectives carry traditional names:

$\wedge$	- and	- conjunction
$\vee$	- or	- disjunction
$\rightarrow$	- if ..., then ...	- implication
$\neg$	- not	- negation
$\leftrightarrow$	- iff	- equivalence, bi-implication
$\perp$	- falsity	- falsum, absurdum

The proposition symbols and $\perp$ stand for the indecomposable propositions, which we call *atoms*, or *atomic propositions*.

1.1.2. Definition. The set PROP of propositions is the smallest set X with the properties.

(i) $p_i \in X \quad (i \in N)$, $\perp \in X$,

(ii) $\varphi, \psi \in X \Rightarrow (\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \to \psi)$, $(\varphi \leftrightarrow \psi) \in X$,

(iii) $\varphi \in X \Rightarrow (\neg \varphi) \in X$.

The clauses describe exactly the possible ways of building propositions. In order to simplify clause (ii) we write $\varphi, \psi \in X \Rightarrow (\varphi \square \psi) \in X$, where $\square$ is one of the connectives $\wedge$, $\vee$, $\to$, $\leftrightarrow$.

A warning to the reader is in order here. We have used Greek letters φ,ψ in the definition; are they propositions?
Clearly we did not intend them to be so, as we want only those strings of symbols obtained by combining symbols of the alphabet in a correct way. Evidently no Greek letters come in at all! The explanation is as follows: φ and ψ are used as variables for propositions. Since we want to study logic, we must use a language to discuss it in. As a rule this language is plain, everyday English. We call the language used to discuss logic our *meta-language* and φ and ψ are *meta-variables* for propositions. We could do without meta-variables by handling (ii) and (iii) verbally: if two propositions are given, then a new proposition is obtained by placing the connective $\wedge$ between them and by adding brackets in front and at the end, etc. This verbal version should suffice to convince the reader of the advantage of the mathematical machinery.

Note that we have added a rather unusual connective, $\perp$. Unusual, in the sense that it does not connect anything. *Logical constant* would be a better name. For uniformity we stick to our present usage. $\perp$ is added for convenience, one could very well do without it, but it has certain advantages.

Examples. $(p_7 \to p_0)$, $((\perp \vee p_{32}) \wedge (\neg p_2)) \in \text{PROP}$.
$p_1 \leftrightarrow p_7$, $\neg\neg \perp$, $((\to \wedge \notin \text{PROP}$.

It is easy to show that something belongs to PROP (just carry out the construction according to 1.1.2); it is much harder to show that something does not belong to PROP. We will do one example:

$$\neg\neg \perp \notin \text{PROP}.$$

Suppose $\neg\neg \perp \in X$ and X satisfies (i), (ii), (iii) of definition 1.1.2.

We claim that $Y = X - \{\neg\neg \bot\}$ also satisfies (i), (ii) and (iii). Since $\bot, p_i \in X$, also $\bot, p_i \in Y$. If $\varphi, \psi \in Y$, then $\varphi, \psi \in X$. Since X satisfies (ii) $(\varphi \square \psi) \in X$. From the form of the expressions it is clear that $(\varphi \square \psi) \neq \neg\neg \bot$ (look at the brackets), so $(\varphi \square \psi) \in X - \{\neg\neg \bot\} = Y$. Likewise one shows that Y satisfies (iii). Hence X is not the smallest set satisfying (i), (ii) and (iii), so $\neg\neg \bot$ cannot belong to PROP.

Properties of propositions are established by an inductive procedure analogous to definition 1.1.2: first deal with the atoms, and then go from the parts to the composite propositions. This is made precise in

1.1.3. Theorem. Let A be a property, then $A(\varphi)$ holds for all $\varphi \in$ PROP if

(i) $A(p_i)$, for all i, and $A(\bot)$,

(ii) $A(\varphi), A(\psi) \Rightarrow A((\varphi \square \psi))$,

(iii) $A(\varphi) \Rightarrow A((\neg \varphi))$.

Proof. Let $X = \{\varphi \in \text{PROP} \mid A(\varphi)\}$, then X satisfies (i), (ii) and (iii) of definition 1.1.2. So $\text{PROP} \subseteq X$, i.e. for all $\varphi \in$ PROP $A(\varphi)$ holds. □

We call an application of theorem 1.1.3 a *proof by induction on* φ. The reader will note an obvious similarity between the above theorem and the principle of complete induction in arithmetic.

1.1.4. Definition. (a) A sequence $\varphi_1, \ldots, \varphi_n$ is called a *formation sequence* of φ if $\varphi_n = \varphi$ and for all $i \leq n$

(i) φ_i = atomic, or

(ii) $\varphi_i = (\varphi_j \square \varphi_k)$ for certain $j, k < i$, or

(iii) $\varphi_i = (\neg \varphi_j)$ for certain $j < i$.

(b) φ is a *subformula* (cf. exercise 9) of ψ if

(i) $\varphi = \psi$, or

(ii) $\psi = (\psi_1 \square \psi_2)$ and φ is a subformula of ψ_1 or of ψ_2, or

(iii) $\psi = (\neg \psi_1)$ and φ is a subformula of ψ_1.

Examples. (a) $\bot, p_2, p_3, (\bot \vee p_2), (\neg (\bot \vee p_2)), (\neg p_3)$ and $p_3, (\neg p_3)$ are both formation sequences of $(\neg p_3)$. Note that formation sequences may contain 'garbage'.
(b) p_2 is a subformula of $((p_7 \vee (\neg p_2)) \rightarrow p_1)$. $(p_1 \rightarrow \bot)$ is a subformula of $(((p_2 \vee (p_1 \wedge p_0)) \leftrightarrow (p_1 \rightarrow \bot))$.

We now give some trivial examples of proof by induction.

1. *Each proposition has an even number of brackets.*

Proof. (i) Each atom has 0 brackets and 0 is even.

(ii) Suppose φ and ψ have 2n, resp. 2m brackets, then $(\varphi \square \psi)$ has 2(n+m+1) brackets.

(iii) Suppose φ has 2n brackets, then $(\neg \varphi)$ has 2(n+1) brackets.

2. *Each proposition has a formation sequence.*

Proof. (i) If φ is an atom, then the sequence consisting of just φ is a formation sequence of φ.

(ii) Let $\varphi_1,\ldots,\varphi_n$ and $\psi_1,\ldots,\psi_m$ be formation sequences of φ and ψ, then one easily sees that $\varphi_1,\ldots,\varphi_n, \psi_1,\ldots,\psi_m, (\varphi_n \square \psi_m)$ is a formation sequence of $(\varphi \square \psi)$.

(iii) Left to the reader.

We can improve on 2:

1.1.5. Theorem. PROP is the set of all expressions having formation sequences.

Proof. Let F be the set of all expressions (i.e. strings of symbols) having formation sequences. We have shown that PROP $\subseteq$ F.
Let φ have a formation sequence $\varphi_1,\ldots,\varphi_n$, we show $\varphi \in$ PROP by induction on n.
$n = 1$: $\varphi = \varphi_1$ and by definition φ is atomic, so $\varphi \in$ PROP. Suppose that all expressions with formation sequences of length $m < n$ are in PROP. By definition $\varphi_n = (\varphi_i \square \varphi_j)$ for $i,j < n$, or $\varphi_n = (\neg \varphi_i)$ for $i < n$, or φ_n is atomic. In the first case φ_i and φ_j have formation sequences of length $i,j < n$, so by induction hypothesis $\varphi_i, \varphi_j \in$ PROP. As PROP satisfies the clauses of definition 1.1.2, also $(\varphi_i \square \varphi_j) \in$ PROP. Treat negation likewise. The atomic case is trivial. Conclusion F $\subseteq$ PROP. □

Theorem 1.1.5 is in a sense a justification of the definition of formation sequence. It also enables us to establish properties of propositions by ordinary induction on the length of formation sequences.

In arithmetic one often defines functions by recursion, e.g. exponentiation is defined by $x^0 = 1$ and $x^{y+1} = x^y \cdot x$, or the factorial function by $0! = 1$ and $(x + 1)! = x! \cdot (x + 1)$.
The justification is rather immediate: each value is obtained by using the preceding values (for positive arguments). There is an analogous principle in our

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