A. Pazy

Semigroups of Linear Operators and Applications to Partial Differential Equations

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Preface

The aim of this book is to give a simple and self-contained presentation of the theory of semigroups of bounded linear operators and its applications to partial differential equations.

The book is a corrected and expanded version of a set of lecture notes which I wrote at the University of Maryland in 1972–1973. The first three chapters present a short account of the abstract theory of semigroups of bounded linear operators. Chapters 4 and 5 give a somewhat more detailed study of the abstract Cauchy problem for autonomous and nonautonomous linear initial value problems, while Chapter 6 is devoted to some abstract nonlinear initial value problems. The first six chapters are self contained and the only prerequisite needed is some elementary knowledge of functional analysis. Chapters 7 and 8 present applications of the abstract theory to concrete initial value problems for linear and nonlinear partial differential equations. Some of the auxiliary results from the theory of partial differential equations used in these chapters are stated without proof. References where the proofs can be found are given in the bibliographical notes to these chapters.

I am indebted to many good friends who read the lecture notes on which this book is based, corrected errors, and suggested improvements. In particular I would like to express my thanks to H. Brezis, M. G. Crandall, and P. Rabinowitz for their valuable advice, and to Danit Sharon for the tedious work of typing the manuscript.

A. PAZY

Contents

CHAPTER 1

Generation and Representation

1.1. Uniformly Continuous Semigroups of Bounded Linear Operators

Definition 1.1. Let X be a Banach space. A one parameter family $T(t)$, $0 \le t < \infty$, of bounded linear operators from X into X is a *semigroup of bounded linear operator on* X if

(i) $T(0) = I$, (I is the identity operator on X).
(ii) $T(t+s) = T(t)T(s)$ for every $t, s \ge 0$ (the semigroup property).

A semigroup of bounded linear operators, $T(t)$, is *uniformly continuous* if

$$\lim_{t \downarrow 0} \|T(t) - I\| = 0. \tag{1.1}$$

The linear operator A defined by

$$D(A) = \left\{ x \in X : \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\} \tag{1.2}$$

and

$$Ax = \lim_{t \downarrow 0} \frac{T(t)x - x}{t} = \left. \frac{d^+ T(t)x}{dt} \right|_{t=0} \quad \text{for} \quad x \in D(A) \tag{1.3}$$

is the *infinitesimal generator* of the semigroup $T(t)$, $D(A)$ is the domain of A.

This section is devoted to the study of uniformly continuous semigroups of bounded linear operators. From the definition it is clear that if $T(t)$ is a uniformly continuous semigroup of bounded linear operators then

$$\lim_{s \to t} \|T(s) - T(t)\| = 0. \tag{1.4}$$

Theorem 1.2. *A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.*

PROOF. Let A be a bounded linear operator on X and set

$$T(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}. \tag{1.5}$$

The right-hand side of (1.5) converges in norm for every $t \geq 0$ and defines, for each such t, a bounded linear operator $T(t)$. It is clear that $T(0) = I$ and a straightforward computation with the power series shows that $T(t + s) = T(t)T(s)$. Estimating the power series yields

$$\|T(t) - I\| \leq t\|A\|e^{t\|A\|}$$

and

$$\left\|\frac{T(t) - I}{t} - A\right\| \leq \|A\| \, \|T(t) - I\|$$

which imply that $T(t)$ is a uniformly continuous semigroup of bounded linear operators on X and that A is its infinitesimal generator.

Let $T(t)$ be a uniformly continuous semigroup of bounded linear operators on X. Fix $\rho > 0$, small enough, such that $\|I - \rho^{-1}\int_0^\rho T(s)\,ds\| < 1$. This implies that $\rho^{-1}\int_0^\rho T(s)\,ds$ is invertible and therefore $\int_0^\rho T(s)\,ds$ is invertible. Now,

$$\begin{aligned} h^{-1}(T(h) - I)\int_0^\rho T(s)\,ds &= h^{-1}\left(\int_0^\rho T(s+h)\,ds - \int_0^\rho T(s)\,ds\right) \\ &= h^{-1}\left(\int_\rho^{\rho+h} T(s)\,ds - \int_0^h T(s)\,ds\right) \end{aligned}$$

and therefore

$$h^{-1}(T(h) - I) = \left(h^{-1}\int_\rho^{\rho+h} T(s)\,ds - h^{-1}\int_0^h T(s)\,ds\right)\left(\int_0^\rho T(s)\,ds\right)^{-1} \tag{1.6}$$

Letting $h \downarrow 0$ in (1.6) shows that $h^{-1}(T(h) - I)$ converges in norm and therefore strongly to the bounded linear operator $(T(\rho) - I)(\int_0^\rho T(s)\,ds)^{-1}$ which is the infinitesimal generator of $T(t)$. □

From Definition 1.1 it is clear that a semigroup $T(t)$ has a unique infinitesimal generator. If $T(t)$ is uniformly continuous its infinitesimal generator is a bounded linear operator. On the other hand, every bounded linear operator A is the infinitesimal generator of a uniformly continuous semigroup $T(t)$. Is this semigroup unique? The affirmative answer to this question is given next.

Theorem 1.3. *Let $T(t)$ and $S(t)$ be uniformly continuous semigroups of bounded linear operators. If*

$$\lim_{t\downarrow 0}\frac{T(t)-I}{t}=A=\lim_{t\downarrow 0}\frac{S(t)-I}{t} \tag{1.7}$$

then $T(t)=S(t)$ for $t\ge 0$.

PROOF. We will show that given $T>0$, $S(t)=T(t)$ for $0\le t\le T$. Let $T>0$ be fixed, since $t\to\|T(t)\|$ and $t\to\|S(t)\|$ are continuous there is a constant C such that $\|T(t)\|\,\|S(s)\|\le C$ for $0\le s,t\le T$. Given $\varepsilon>0$ it follows from (1.7) that there is a $\delta>0$ such that

$$h^{-1}\|T(h)-S(h)\|<\varepsilon/TC \qquad \text{for} \quad 0\le h\le\delta. \tag{1.8}$$

Let $0\le t\le T$ and choose $n\ge 1$ such that $t/n<\delta$. From the semigroup property and (1.8) it then follows that

$$\|T(t)-S(t)\|=\left\|T\left(n\frac{t}{n}\right)-S\left(n\frac{t}{n}\right)\right\|$$

$$\le\sum_{k=0}^{n-1}\left\|T\left((n-k)\frac{t}{n}\right)S\left(\frac{kt}{n}\right)-T\left((n-k-1)\frac{t}{n}\right)S\left(\frac{(k+1)t}{n}\right)\right\|$$

$$\le\sum_{k=0}^{n-1}\left\|T\left((n-k-1)\frac{t}{n}\right)\right\|\left\|T\left(\frac{t}{n}\right)-S\left(\frac{t}{n}\right)\right\|\left\|S\left(\frac{kt}{n}\right)\right\|\le Cn\frac{\varepsilon}{TC}\frac{t}{n}\le\varepsilon.$$

Since $\varepsilon>0$ was arbitrary $T(t)=S(t)$ for $0\le t\le T$ and the proof is complete. □

Corollary 1.4. *Let $T(t)$ be a uniformly continuous semigroup of bounded linear operators. Then*

a) *There exists a constant $\omega\ge 0$ such that $\|T(t)\|\le e^{\omega t}$.*
b) *There exists a unique bounded linear operator A such that $T(t)=e^{tA}$.*
c) *The operator A in part* (b) *is the infinitesimal generator of $T(t)$.*
d) *$t\to T(t)$ is differentiable in norm and*

$$\frac{dT(t)}{dt}=AT(t)=T(t)A \tag{1.9}$$

PROOF. All the assertions of Corollary 1.4 follow easily from (b). To prove (b) note that the infinitesimal generator of $T(t)$ is a bounded linear operator A. A is also the infinitesimal generator of e^{tA} defined by (1.5) and therefore, by Theorem 1.3, $T(t)=e^{tA}$. □

1.2. Strongly Continuous Semigroups of Bounded Linear Operators

Throughout this section X will be a Banach space.

Definition 2.1. A semigroup $T(t)$, $0 \le t < \infty$, of bounded linear operators on X is a *strongly continuous* semigroup of bounded linear operators if

$$\lim_{t \downarrow 0} T(t)x = x \qquad \text{for every} \quad x \in X. \tag{2.1}$$

A strongly continuous semigroup of bounded linear operators on X will be called a *semigroup of class* C_0 or simply a C_0 *semigroup*.

Theorem 2.2. *Let $T(t)$ be a C_0 semigroup. There exist constants $\omega \ge 0$ and $M \ge 1$ such that*

$$\|T(t)\| \le Me^{\omega t} \qquad \textit{for} \quad 0 \le t < \infty. \tag{2.2}$$

Proof. We show first that there is an $\eta > 0$ such that $\|T(t)\|$ is bounded for $0 \le t \le \eta$. If this is false then there is a sequence $\{t_n\}$ satisfying $t_n \ge 0$, $\lim_{n \to \infty} t_n = 0$ and $\|T(t_n)\| \ge n$. From the uniform boundedness theorem it then follows that for some $x \in X$, $\|T(t_n)x\|$ is unbounded contrary to (2.1). Thus, $\|T(t)\| \le M$ for $0 \le t \le \eta$. Since $\|T(0)\| = 1$, $M \ge 1$. Let $\omega = \eta^{-1} \log M \ge 0$. Given $t \ge 0$ we have $t = n\eta + \delta$ where $0 \le \delta < \eta$ and therefore by the semigroup property

$$\|T(t)\| = \|T(\delta)T(\eta)^n\| \le M^{n+1} \le MM^{t/\eta} = Me^{\omega t}. \qquad \square$$

Corollary 2.3. *If $T(t)$ is a C_0 semigroup then for every $x \in X$, $t \to T(t)x$ is a continuous function from $\mathbb{R}_0^+$ (the nonnegative real line) into X.*

Proof. Let $t, h \ge 0$. The continuity of $t \to T(t)x$ follows from

$$\|T(t+h)x - T(t)x\| \le \|T(t)\|\,\|T(h)x - x\| \le Me^{\omega t}\|T(h)x - x\|$$

and for $t \ge h \ge 0$

$$\begin{aligned}\|T(t-h)x - T(t)x\| &\le \|T(t-h)\|\,\|x - T(h)x\| \\ &\le Me^{\omega t}\|x - T(h)x\|. \qquad \square\end{aligned}$$

Theorem 2.4. *Let $T(t)$ be a C_0 semigroup and let A be its infinitesimal generator. Then*

a) *For $x \in X$,*

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(s)x\,ds = T(t)x. \tag{2.3}$$

b) *For* $x \in X$, $\int_0^t T(s)x\,ds \in D(A)$ *and*

$$A\left(\int_0^t T(s)x\,ds\right) = T(t)x - x. \tag{2.4}$$

c) *For* $x \in D(A)$, $T(t)x \in D(A)$ *and*

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax. \tag{2.5}$$

d) *For* $x \in D(A)$,

$$T(t)x - T(s)x = \int_s^t T(\tau)Ax\,d\tau = \int_s^t AT(\tau)x\,d\tau. \tag{2.6}$$

PROOF. Part (a) follows directly from the continuity of $t \to T(t)x$. To prove (b) let $x \in X$ and $h > 0$. Then,

$$\begin{aligned}\frac{T(h)-I}{h}\int_0^t T(s)x\,ds &= \frac{1}{h}\int_0^t (T(s+h)x - T(s)x)\,ds\\ &= \frac{1}{h}\int_t^{t+h} T(s)x\,ds - \frac{1}{h}\int_0^h T(s)x\,ds\end{aligned}$$

and as $h \downarrow 0$ the right-hand side tends to $T(t)x - x$, which proves (b). To prove (c) let $x \in D(A)$ and $h > 0$. Then

$$\frac{T(h)-I}{h}T(t)x = T(t)\left(\frac{T(h)-I}{h}\right)x \to T(t)Ax \qquad \text{as} \quad h \downarrow 0. \tag{2.7}$$

Thus, $T(t)x \in D(A)$ and $AT(t)x = T(t)Ax$. (2.7) implies also that

$$\frac{d^+}{dt}T(t)x = AT(t)x = T(t)Ax,$$

i.e., that the right derivative of $T(t)x$ is $T(t)Ax$. To prove (2.5) we have to show that for $t > 0$, the left derivative of $T(t)x$ exists and equals $T(t)Ax$. This follows from,

$$\begin{aligned}&\lim_{h\downarrow 0}\left[\frac{T(t)x - T(t-h)x}{h} - T(t)Ax\right]\\ &\quad = \lim_{h\downarrow 0} T(t-h)\left[\frac{T(h)x - x}{h} - Ax\right] + \lim_{h\downarrow 0}(T(t-h)Ax - T(t)Ax),\end{aligned}$$

and the fact that both terms on the right-hand side are zero, the first since $x \in D(A)$ and $\|T(t-h)\|$ is bounded on $0 \le h \le t$ and the second by the strong continuity of $T(t)$. This concludes the proof of (c). Part (d) is obtained by integration of (2.5) from s to t. □

Corollary 2.5. *If* A *is the infinitesimal generator of a* C_0 *semigroup* $T(t)$ *then* $D(A)$, *the domain of* A, *is dense in* X *and* A *is a closed linear operator.*

PROOF. For every $x \in X$ set $x_t = 1/t\int_0^t T(s)x\,ds$. By part (b) of Theorem 2.4, $x_t \in D(A)$ for $t > 0$ and by part (a) of the same theorem $x_t \to x$ as $t \downarrow 0$. Thus $\overline{D(A)}$, the closure of $D(A)$, equals X. The linearity of A is evident. To prove its closedness let $x_n \in D(A)$, $x_n \to x$ and $Ax_n \to y$ as $n \to \infty$. From part (d) of Theorem 2.4 we have

$$T(t)x_n - x_n = \int_0^t T(s)Ax_n\,ds. \tag{2.8}$$

The integrand on the right-hand side of (2.8) converges to $T(s)y$ uniformly on bounded intervals. Consequently letting $n \to \infty$ in (2.8) yields

$$T(t)x - x = \int_0^t T(s)y\,ds. \tag{2.9}$$

Dividing (2.9) by $t > 0$ and letting $t \downarrow 0$, we see, using part (a) of Theorem 2.4, that $x \in D(A)$ and $Ax = y$. □

Theorem 2.6. *Let $T(t)$ and $S(t)$ be C_0 semigroups of bounded linear operators with infinitesimal generators A and B respectively. If $A = B$ then $T(t) = S(t)$ for $t \geq 0$.*

PROOF. Let $x \in D(A) = D(B)$. From Theorem 2.4 (c) it follows easily that the function $s \to T(t-s)S(s)x$ is differentiable and that

$$\begin{aligned}\frac{d}{ds}T(t-s)S(s)x &= -AT(t-s)S(s)x + T(t-s)BS(s)x\\ &= -T(t-s)AS(s)x + T(t-s)BS(s)x = 0.\end{aligned}$$

Therefore $s \to T(t-s)S(s)x$ is constant and in particular its values at $s = 0$ and $s = t$ are the same, i.e., $T(t)x = S(t)x$. This holds for every $x \in D(A)$ and since, by Corollary 2.5, $D(A)$ is dense in X and $T(t)$, $S(t)$ are bounded, $T(t)x = S(t)x$ for every $x \in X$. □

If A is the infinitesimal generator of a C_0 semigroup then by Corollary 2.5, $\overline{D(A)} = X$. Actually, a much stronger result is true. Indeed we have,

Theorem 2.7. *Let A be the infinitesimal generator of the C_0 semigroup $T(t)$. If $D(A^n)$ is the domain of A^n, then $\bigcap_{n=1}^{\infty} D(A^n)$ is dense in X.*

PROOF. Let $\mathfrak{D}$ be the set of all infinitely differentiable compactly supported complex valued functions on $]0,\infty[$. For $x \in X$ and $\varphi \in \mathfrak{D}$ set

$$y = x(\varphi) = \int_0^\infty \varphi(s)T(s)x\,ds. \tag{2.10}$$

If $h > 0$ then

$$\begin{aligned}\frac{T(h)-I}{h}y &= \frac{1}{h}\int_0^\infty \varphi(s)[T(s+h)x - T(s)x]\,ds\\ &= \int_0^\infty \frac{1}{h}[\varphi(s-h) - \varphi(s)]T(s)x\,ds.\end{aligned} \tag{2.11}$$

The integrand on the right-hand side of (2.11) converges as $h \downarrow 0$ to $-\varphi'(s)T(s)x$ uniformly on $[0, \infty[$. Therefore $y \in D(A)$ and

$$Ay = \lim_{h \downarrow 0} \frac{T(h) - I}{h} y = -\int_0^\infty \varphi'(s)T(s)x\,ds.$$

Clearly, if $\varphi \in \mathfrak{D}$ then $\varphi^{(n)}$, the n-th derivative of φ, is also in $\mathfrak{D}$ for $n = 1, 2, \ldots$. Thus, repeating the previous argument we find that $y \in D(A^n)$

$$A^n y = (-1)^n \int_0^\infty \varphi^{(n)}(s)T(s)x\,ds \qquad \text{for} \quad n = 1, 2, \ldots$$

and consequently $y \in \bigcap_{n=1}^\infty D(A^n)$. Let $Y = \{x(\varphi) : x \in X, \varphi \in \mathfrak{D}\}$. Y is clearly a linear manifold. From what we have proved so far it follows that $Y \subseteq \bigcap_{n=1}^\infty D(A^n)$. To conclude the proof we will show that Y is dense in X. If Y is not dense in X, then by Hahn-Banach's theorem there is a functional $x^* \in X^*$, $x^* \neq 0$ such that $x^*(y) = 0$ for every $y \in Y$ and therefore

$$\int_0^\infty \varphi(s)x^*(T(s)x)\,ds = x^*\left(\int_0^\infty \varphi(s)T(s)x\,ds\right) = 0 \tag{2.12}$$

for every $x \in X$, $\varphi \in \mathfrak{D}$. This implies that for $x \in X$ the continuous function $s \to x^*(T(s)x)$ must vanish identically on $[0, \infty[$ since otherwise, it would have been possible to choose $\varphi \in \mathfrak{D}$ such that the left-hand side of (2.12) does not vanish. Thus in particular for $s = 0$, $x^*(x) = 0$. This holds for every $x \in X$ and therefore $x^* = 0$ contrary to the choice of x^*. □

We conclude this section with a simple application of Theorem 2.4.

Lemma 2.8. *Let A be the infinitesimal generator of a C_0 semigroup $T(t)$ satisfying $\|T(t)\| \leq M$ for $t \geq 0$. If $x \in D(A^2)$ then*

$$\|Ax\|^2 \leq 4M^2\|A^2x\|\,\|x\|. \tag{2.13}$$

PROOF. Using (2.6) it is easy to check that for $x \in D(A^2)$

$$T(t)x - x = tAx + \int_0^t (t-s)T(s)A^2x\,ds.$$

Therefore,

$$\begin{aligned}\|Ax\| &\leq t^{-1}(\|T(t)x\| + \|x\|) + t^{-1}\int_0^t (t-s)\|T(s)A^2x\|ds \\ &\leq \frac{2M}{t}\|x\| + \frac{Mt}{2}\|A^2x\|.\end{aligned} \tag{2.14}$$

Here we used that $M \geq 1$ (since $\|T(0)\| = 1$). If $A^2x = 0$ then (2.14) implies $Ax = 0$ and (2.13) is satisfied. If $A^2x \neq 0$ we substitute $t = 2\|x\|^{1/2}\|A^2x\|^{-1/2}$ in (2.14) and (2.13) follows. □

EXAMPLE 2.9. Let X be the Banach space of bounded uniformly continuous functions on $]-\infty, \infty[$ with the supremum norm. For $f \in X$ we define

$$(T(t)f)(s) = f(t+s).$$

It is easy to check that $T(t)$ is a C_0 semigroup satisfying $\|T(t)\| \leq 1$ for $t \geq 0$. The infinitesimal generator of $T(t)$ is defined on $D(A) = \{f: f \in X, f' \text{ exists}, f' \in X\}$ and $(Af)(s) = f'(s)$ for $f \in D(A)$. From Lemma 2.8 we obtain Landau's inequality

$$(\sup|f'(s)|)^2 \leq 4(\sup|f''(s)|)(\sup|f(s)|) \tag{2.15}$$

where the sup are taken over $]-\infty, \infty[$. Example 2.9 can be easily modified to the case where $X = L^p(-\infty, \infty)$, $1 < p < \infty$.

1.3. The Hille-Yosida Theorem

Let $T(t)$ be a C_0 semigroup. From Theorem 2.2 it follows that there are constants $\omega \geq 0$ and $M \geq 1$ such that $\|T(t)\| \leq Me^{\omega t}$ for $t \geq 0$. If $\omega = 0$, $T(t)$ is called *uniformly bounded* and if moreover $M = 1$ it is called a C_0 *semigroup of contractions*. This section is devoted to the characterization of the infinitesimal generators of C_0 semigroups of contractions. Conditions on the behavior of the resolvent of an operator A, which are necessary and sufficient for A to be the infinitesimal generator of a C_0 semigroup of contractions, are given.

Recall that if A is a linear, not necessarily bounded, operator in X, the resolvent set $\rho(A)$ of A is the set of all complex numbers λ for which $\lambda I - A$ is invertible, i.e., $(\lambda I - A)^{-1}$ is a bounded linear operator in X. The family $R(\lambda : A) = (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$ of bounded linear operators is called the resolvent of A.

Theorem 3.1 (Hille-Yosida). *A linear (unbounded) operator A is the infinitesimal generator of a C_0 semigroup of contractions $T(t)$, $t \geq 0$ if and only if*

(i) *A is closed and $\overline{D(A)} = X$.*
(ii) *The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and for every $\lambda > 0$*

$$\|R(\lambda : A)\| \leq \frac{1}{\lambda}. \tag{3.1}$$

Proof of Theorem 3.1 (Necessity). If A is the infinitesimal generator of a C_0 semigroup then it is closed and $\overline{D(A)} = X$ by Corollary 2.5. For $\lambda > 0$ and $x \in X$ let

$$R(\lambda)x = \int_0^\infty e^{-\lambda t}T(t)x\,dt. \tag{3.2}$$

Since $t \to T(t)x$ is continuous and uniformly bounded the integral exists as an improper Riemann integral and defines a bounded linear operator $R(\lambda)$ satisfying

$$\|R(\lambda)x\| \leq \int_0^\infty e^{-\lambda t}\|T(t)x\|\,dt \leq \frac{1}{\lambda}\|x\|. \tag{3.3}$$

Furthermore, for $h > 0$

$$\frac{T(h)-I}{h}R(\lambda)x = \frac{1}{h}\int_0^\infty e^{-\lambda t}(T(t+h)x - T(t)x)\,dt$$

$$= \frac{e^{\lambda h}-1}{h}\int_0^\infty e^{-\lambda t}T(t)x\,dt - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t}T(t)x\,dt. \tag{3.4}$$

As $h \downarrow 0$, the right-hand side of (3.4) converges to $\lambda R(\lambda)x - x$. This implies that for every $x \in X$ and $\lambda > 0$, $R(\lambda)x \in D(A)$ and $AR(\lambda) = \lambda R(\lambda) - I$, or

$$(\lambda I - A)R(\lambda) = I. \tag{3.5}$$

For $x \in D(A)$ we have

$$R(\lambda)Ax = \int_0^\infty e^{-\lambda t}T(t)Ax\,dt = \int_0^\infty e^{-\lambda t}AT(t)x\,dt$$

$$= A\left(\int_0^\infty e^{-\lambda t}T(t)x\,dt\right) = AR(\lambda)x. \tag{3.6}$$

Here we used Theorem 2.4 (c) and the closedness of A. From (3.5) and (3.6) it follows that

$$R(\lambda)(\lambda I - A)x = x \qquad \text{for} \quad x \in D(A). \tag{3.7}$$

Thus, $R(\lambda)$ is the inverse of $\lambda I - A$, it exists for all $\lambda > 0$ and satisfies the desired estimate (3.1). Conditions (i) and (ii) are therefore necessary. □

In order to prove that the conditions (i) and (ii) are sufficient for A to be the infinitesimal generator of a C_0 semigroup of contractions we will need some lemmas.

Lemma 3.2. *Let A satisfy the conditions* (i) *and* (ii) *of Theorem* 3.1 *and let* $R(\lambda : A) = (\lambda I - A)^{-1}$. *Then*

$$\lim_{\lambda\to\infty} \lambda R(\lambda : A)x = x \qquad \textit{for} \quad x \in X. \tag{3.8}$$

PROOF. Suppose first that $x \in D(A)$. Then

$$\|\lambda R(\lambda : A)x - x\| = \|AR(\lambda : A)x\|$$

$$= \|R(\lambda : A)Ax\| \le \frac{1}{\lambda}\|Ax\| \to 0 \qquad \text{as} \quad \lambda \to \infty.$$

But $D(A)$ is dense in X and $\|\lambda R(\lambda : A)\| \le 1$. Therefore $\lambda R(\lambda : A)x \to x$ as $\lambda \to \infty$ for every $x \in X$. □

We now define, for every $\lambda > 0$, the *Yosida approximation* of A by

$$A_\lambda = \lambda AR(\lambda : A) = \lambda^2 R(\lambda : A) - \lambda I. \tag{3.9}$$