GEOMETRIES AND GROUPS

Edited by M. ASCHBACHER, A. M. COHEN, and W. M. KANTOR

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Edited by
M. ASCHBACHER (CalTech, Pasadena)
A. M. COHEN (CWI, Amsterda
and
W. M. KANTOR (IAS, Princeton)

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Geometries and Groups, Finite and Algebraic

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MEIXNER, *Some groups with Coxeter diagrams*
HEWITT, $\cdot 1$ *and the geometries of type* B_4
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ZARA, *Generic modules for Fischer Groups*

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PREFACE

The workshop was set up in order to stimulate the interaction between (finite and algebraic) geometries and groups. Five areas of concentrated research were chosen on which attention would be focused, namely: diagram geometries and chamber systems with transitive automorphism groups, geometries viewed as incidence systems, properties of finite groups of Lie type, geometries related to finite simple groups, and algebraic groups. The list of talks (cf. page iii) illustrates how these subjects were represented during the workshop.

The contributions to these proceedings mainly belong to the first three areas; therefore, (i) diagram geometries and chamber systems with transitive automorphism groups, (ii) geometries viewed as incidence systems, and (iii) properties of finite groups of Lie type occur as section titles. The fourth and final section of these proceedings has been named graphs and groups; besides some graph theory, this encapsules most of the work related to finite simple groups that does not (explicitly) deal with diagram geometry. A few more words about the content:

(i). Diagram geometries and chamber systems with transitive automorphism groups. As a consequence of Tits' seminal work on the subject, all finite buildings are known. But usually, in a situation where groups are to be characterized by certain data concerning subgroups, a lot less is known than the full parabolic picture corresponding to the building. In this context the question was raised whether, for an arbitrary Coxeter diagram $M = (m_{ij})_{ij}$ over an index set I, all finite geometries over I with chamber transitive groups whose residues of type ij are isomorphic to a classical generalized m_{ij} -gon (coming from the parabolic structure of a finite Chevalley group of Lie rank 2) can be determined. The papers by Kantor, Meixner, Stroth and Timmesfeld show that the final answer may be expected soon.
A slight variation of the diagram, e.g., letting rank 2 residues be triple covers of a generalized quadrangle instead of the generalized quadrangle itself, allows for sporadic groups to occur as automorphism groups. The Monster is a most desired example; Segev has come as close to a geometric characterization as the sporadic group Co_1 involved in the Monster.

(ii). Geometries as incidence systems. Here the focus is on the characterization of geometries whose objects are point and lines (and possibly other types) for which natural axioms hold. Shult comes back to an early and already famous result of his, obtained jointly with Buekenhout, and relates on connections with new lines of research, such as the geometries related to the diagram C_3. Hanssens surveys the synthetic theory for point,line incidence systems coming from spherical buildings, ranging from the Buekenhout-Shult theorem to the latest results of his own. Buekenhout & Buset generalize the existing notion of diagram geometry further and come very close to notions for arbitrary graphs. Going in the other direction, Pasini specializes to geometries with diagram C_3, one of those hard low rank cases where geometries whose diagram is that of a building need not be a (decent quotient of a) building. Starting from the observation that highly transitive permutation groups provide geometries of matroid type, Cameron studies a version, called permutation geometries, in which the permutations involved need no longer constitute a group but in which many of the other properties relating to a matroid-type geometry are preserved.

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(iii). Chevalley groups. There are two surveys in this section. In the first, Smith expounds on various constructions of group representations starting from geometries, and muses on lines of future research. In the second, Kleidman & Liebeck survey the present knowledge on maximal subgroups of finite simple groups; though this is mainly a Chevalley group affair, they have included very recent theorems on the alternating and sporadic simple groups. Seitz gives powerful results relating the module structure of algebraic groups over an algebraically closed field of strictly positive characteristic to modules for the corresponding finite groups. In joint work with Gilkey , he also presents tables for dimensions of low degree representations of simple Lie algebras of characteristic p.

For many purposes, the classical groups are best studied on their standard module where they preserve a form of degree 2 (or, in the twisted version, a sesquilinear form). Aschbacher revives interest in 'classical' (non Lie-algebraic) definitions of several groups of exceptional Lie type and initiates a study of standard modules for them, replacing the form of degree 2 by a suitable form of higher degree (and, in the twisted case, similarly for the sesquilinear forms). One example is the 27-dimensional module for $E_6(q)$. Cohen & Cooperstein also deal with this standard module and, using a well-known point-line geometry, describe the orbits of 2-dimensional subspaces.

(iv). Graphs and groups. Many of the 2,3-transposition groups are Chevalley groups. But the mere fact that there are sporadic examples (the Fischer groups) implies that the geometric structure, known as Fischer graphs, is different from that of buildings. Norton and Zara study particular aspects of these Fischer graphs. Norton shows how these graphs lead to algebras and representations, Zara shows how they may be used to understand Fischer 's work and relax some of his conditions. Using the similarity to parabolics in a genuine Chevalley group, Stroth & Weiss set up generators and relations for one of the biggest sporadic group J_4. In the paper by Blokhuis & Brouwer, the emphasis is on graphs; they pin down the structure of a finite graph in which every two nonadjacent vertices are joined by a unique path of length 2. Finally, Surowski introduces a topological view of graphs obtained from specific Chevalley groups by use of tori and involutions, leading to many questions (some of which are number theoretic!).

So much for a description of topics. The workshop was located at the Leeuwenhorst Conference Center in Noordwijkerhout, The Netherlands. Financial support came from NATO; this is gratefully acknowledged. The workshop formed an 'Advanced Research Workshop' of their International Scientific Exchange Programme. The organizing committee consisted of M. Hazewinkel, W.M. Kantor, F. Timmesfeld, J. Tits, and A.M. Cohen. They would especially like to thank Ms. E. Both for her excellent administrative support before and during the workshop.

Arjeh M. COHEN
Workshop Director

(i) DIAGRAM GEOMETRIES AND CHAMBER SYSTEMS WITH TRANSITIVE GROUPS

On Amalgamation of Rank 1 parabolic Groups

by

F.G. Timmesfeld

§ 1 Introduction.

Abusing the notation of parabolic subgroups, we call a finite group P a rank n parabolic group of char.p, if and only if $\overline{P} = O^{p'}(P/O_p(P))$ is a perfect central extension of a finite simple rank n Lie-type group in char.p or one of the following exceptions:

$PSL_2(3)$ and ${}^2G_2(3)$ if $n = 1$, $p = 3$

$PSL_2(2)$, $PSU_3(2)$ and $Sz(2) \simeq F_{20}$ if $n = 1$ and $p = 2$

or

A_6, $\Sigma_6 \simeq Sp(4,2)$, $G_2(2)' \simeq U_3(3)$, $G_2(2)$, ${}^2F_4(2)'$ and ${}^2F_4(2)$ if $n = 2$ and $p = 2$.

If P is a rank n parabolic group in char.p, then a Borel subgroup B of P is just the normalizer of a p-Sylow subgroup. It is obvious that $B \cap O^{p'}(P)$ projects onto a Borel subgroup of the Lie-type group $\overline{P}$. In this paper we are mainly concerned with the embeddings of a "possible" Borel subgroup B into rank 1 parabolic groups. For this end remember that a group B is p-closed if it has a normal p-Sylow subgroup. We are now already in the position to formulate our:

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Theorem 1. Let B be a p-closed group and $\{P_i \mid i \in I\}$ a set of rank 1 parabolic groups of char.p. Suppose there exists a family $\{\chi_j \mid j \in J\}$ of monomorphisms:

$$\chi_j : B \to P_{i(j)}; \text{ where } j \to i(j) \text{ is a map from J in I}$$

such that, $\chi_j(B)$ is a Borel subgroup of $P_{i(j)}$ for each $j \in J$. Let $M_i = O_p(P_i)$, $\widetilde{P}_i = O^{p'}(P_i)$ and $N_i = [M_i, O^p(\widetilde{P}_i)]$ for $i \in I$. Then one of the following holds:

(1) $|J| \leq 2$

or (2) There exists a pair $j \neq k \in J$ such that

$$\chi_j^{-1}(N_{i(j)}) \leq \chi_k^{-1}(M_{i(k)}).$$

Theorem 1 is somewhat abstract. To make it more explicit we state some "concrete" corollaries and also an equivalent more "geometric" version of Theorem 1, which actually was the origin of that theorem. The proof of Theorem 1 depends on the classification of weak BN-pairs of rank 2 by Delgado-Stellmacher in [4], a forthcoming paper of Delgado describing the failure of factorization modules for the groups with a weak BN-pair of rank 2 and on certain "amalgam-type" arguments.

Before we can state the more "geometric" version of Theorem 1 we need some further notation.

We say the group G satisfies P_n if and only if the following holds:

(1) $G = \langle P_i \mid i \in I \rangle$, $|I| = n$; where the P_i are pairwise different rank 1 parabolic groups of the same char.p.

(2) The P_i have a common Borel-subgroup B.

We say G satisfies P_n^+ if it satisfies in addition:

(3) Let $P_{ij} = \langle P_i, P_j \rangle$ for $i \neq j \in I$. Then either P_{ij} is a rank 2 parabolic group of char.p with Borel subgroup B or $P_{ij} = P_i P_j$ and $B = P_i \cap P_j$.

It has been shown in [14] and [17] that the notion of groups satisfying P_n^+ is "more or less" equivalent to the notion of a classical, locally finite Tits chamber system C of rank n, with discrete transitive automorphism group G. If G satisfies P_n^+, then the diagram $\Delta = \Delta(I)$ is defined in the usual way. (This is also the diagram of the corresponding Tits chamber system C !) Generalizing this concept one can define a graph $\Gamma(I)$, if the group G just satisfies P_n, in the following way:

(a) I is the vertex set of $\Gamma(I)$.

(b) Let $M_i = O_p(P_i)$, $\widetilde{P}_i = O^{p'}(P_i)$. Then (i,j) is an edge of $\Gamma(I)$ if and only if

$$M_i \cap M_j \text{ is not normal in } \widetilde{P}_i \text{ and in } \widetilde{P}_j.$$

It will be shown in (2.3) that, if G satisfies P_n^+, i and j are connected in $\Gamma(I)$ if and only if they are connected in $\Delta(I)$. Now the "geometric" version of Theorem 1.

Theorem 2. Suppose the group G satisfies P_n. Then $\Gamma(I)$ contains no triangles.

Theorem 2 can be considered as a generalization of the non-existence-theorem for classical locally finite Tits chamber systems of generalized triangular type (i.e. $m_{ij} \geq 3$ for all $i \neq j \leq 3$!) with discrete transitive automorphism group, which is in arbitrary characteristic a consequence of [12], to arbitrary chamber systems which are just built-up from rank 1 cells, which resemble rank 1 groups of Lie-type. It also covers most of the geometries which could be obtained from rank 1 and 2 p-local subgroups of some sporadic group G.

The above mentioned papers [14],[17] and [18] together with the work of G.Stroth (see these proceedings!) show that the problem of classifying (at least locally) classical locally finite Tits chamber systems with discrete transitive automorphism group is near to completion. Theorem 2 of this paper and Corollary 2 of [19], where a pushing up-type result was obtained under the condition P_n and $\Gamma(I)$ connected, can be considered as the beginning of a more general theory, which should include the geometries of the sporadic groups. In my opinion, the main problem here is to find suitable conditions, which allow one to show that a p-local geometry of a simple group is either a Tits geometry or just has those rank 2 residues which occur in the sporadic groups. The connectedness of $\Gamma(I)$ might be a first ingredient of such conditions.

The first corollary we state is actually a sharpening of Theorem 1 in the special case, when $M_i = [M_i,\widetilde{P}_i]$ for all $i \in I$.

<u>Corollary A</u>. Let B be a p-closed group and $\{P_i \mid i \in I\}$ be a collection of rank 1 parabolic groups of char.p, satisfying:

$$(*) \qquad M_i = [M_i,\widetilde{P}_i] \quad \text{for all} \quad i \in I$$

where $\widetilde{P}_i = O^{p'}(P_i)$, $M_i = O_p(P_i)$. Suppose there exist monomorphisms

$$\chi_j : B \to P_{i(j)}; \quad j = 1,\dots,n, \quad i(j) \in I$$

such that $\chi_j(B)$ is a Borel subgroup of $P_{i(j)}$. Then either $n \leq 2$ or $\chi_j^{-1}(M_{i(j)}) = \chi_k^{-1}(M_{i(k)})$ for some pair $j \neq k \leq n$.

Of course Corollary A can be stated also in language of groups satisfying P_n:

Corollary A'. Suppose $G = \langle P_i \mid i \in I \rangle$, $n = |I| \geq 3$ satisfies P_n and in addition

(*) $M_i = [M_i, \widetilde{P}_i]$ for each $i \in I$.

Then $M_i = M_j$ for some $i \neq j \in I$.

If we specialize Corollary A' to the case where all the P_i are permuted by the automorphism group of G we obtain a corollary, which also can be obtained from the Baumann-Glauberman-Niles theorem.

Corollary B. Let P be a rank 1 parabolic group in char.p with Borel subgroup B, $M = O_p(P)$ and $\widetilde{P} = O^{p'}(P)$. Suppose

(*) $M = [M, \widetilde{P}]$

and let $A = \mathrm{Aut}(B)$. Then $|A : N_A(M)| \leq 2$.

Finally a similar corollary in the more general case, when $M \neq [M, \widetilde{P}]$.

Corollary C. Let P be a rank 1 parabolic group in char.p with Borel subgroup B, $\widetilde{P} = O^{p'}(P)$, $M = O_p(P)$ and $N = [M, O^p(\widetilde{P})]$. Suppose there exist automorphisms α_i, $i = 1,2$ of B such that

$$N^{\alpha_i^{\pm 1}} \nleq M \quad \text{for} \quad i = 1,2.$$

Then $N^{\alpha_1 \alpha_2^{-1}} \leq M$ or $N^{\alpha_2 \alpha_1^{-1}} \leq M$.

Some remarks on the history of Theorem 1 or (equivalently Theorem 2!). The first result in this direction was the nonexistence proof for groups satisfying P_3^+ with $\overline{P}_{i,j} \simeq L_3(2)$ for $i,j \leq 3$ by A.Chermak in [1]. This has been generalized in [10] to groups satisfying P_3^+ with $\overline{P}_{i,j} \simeq L_3(p^{n_{ij}})$ for $i,j \leq 3$ and in [16] to groups satisfying P_3^+ with $P_{i,j}$ arbitrary rank 2 parabolics in char.2. The same result in arbitrary characteristic is a consequence of the main theorem of [12]. The only known result which moves slightly away from the +- condition is the nonexistence of groups satisfying P_3 with $\overline{P}_i \simeq L_2(2)$ and $\overline{P}_{i,j} \simeq A_7$ proved in [10].

Since for our Theorem 2 we do not need to know the structure of $\overline{P}_{i,j}$, it is obvious that Theorem 2 is a far reaching generalization of all these results. But in fact it should be viewed as a result about possible embeddings of a p-closed group B into rank 1 parabolics P_i. If more than two such embeddings are given, one can take for G just the free amalgamated product of the P_i over B and one obtains certain restrictions on the embeddings. It should also be mentioned that A.Chermak has classified the groups satisfying the conclusion of Corollary A over F_2 in [2].

§ 2 contains the proof of the equivalence of Theorem 1 and 2 and of the connectedness in $\Gamma(I)$ and $\Delta(I)$ if G satisfies P_n^+. In § 3 we start with the proof of Theorem 2. It contains some further notation and properties of groups with a weak BN-pair of rank 2. From § 4 on we assume by way of contradiction that G satisfies P_3 and $\Gamma(I)$ is a triangle. We show that $G_i = \langle P_j, P_k \rangle \leq N(Z)$ in the constrained case for some $i = o,1$ or 2; where $Z = \Omega_1(Z(S))$ and S is the p-Sylow-subgroup of B. § 5 contains the main reduction. Here we assume that G is the free amalgamated product of P_o, P_1, P_2 over B, $G_2 = \langle P_o, P_1 \rangle \leq N(Z)$ and that we are in the constrained case. Then by [6,(7.9)] $G_o \cap G_1 = P_2$ and we can show, considering the coset graph $\Gamma = \Gamma(G_o, G_1)$, that a certain parameter usually called d or b is smaller or equal to 2. § 6 contains the final contradiction in the constrained case. Here the fact that $d \leq 2$ shows that S is "small". Especially if $Z_i = \langle Z^{G_i} \rangle$, $i = o,1$ and $V_2 = \langle (Z_o \cap Z_1)^{G_2} \rangle$, then V_2/Z is a $GF(p)G_2$-module on which M_o or M_1 acts quadratically. But by (3.11), (3.12) such a module does not exist.

Finally § 7 contains the contradiction in the non-constrained case, which is easy but does depend very much on the structure of the parabolics in groups with a weak BN-pair of rank 2.

Finally a remark on the references. Except the already mentioned papers of Delgado and Delgado-Stellmacher all other papers are either only quoted in the introduction or to obtain results on the structure and on GF(p)-representations of rank 1 or 2 Lie-type groups.