Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1208

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Interpolation Functors and Duality

Springer-Verlag

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Mathematics Subject Classification (1980): 46M15, 46M35

ISBN 3-540-16790-0 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-16790-0 Springer-Verlag New York Berlin Heidelberg

Library of Congress Cataloging-in-Publication Data. Kaijser, Sten. Interpolation functors and duality. (Lecture notes in mathematics; 1208) Bibliography: p. Includes index. 1. Linear topological spaces. 2. Functor theory. I. Pelletier, Joan Wick, 1942-. II. Title. III. Series: Lecture notes in mathematics (Springer-Verlag); 1208.
QA3.L28 no. 1208 510 s 86-20242 [QA322] [515.7'3]
ISBN 0-387-16790-0 (U.S.)

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

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CHAPTER 0

INTRODUCTION

Duality is one of the most important notions of functional analysis (and of modern Mathematics in general). It is thus not surprising that in the theory of interpolation spaces much attention has been devoted to duality questions. There are, however, some intrinsic obstacles that prevent the formulation of a good duality theory in the present setting of interpolation theory, which is the category of Banach couples. The first difficulty is that if $\overline{X} = (X_0, X_1)$ is a Banach couple, then the dual spaces (X_0', X_1') need not be a "dual couple". A necessary condition for the dual spaces to form a Banach couple is that $\overline{X}$ be what is usually called a "regular" couple, meaning that the intersection, $\Delta\overline{X} = X_0 \cap X_1$, is dense in both X_0 and X_1. However, as examples show, even if one restricts attention to regular couples, it turns out that the dual couple need not be regular, so that the bidual is not a Banach couple. The second difficulty is that if X is an interpolation space for $\overline{X}$, then, even if $\overline{X}$ is regular, the dual space X' need not be an intermediate space for $\overline{X}$, much less an interpolation space. A necessary condition for X' to be an intermediate space is that $\Delta\overline{X}$ be dense also in X. This does not, however, insure that X' is an interpolation space for $\overline{X}'$. The third difficulty is that there is no general rule for what a "dual method" should be for the construction of interpolation spaces, although general intuition and experience has usually led to the right constructions.

In this paper we are proposing a slightly different setting for interpolation theory. We propose to work in the somewhat larger category of doolittle diagrams (see Freyd [8] for the name) of Banach spaces, which we shall denote by $\bar{\mathfrak{B}}$ (mimicking the standard notation for the category of Banach couples). Our category $\bar{\mathfrak{B}}$ is the smallest (natural) category, containing the category of Banach couples while being closed under duality. $\bar{\mathfrak{B}}$ also enjoys some other interesting properties: it has a Ban-valued hom-functor (which simply means that the set of morphisms from $\bar{X}$ to $\bar{Y}$ is a Banach space under the natural norm) and also a very useful Ban-valued tensor product.

As we have just pointed out, the new setting of $\bar{\mathfrak{B}}$ takes care of the first of the traditional difficulties mentioned above. However, it would be naive to think that there are no difficulties inherent in this setting. The most important new difficulty that arises is that it is no longer completely obvious what an interpolation space should be. We have chosen to say that the "intersection" $\Delta\bar{X}$ (in our theory, the pullback) and the "sum-space" $\Sigma\bar{X}$ (the pushout) should be interpolation spaces, and then we consider two classes of interpolation spaces modelled on these paradigms. On the one hand we consider smaller (semi-) norms on $\Delta\bar{X}$ so that we get spaces that are completions of $\Delta\bar{X}$; we call such spaces Δ-interpolation spaces. On the other hand we consider larger (extended) norms on $\Sigma\bar{X}$ so that we get subspaces of $\Sigma\bar{X}$; these, which are the only interpolation spaces considered in the classical theory, we call Σ-interpolation spaces.

As a first result, we show that the most important classical methods, i.e. the real and the complex methods, have very natural definitions in our theory. Moreover, we show that even if some of them (the J-methods and to some extent the C_θ-method) are intrinsically Δ-methods, they actually turn out also to be Σ-interpolation methods.

When we begin to study duality questions in our theory, we run into part of the second traditional difficulty that unless $\Delta\bar{X}$ is dense in X, the dual space may be too large. Since this problem also exists in the $\bar{\mathfrak{B}}$-setting, our theory is quite satisfactory for Δ-interpolation spaces, while it is much less satisfactory for Σ-interpolation methods.

The other part of the second difficulty - that of insuring the duals of Δ-interpolation spaces for regular couples are interpolation spaces - is overcome in our theory with the aid of the tensor product. The problem arises because not all maps on a dual space are adjoints, so even if the space is preserved by all adjoints, it need not be preserved by all $\bar{X}'$-maps. Our tensor product makes it possible to consider a somewhat smaller substitute for the dual space which is preserved by $\bar{X}'$-maps (or in our terminology is a module over $L(\bar{X}') = L(\bar{X}',\bar{X}')$). Many of our results are formulated in terms of this "natural dual". The same definition can be applied also to Σ-interpolation spaces, but we have not been able to determine whether the "natural duals" are interpolation spaces in this case.

Finally, we come to the last difficulty, namely that there is no general rule for obtaining the "dual method" except that, as far as possible, it should give rise to the dual space when applied to the dual couple. We overcome this difficulty here by using the notion of a "dual functor", first defined by Fuks [9] and applied to Banach spaces by Mityagin and Švarc [19]. Since this notion is based on the "natural duality" between tensor products and hom-functors, which our category $\bar{\mathfrak{B}}$ is also endowed with, it is possible to define the dual functor for any Ban-valued functor F on $\bar{\mathfrak{B}}$. This dual functor has the property that the dual of a Δ-interpolation functor is a Σ-interpolation functor, while the dual of a Σ-functor is in some algebraic sense still some kind of interpolation functor. The most important classical methods that are not Δ-methods are the real $K(\theta,\infty)$-method and the complex C^{θ}-method.

The purpose of this paper is to construct a theory of interpolation which contains the classical theory and which is suitable for duality. We have not, however, tried to prove everything in the classical theory; e.g. we have made no efforts to prove compactness results or to generalize recent developments like the interpolation of more than two spaces or the notions of Calderón pairs or K-divisibility. On the other hand we have included the recent development by Janson [11] and Brudnyi-Krugljak [3] which has merged the important Aronszajn-Gagliardo paper [1] with the classical papers of Calderón [4] and Lions-Peetre [17] by showing that the real and complex methods are actually minimal methods in the sense of Aronszajn-Gagliardo. We have in fact strengthened these results somewhat by proving that the methods are not only minimal as interpolation functors but are minimal among all functors F such that for some couple $\bar{A}$, $F\bar{A} = A$.

In spite of the fact that one of the main features of our theory is the thesis that the ordinary Banach space dual is not the natural dual to consider in interpolation theory, we have made efforts to prove that in certain cases our dual is actually the ordinary dual. These efforts involve introducing a certain notion of "computability", for interpolation functors, which is related to the notion of computability for functors on Banach spaces introduced by Herz-Pelletier [10]. The "computable" interpolation functors behave much the way they are expected classically to behave.

Compared to most other papers on interpolation theory, even to those which are categorical "in spirit", ours is probably the most categorical. We have used several important ideas from category theory. To some extent this is unavoidable because the definition of the dual functor requires some sophisticated ideas from category theory. However, for the most part our use of category theory is intentional, because we feel that interpolation theory is so functorial in nature that category theory will lead to the correct notions. For example, we dc feel that our notion of duality is the correct one for interpolation spaces while ordinary dual spaces are not sufficiently adapted to this situation. Along the same line of thinking, we feel that the notion of a Banach module (over the algebra $L(\overline{X})$) should be taken as the starting point for the idea of an interpolation space rather than the notion of an intermediate space. We believe that this more "algebraic" approach will be important for the study of questions arising out of interpolation theory, such as the "interpolation" of many (perhaps infinitely many) spaces.

Categorical methods have by now influenced most parts of Mathematics - except analysis. One reason why analysts have been unwilling to use categorical methods is probably that the languages of analysis and category theory are as difficult to translate as English and Swedish. In this paper we have occasionally had to choose either an analytical notation or a categorical one. We have made our compromise, and we hope that our paper will be readable for all.

Among our possibly diverse audience we anticipate that there will be functional analysts who merely wish to see that the real and complex methods are contained in a theory with better duality than hitherto present. We have organized the paper in such a way that Part I presents these results largely in terms of analysis. The experts in interpolation theory will, we hope, continue further in the paper to read about general interpolation functors in the $\overline{\mathfrak{B}}$-setting and their duality. We also expect that category theorists who are interested in applications of categorical methods to analysis will be among our readers. They may wish to begin directly with Part II, which is meant to be self-contained. In Chapter IV we have introduced and investigated all the categorical properties that are natural in $\overline{\mathfrak{B}}$, not all of which are actually used in the paper. We have also tried to indicate from time to time the extent of categorical generality inherent in our constructions in the hope that applications of these ideas may arise in other areas. Finally, we hope our readership will include mathematicians who are interested in the interplay of various branches of Mathematics. They should be particularly interested in Part III, which is our attempt to tie together the concrete applications of Chapters II and III with the more abstract theory of Chapters VI and VII.

Parts of this work have been presented previously. The real method à la Chapter II was presented at a conference on interpolation theory held in Lund, Sweden in August 1983, and a paper [14] based on this presentation is contained in the Conference Proceedings. Preliminary versions of the more categorical aspects of the paper have been presented at conferences in Sussex, England, Denver, Colorado and Murten, Switzerland, and the articles [15] and [23] have emerged.

In closing we wish to make some acknowledgments. Several institutions have hosted us during some part of our four years' collaboration and we are grateful for their hospitality: our home universities - York and Uppsala - the University of Connecticut, and McGill University. We wish to single out for thanks Ms. P. Ferguson of McGill University and Ms. Paula Panaro of York University for their superb typing of the preliminary and final versions of the manuscript, respectively. We are grateful to the Natural Sciences and Engineering Research Council for its support, without which this collaboration would not have been possible. We also acknowledge the interest and encouragement of several mathematicians, in particular, J.W. Gray, J. Peetre, and S. Janson. Finally, we thank C. Herz, who, anticipating our common interests, introduced us to one another.

PART I

CHAPTER I

PRELIMINARIES

1. The Setting.

As we have explained at length in the introduction, we feel that the category of Banach couples is not the best setting for interpolation theory. We are proposing to work in a larger category - the category of doolittle diagrams of Banach spaces - which is a simple extension of the category of Banach couples enjoying the property of being closed under duality. We believe that, despite certain difficulties arising in this setting, this is the "right category" for studying interpolation.

We begin by giving our basic definitions.

1.1 Definition. A doolittle diagram $\overline{X}$ of Banach spaces is a commutative diagram (of Banach spaces)

$$\overline{X} = \begin{array}{ccc} \Delta\overline{X} & \xrightarrow{\delta_0} & X_0 \\ {\scriptstyle \delta_1}\downarrow & & \downarrow{\scriptstyle \sigma_0} \\ X_1 & \xrightarrow[\sigma_1]{} & \Sigma\overline{X} \end{array}$$

such that

(i) all maps are continuous linear maps and

(ii) $\overline{X}$ is both a pullback and pushout.

Condition (ii) means the above diagram is commutative and that $\Delta\overline{X}$ and $\Sigma\overline{X}$ are "universal" in the following sense: if Y is a "candidate" for the top left corner, i.e. if there are maps $f_i : Y \to X_i$, $i=0,1$, such that $\sigma_0 \circ f_0 = \sigma_1 \circ f_1$, then Y factors uniquely through $\Delta\overline{X}$, i.e. there is a unique map $f : Y \to \Delta\overline{X}$ such that $f \circ \delta_i = f_i$, $i=0,1$, and similarly for $\Sigma\overline{X}$.

In practice we can give the following concrete description of doolittle diagrams of Banach spaces. First for a pair (X_0, X_1) of Banach spaces we denote by $X_0 \pi X_1$ and $X_0 \amalg X_1$ the product and sum (or coproduct) spaces, respectively, where

$$\|(x_0, x_1)\|_{X_0 \pi X_1} = \sup(\|x_0\|, \|x_1\|)$$

and

$$\|x\|_{X_0 \amalg X_1} = \inf(\|x_0\| + \|x_1\| \mid x = x_0 + x_1) .$$

1.2. <u>Proposition</u>. A doolittle diagram of Banach spaces is determined by a pair (X_0, X_1) of Banach spaces and a closed subspace $\Delta\overline{X}$ of $X_0 \pi X_1$.

<u>Proof</u>: Let

$$D = \begin{array}{ccc} P & \xrightarrow{\;u\;} & X_0 \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle f} \\ X_1 & \xrightarrow[\;g\;]{} & Q \end{array}$$

be a doolittle diagram of Banach spaces and let $\varphi : P \to X_0 \pi X_1$ be defined by $\varphi = (u, v)$. Then by the definition of the pullback, one sees that φ is an isometry, so P may be considered a closed subspace of $X_0 \Pi X_1$. More precisely, we see that P is isomorphic to the subset of $X_0 \Pi X_1$ consisting of those (x_0, x_1) such that $fx_0 = gx_1$.

Conversely, if $\Delta\bar{X}$ is a closed subspace of $X_0 \Pi X_1$, we denote by $\delta_i : \Delta\bar{X} \to X_i$, the projection of $\Delta\bar{X}$ to X_i. Then the pushout Q in the diagram

$$\begin{array}{ccc} \Delta\bar{X} & \xrightarrow{\delta_0} & X_0 \\ {\scriptstyle\delta_1}\downarrow & & \downarrow{\scriptstyle\sigma_0} \\ X_1 & \xrightarrow[\sigma_1]{} & Q \end{array}$$

can be described as a quotient of $X_0 \amalg X_1$ over the subspace

$$\Delta\bar{X}^- = \{(x_0, x_1) \mid \exists x \in \Delta\bar{X},\ x_0 = \delta_0 x,\ x_1 = -\delta_1 x\};$$

$\sigma_i : X_i \to Q$ are the canonical maps. It is easy to verify that $\Delta\bar{X}$ is the pullback of the above diagram, and hence, that it is a doolittle diagram. □

The general doolittle diagram in our paper will be denoted by $\bar{X}$ or $(X_0, X_1, \Delta\bar{X})$, where $\Delta\bar{X}$ is understood to be a closed subspace of $X_0 \Pi X_1$, and it will be equipped with morphisms as follows:

$$\begin{array}{ccc} \Delta\bar{X} & \xrightarrow{\delta_0} & X_0 \\ {\scriptstyle\delta_1}\downarrow & & \downarrow{\scriptstyle\sigma_0} \\ X_1 & \xrightarrow[\sigma_1]{} & \Sigma\bar{X} \end{array} .$$

Since the diagram is commutative, we have $\sigma_0 \circ \delta_0 = \sigma_1 \circ \delta_1$. We shall denote this frequently used map by j and call $\overline{X}$ <u>non-trivial</u> if $j \neq 0$.

1.3. <u>Examples</u>. 1. A sum diagram and a product diagram

$$\begin{array}{ccc} 0 & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_0 \amalg X_1, \end{array} \qquad \begin{array}{ccc} X_0 \sqcap X_1 & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & 0 \end{array}$$

are both (trivial) doolittle diagrams. 2. Every Banach couple is a doolittle diagram such that all the maps are injective; conversely, a doolittle diagram the maps of which are injective is simply a Banach couple.

Since the purpose of interpolation theory is to interpolate operators, we have to know what an operator between doolittle diagrams is.

1.4. <u>Definition</u>. Let $\overline{X}$ and $\overline{Y}$ be doolittle diagrams. A map T from $\overline{X}$ to $\overline{Y}$ is a pair (T_0, T_1) of continuous linear maps such that the following diagram commutes:

$$\begin{array}{ccccccc} & \overset{\delta_0}{\nearrow} & X_0 & \xrightarrow{T_0} & Y_0 & \overset{\sigma_0}{\searrow} & \\ \Delta\overline{X} & & & & & & \Sigma\overline{Y} \\ & \underset{\delta_1}{\searrow} & X_1 & \xrightarrow[T_1]{} & Y_1 & \underset{\sigma_1}{\nearrow} & \end{array}$$

(We are deliberately avoiding notation like $\delta_0(\overline{X})$, $\sigma_1(\overline{Y})$ which is cumbersome.)

1.5. <u>Remarks</u>. 1. We note that when $\overline{X}$ and $\overline{Y}$ are Banach couples, our definition of morphism is the same as the classical definition.

2. In view of the definition of the pullback, the map $\sigma_0 \circ T_0 \circ \delta_0 = \sigma_1 \circ T_1 \circ \delta_1$ factors through $\Delta\overline{Y}$, so there exists $\Delta T : \Delta\overline{X} \to \Delta\overline{Y}$. Similarly, from the definition of the pushout we get $\Sigma T : \Sigma\overline{X} \to \Sigma\overline{Y}$.

We shall denote by $L(\overline{X}, \overline{Y})$ the set of all maps from $\overline{X}$ to $\overline{Y}$. Actually, $L(\overline{X}, \overline{Y})$ is a Banach space under the norm

$$\|T\| = \max(\|T_0\|, \|T_1\|).$$

We may also observe from our description of pullbacks given in 1.2 that $L(\overline{X}, \overline{Y})$ is the pullback of the diagram

$$\begin{array}{ccc} & & L(X_0, Y_0) \\ & & \downarrow \\ L(X_1, Y_1) & \longrightarrow & L(\Delta\overline{X}, \Sigma\overline{Y}). \end{array}$$

The category of doolittle diagrams of Banach spaces and bounded linear morphisms as described above is denoted by $\overline{\mathfrak{B}}$, while the sub-category of Banach couples is denoted by $\overline{\mathfrak{B}}\mathfrak{C}$; $\mathfrak{B}$ will denote the category of Banach spaces.

Since we have motivated our introduction of $\overline{\mathfrak{B}}$ by a discussion of the better duality properties it enjoys, we should begin at least by showing that $\overline{\mathfrak{B}}$ is closed under duals.

1.6 <u>Proposition</u>. Let $\overline{X}$ be a doolittle diagram and let $\overline{X}'$ be the diagram

$$\begin{array}{ccc}
(\Sigma\overline{X})' & \xrightarrow{\sigma_0'} & X_0' \\
{\scriptstyle \sigma_0'}\downarrow & & \downarrow{\scriptstyle \delta_0'} \\
X_1' & \xrightarrow[\delta_1']{} & (\Delta\overline{X})' \;.
\end{array}$$

Then $\overline{X}'$ is a doolittle diagram, i.e. $\Delta\overline{X}'=(\Sigma\overline{X})'$ and $\Sigma\overline{X}'=(\Delta\overline{X})'$.

<u>Proof</u>: The commutativity of the above diagram is obvious. That it is a pullback follows from the pushout property of $\overline{X}$ directly. That it is a pushout as well is a fact, non-trivial only in the sense that it depends on a deep theorem, namely the Hahn-Banach theorem. □

2. <u>Doolittle diagrams, Couples, and Regular Couples</u>.

We have observed above that a Banach couple is merely a doolittle diagram such that all maps are injective. The main difference then between our category $\overline{\mathfrak{B}}$ and the traditional category $\overline{\mathfrak{BC}}$ is that in an arbitrary doolittle diagram the maps need not be injective. It is natural, therefore, to consider the kernels of the maps δ_i, σ_i in $\overline{X}$, at least one of which will be a non-trivial space if $\overline{X}$ is not a Banach couple.

Let us denote by $K_i\overline{X}$ the space $\ker(\sigma_i)$ $(\subset X_i)$. We may prove the following proposition.

2.1. <u>Proposition</u>. Let $\overline{X}$ be a doolittle diagram. Then $\ker(\delta_0) = \ker(\sigma_1)(=K_1\overline{X})$ and $\ker(\delta_1) = \ker(\sigma_0)(=K_0\overline{X})$.

Proof: Recall from 1.2 that $\Delta\overline{X}$ may be interpreted as the following subspace of $X_0 \Pi X_1$:

$$\{(x_0,x_1) \mid \sigma_0 x_0 = \sigma_1 x_1\}.$$

Let $x_1 \in K_1\overline{X}$. Then $\sigma_1 x_1 = 0$, so $x = (0,x_1) \in \Delta\overline{X}$. Therefore, $\delta_0 x = 0$, so $K_1\overline{X} \subset \ker(\delta_0) \subset \Delta\overline{X}$. Conversely, if $y \in \ker(\delta_0)$, then $y = (0,y_1)$, where y_1 is such that $\sigma_1 y_1 = 0$, so $y_1 \in K_1\overline{X}$. Hence, $\ker(\delta_0) \subset K_1\overline{X}$. The same argument proves that $K_0\overline{X} = \ker(\delta_1)$. □

We shall write $K\overline{X} = K_0\overline{X}\Pi K_1\overline{X} \subset \Delta\overline{X}$ and observe that $K\overline{X} = \ker(j)$. $\overline{K\overline{X}}$ will denote the trivial doolittle diagram

$$\overline{K\overline{X}} = \begin{array}{ccc} K\overline{X} & \longrightarrow & K_0\overline{X} \\ \downarrow & & \downarrow \\ K_1\overline{X} & \longrightarrow & 0 \end{array} .$$

Letting $Y_i = X_i/K_i\overline{X}$, we can define a pullback diagram

$$\begin{array}{ccc} \Delta\overline{Y} & \longrightarrow & Y_0 \\ \downarrow & & \downarrow \\ Y_1 & \longrightarrow & \Sigma\overline{X} \end{array} .$$

Then by the property of the pullback, $j:\Delta\overline{X} \to \Sigma\overline{X}$ must factor through $\Delta\overline{Y} \to \Sigma\overline{X}$, which is obviously injective. It is easy to conclude that $\Delta\overline{Y} = \Delta\overline{X}/K\overline{X}$ and that