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PARTICLES AND SYMMETRIES

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PARTICLES AND SYMMETRIES

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CHAPTER I

PARTICLES AND FIELDS

1. Vacuum, particle states and transition amplitudes

Any portion of the world, defined, for instance, as the enclosure of a box, can be described in many different ways. For our purpose it is convenient to give the number and nature of the particles present, together with the momentum of each. Further degrees of freedom may have to be specified if the particles possess an internal structure, and occasionally the angular momentum will be given instead of the momentum.

The formalism embodying this mode of description defines first the vacuum state, denoted by the ket $|0\rangle$. Particle states are constructed by applying to the vacuum certain creation operators $\mathbf{c}^*(\mathbf{k}, \lambda)$, where $\mathbf{k}$ stands for the momentum of a particle and λ for any additional information, e.g. charge, spin, needed to make the representation complete. Thus

$$\mathbf{c}^*(\mathbf{k}_1, \lambda_1)\mathbf{c}^*(\mathbf{k}_2, \lambda_2) \ldots \mathbf{c}^*(\mathbf{k}_m, \lambda_m) |0\rangle \tag{1}$$

is a state with m particles, each having a given momentum and additional degrees of freedom.

Multiplication of (1) by a function $\varphi(\mathbf{k}_1, \mathbf{k}_2 \ldots \mathbf{k}_m)$, followed by integration over the momenta $\mathbf{k}_1, \mathbf{k}_2 \ldots \mathbf{k}_m$, produces a state in which the particles are localized in space. Whereas the function φ can then be regarded as a momentum wave function, in the non-relativistic limit its Fourier transform can be identified with the ordinary Schroedinger wave function $\psi(\mathbf{r}_1, \mathbf{r}_2 \ldots \mathbf{r}_m)$ ($\mathbf{r}_i$ denoting the position of the i-th particle).

Interactions are the agents which induce transitions from one state to another. The probability of a certain transition taking place is proportional to the square of the modulus of the transition amplitude

$$(0|\, c(\mathbf{k}'_n, \lambda'_n) \ldots c(\mathbf{k}'_1, \lambda'_1) U c^* (\mathbf{k}_1, \lambda_1) \ldots c^* (\mathbf{k}_m, \lambda_m)\, |0). \tag{2}$$

The state vector to the left of the transition operator U is a bra, the conjugate of the ket $c^* (\mathbf{k}'_1, \lambda'_1) \ldots c^* (\mathbf{k}'_n, \lambda'_n) |0)$. It results from the latter by the operation of Hermitian conjugation, which replaces the ket vacuum $|0)$ by the bra vacuum $(0|$, each creation operator by its Hermitian conjugate, which is a destruction operator, and reverses the order of all the factors †.

U is a Lorentz invariant functional of the interaction Lagrangian $\mathscr{L}(x)$, which is always Lorentz invariant of its own accord and a product of field operators $\psi(x)\,(x \equiv \mathbf{r}, t)$. These are related to the creation and destruction operators before mentioned in much the same way as a function is to the coefficients of its Fourier expansion. The relationship will be explicitly written in the following sections, and lists of Lorentz invariant interactions given in appropriate places.

Expanded in powers of the interaction, the transition operator it given by

$$U = \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4 x_1 \ldots d^4 x_n \, (\mathscr{L}(x_1) \ldots \mathscr{L}(x_n))_+ , \tag{3}$$

where $d^4 x \equiv d\mathbf{r}\, dt$, and the integrations cover the whole space-time. The chronological ordering $(\quad)_+$ signifies that for any two factors $\mathscr{L}(x_j)$ and $\mathscr{L}(x_k)$, $\mathscr{L}(x_j)$ must take the left of $\mathscr{L}(x_k)$ if $t_j > t_k$, and the right if $t_j < t_k$.

2. Spinless particles

2.1. CREATION AND DESTRUCTION OPERATORS

For neutral †† spinless particles the creation and destruction operators $c^*(\mathbf{k})$ and $c(\mathbf{k})$ are functions of the momentum only. The field operator $\phi(x)$ is Hermitian and is given by

$$\phi(x) = \sum [c(\mathbf{k}) e^{i(k,x)} + c^*(\mathbf{k}) e^{-i(k,x)}], \tag{4}$$

where $(k, x) \equiv \mathbf{k} \cdot \mathbf{r} - \omega t$, and $\omega = \sqrt{\mathbf{k}^2 + \mu^2}$, μ being the mass of the particles.

We have introduced the abbreviation $\sum \equiv \sum_{\mathbf{k}} (2V\omega)^{-\frac{1}{2}}$, V being the volume of the box in which the particles are contained.

† The Hermitian conjugate of a c-number $\gamma = \alpha + i\beta$ (α, β real) is the complex conjugate $\gamma^* = \alpha - i\beta$. If c_1 and c_2 are destruction operators, the Hermitian conjugate of the operator $c = c_1 + ic_2$ is $c^* = c_1^* - ic_2^*$. An operator is said to be Hermitian when it is equal to its Hermitian conjugate.

†† Or rather, particles which are in all respects indistinguishable from their antiparticles. In Ch. VII we shall mention an uncharged spinless particle, the θ_0 meson, which is not identical with its antiparticle $\overline{\theta}_0$. Therefore the field operator of $(\theta_0, \overline{\theta}_0)$ will be non-Hermitian, and must be expanded according to eq. (5).

For charged spinless particles, two kinds of creation and destruction operators must be introduced, $\mathsf{a}^*(\mathbf{k})$ and $\mathsf{a}(\mathbf{k})$ for positive particles (particles) $\mathsf{b}^*(\mathbf{k})$ and $\mathsf{b}(\mathbf{k})$ for negative particles (antiparticles). The expansion of the field operator $\boldsymbol{\phi}(x)$ contains both,

$$\boldsymbol{\phi}(x) = \sum [\mathsf{a}(\mathbf{k})e^{i(k,x)}+\mathsf{b}^*(\mathbf{k})e^{-i(k,x)}]. \tag{5}$$

In contrast with the case of neutral particles, (5) is not Hermitian. This is true of any field of charged particles, the reason being that only with a non-Hermitian field is it possible to construct a non-vanishing interaction † with the electromagnetic field.

2.2. ANGULAR MOMENTUM REPRESENTATION

If the energy and the angular momentum are chosen as degrees of freedom for the individual particles, appropriate creation and destruction operators $\mathsf{c}^*(k, l, m)$, $\mathsf{c}(k, l, m)$, . . . must be introduced [3, 10]. Then, for instance,

$$\mathsf{c}^*(k, l, m)|0), \qquad \mathsf{a}^*(k, l, m)|0), \qquad \mathsf{b}^*(k, l, m)\,|0)$$

are states with one (neutral, positive, negative) spinless particle of energy $\omega = \sqrt{k^2+\mu^2}$ and angular momentum quantum numbers $\mathbf{l}^2 = l(l+1)$, $l_z = m$. An expansion for the field operator equivalent to (5) is [3, 10])

$$\boldsymbol{\phi}(x) = \sum_{klm} [\mathsf{a}(k, l, m)\, Y_l^{(m)}(\theta, \phi)e^{-i\omega t}+\mathsf{b}^*(k, l, m)Y_l^{(m)*}(\theta, \phi)e^{i\omega t}] \tag{6}$$

(for neutral particles $\mathsf{a} \to \mathsf{c}$, $\mathsf{b} \to \mathsf{c}$). Here θ and ϕ are the polar angles of $\mathbf{r}$, $Y_l^{(m)}$ (θ, ϕ) is a spherical harmonic defined so that $Y_l^{(m)*} = (-1)^m\, Y_l^{(-m)}$, and $\sum_{klm}$ stands for $\sum_{klm}$ $(2\omega)^{-\frac{1}{2}}\, g_{lk}(r)$, where the radial functions $g_{lk}(r)$ are given by

$$g_{lk}(r) = \sqrt{\frac{\pi k}{rR}}\cdot J_{l+\frac{1}{2}}(kr), \tag{7}$$

and are solutions of the radial Laplace equation.

R is the (large) radius of a spherical box, and $J_{l+\frac{1}{2}}$ is a Bessel function.

The application of the differential operator $(\Box^2-\mu^2)$ to each term of (4) (5) and (6) gives zero, so that

$$(\Box^2-\mu^2)\boldsymbol{\phi}(x) = 0. \tag{8}$$

Comparison of eqs. (5) and (6) yields the relation

$$\mathsf{a}^*(\mathbf{k}) = \frac{\pi}{k}\sqrt{\frac{8R}{V}}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l} i^l\, Y_l^{(m)*}(\mathbf{k})\mathsf{a}^*(k, l, m) \tag{9}$$

between creation operators in the momentum representation and those in

† Which interaction may be a multipole of any order, so long as it discriminates between particles and antiparticles. It is not, however, known whether the distinction between the mesons θ_0 and $\bar{\theta}_0$ is of electromagnetic origin (see footnote p. 3).

the angular momentum representation. Applying the two members of (9) to the vacuum $|0)$, a relation ensues between corresponding one particle states.

The angular momentum representation may also be used for the description of particles acted upon by a central static field. The creation and destruction operators then refer to non-free particles of given angular momentum and energy. The expansion (6) is still valid with g_{lk} replaced by the solutions of a radial equation including the static potential.

2.3. TWO-PARTICLE STATES

Two-particle states are generated by operating on the vacuum with two creation operators. Thus

$$\mathsf{a}^*(\mathbf{k})\mathsf{b}^*(\mathbf{k}')\,|0) \tag{10}$$

is a state with one particle of momentum $\mathbf{k}$ and one antiparticle of momentum $\mathbf{k}'$, and

$$\mathsf{a}^*(k, l, m)\mathsf{b}^*(k', l', m')\,|0) \tag{10'}$$

is a state with one particle with the quantum numbers (k, l, m) and one antiparticle with (k', l', m') in the angular momentum representation.

More interesting is the construction of a particle-antiparticle state of relative angular momentum (L, M) in the centre-of-momentum frame. This can be done by superposing states (10) with $\mathbf{k}' = -\mathbf{k}$. The expression for the state in question is [3])

$$|k, L, M) = \int \mathrm{d}\Omega_{\mathbf{k}}\, Y_L^{(M)}(\mathbf{k})\mathsf{a}^*(\mathbf{k})\mathsf{b}^*(-\mathbf{k})\,|0), \tag{11}$$

where $Y_L^{(M)}(\mathbf{k})$ is a spherical harmonic whose arguments are the polar angles of $\mathbf{k}$, and $\mathrm{d}\Omega_{\mathbf{k}}$ is an element of solid angle around $\mathbf{k}$. Using the Clebsch-Gordan coefficients [1]), states with any number of particles and antiparticles and given total angular momentum can be formed.

2.4. BOSE-EINSTEIN STATISTICS

Particles of spin zero obey the Bose-Einstein statistics, a property which is formally expressed by the commutation relations † among creation and destruction operators,

$$[\mathsf{c}(\mathbf{k}), \mathsf{c}^*(\mathbf{k}')] = [\mathsf{a}(\mathbf{k}), \mathsf{a}^*(\mathbf{k}')] = [\mathsf{b}(\mathbf{k}), \mathsf{b}^*(\mathbf{k}')] = \delta_{\mathbf{k}\mathbf{k}'}$$

all the other commutators being zero.

Let us consider, for instance, neutral particles. There is no restriction on the number of particles of momentum $\mathbf{k}$ in the state $\mathsf{c}^*(\mathbf{k})\mathsf{c}^*(\mathbf{k})\ldots\mathsf{c}^*(\mathbf{k})\,|0)$, since $[\mathsf{c}^*(\mathbf{k}), \mathsf{c}^*(\mathbf{k})] = 0$ is only a trivial identity. On the other hand, this state is an eigenstate of the operator $\mathsf{N}(\mathbf{k}) = \mathsf{c}^*(\mathbf{k})\mathsf{c}(\mathbf{k})$ (number of particles of momentum $\mathbf{k}$) whose eigenvalue is the number of factors $\mathsf{c}^*(\mathbf{k})$ which operate on the vacuum. For example, omitting $\mathbf{k}$ for brevity, $\mathsf{N}\mathsf{c}^*\mathsf{c}^*\,|0) =$

† Here $[a, b] = ab-ba$, $\delta_{\mathbf{k}\mathbf{k}'}$ ($= 1$ for $\mathbf{k} = \mathbf{k}'$, $= 0$ for $\mathbf{k} \neq \mathbf{k}'$).

$c^* cc^* c^* |0) = c^*(1+c^* c)c^* |0) = c^* c^* cc^* |0) + c^* c^* |0) = c^* c^* (1+c^* c)|0) + c^* c^* |0) = 2c^* c^* |0)$ since $c|0) = 0$ (destruction operators give zero when applied to the ket **vacuum**; note, however, that $(0|c \neq 0$, since it results from $c^*|0)$ by Hermitian conjugation).

3. Particles of spin $\frac{1}{2}$

3.1. SPIN STATES AND SPIN MATRICES

We denote by $a^*(\mathbf{k}, \lambda)$, $a(\mathbf{k}, \lambda)$ the creation and destruction operators for a particle of momentum **k** and with spin either parallel, $\lambda = \uparrow$, or antiparallel, $\lambda = \downarrow$, to the momentum. The analogous operators for an antiparticle will be $b^*(\mathbf{k}, \lambda)$ and $b(\mathbf{k}, \lambda)$.

Particles of spin $\frac{1}{2}$ obey the Fermi-Dirac statistics. The creation and destruction operators satisfy anticommutation † relations

$$\{a(\mathbf{k}, \lambda), a^*(\mathbf{k}', \lambda')\} = \{b(\mathbf{k}, \lambda), b^*(\mathbf{k}', \lambda')\} = \delta_{\lambda\lambda'} \delta_{\mathbf{k}\mathbf{k}'}.$$

All the other anticommutatores are zero. E.g.,

$$\{a^*(\mathbf{k}, \lambda), a^*(\mathbf{k}', \lambda')\} = 0,$$

which, for $\mathbf{k} = \mathbf{k}'$ and $\lambda = \lambda'$ reduces to $a^*(\mathbf{k}, \lambda)a^*(\mathbf{k}, \lambda) = 0$, so that states with more than one particle of given $(\mathbf{k}, \lambda)$ do not exist.

The operator $N(\mathbf{k}, \lambda) = a^*(\mathbf{k}, \lambda)a(\mathbf{k}, \lambda)$ represents the number of particles of momentum **k** and spin orientation λ and has the eigenvalues 1 and 0, for the eigenstates $a^*(\mathbf{k}, \lambda)|0)$ and $|0)$, respectively. Similarly $\overline{N}(\mathbf{k}, \lambda) = b^*(\mathbf{k}, \lambda)b(\mathbf{k}, \lambda)$ represents the number of antiparticles of momentum **k** and spin orientation λ.

The expansion of the field operator $\psi(x)$ must be done with care. In analogy with eq. (5)

$$\psi(x) = \sum [a(\mathbf{k}, \lambda)u(\mathbf{k}, \lambda)e^{i(k,x)} + b^*(\mathbf{k}, \lambda)v(\mathbf{k}, \lambda)e^{-i(k,x)}], \tag{12}$$

where now $\sum \equiv \sum_{\mathbf{k}} \sum_{\lambda} (2V\omega)^{-\frac{1}{2}}$ includes a sum over the direction of the spin.

If $u(\mathbf{k}, \lambda)e^{i(k,x)}$ and $v(\mathbf{k}, \lambda)e^{-i(k,x)}$ are solutions of the Dirac equation, one has also

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right) \psi(x) = 0. \tag{13}$$

We here anticipate that, whereas $u(\mathbf{k}, \uparrow)$ and $u(\mathbf{k}, \downarrow)$ are positive energy solutions for momentum **k** and spin parallel and antiparallel to **k**, respectively, $v(\mathbf{k}, \uparrow)$ and $v(\mathbf{k}, \downarrow)$ are negative energy solutions for momentum $-\mathbf{k}$ and spin parallel and antiparallel to $-\mathbf{k}$, respectively. The reason for this is that when we insert eq. (12) into, e.g., the expression for the total momen-

† Here $\{a, b\} = ab + ba$. Cf. footnote on page 5 for the definition of $\delta_{\mathbf{k}\mathbf{k}'}$. That of $\delta_{\lambda\lambda'}$ is $\delta_{\uparrow\uparrow} = \delta_{\downarrow\downarrow} = 1$, $\delta_{\uparrow\downarrow} = \delta_{\downarrow\uparrow} = 0$.

tum **P** of the field, of the form $\int \psi^* \ldots \psi \, d\tau$, by virtue of the anticommutativity of the creation and destruction operators we find

$$\mathbf{P} = \sum_{\mathbf{k}\lambda} [\mathsf{N}(\mathbf{k}, \lambda) \times \text{momentum of } u(\mathbf{k}, \lambda) \\ -\overline{\mathsf{N}}(\mathbf{k}, \lambda) \times \text{momentum of } v(\mathbf{k}, \lambda)]$$

where N and $\overline{\mathsf{N}}$ denote the numbers of particles and antiparticles. Clearly the momentum of $v(\mathbf{k}, \lambda)$ must be $-\mathbf{k}$, if **P** is to be the sum of the momenta of particles and antiparticles present. A similar consideration would hold for the energy and the spin.

Since the γ_μ are 4×4 matrices, $u(\mathbf{k}, \lambda)$ and $v(\mathbf{k}, \lambda)$ are four-component c-numbers, and $\psi(x)$ four-component operators.

It is to be borne in mind that the form of the γ-matrices is not unique [8]. If γ_μ and γ'_μ ($\mu = 1, 2, 3, 4$) are two sets of Hermitian matrices satisfying the anticommutation relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu} \tag{14}$$

a unitary matrix exists connecting them, $\gamma'_\mu = S\gamma_\mu S^{-1}$, and for our purpose are entirely equivalent.

The following are the most usual choices for the γ's (with $\boldsymbol{\gamma} \equiv (\gamma_1, \gamma_2, \gamma_3)$, and we give also $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$):

$$\begin{array}{llll} \text{(Pauli) [3, 7]} & \boldsymbol{\gamma} = \rho_y \times \boldsymbol{\sigma}, & \gamma_4 = \rho_z \times I, & \gamma_5 = -\rho_x \times I; \\ \text{(Kramers) [4, 5]} & \boldsymbol{\gamma}' = -\rho_y \times \boldsymbol{\sigma}, & \gamma'_4 = \rho_x \times I, & \gamma'_5 = -\rho_z \times I; \\ \text{(Majorana) [6]} & \gamma''_1 = -I \times \sigma_z, & \gamma''_2 = -\rho_y \times \sigma_y, & \gamma''_3 = I \times \sigma_x, \\ & & \gamma''_4 = \rho_z \times \sigma_y, & \gamma''_5 = \rho_x \times \sigma_y. \end{array} \tag{15}$$

We have introduced the matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{16}$$

and $\rho_x, \rho_y, \rho_z, I$, which are defined in the same way. If the four components of u, say, are written $u_1 \equiv u_{1,1}, u_2 \equiv u_{-1,1}, u_3 \equiv u_{1,-1}, u_4 \equiv u_{-1,-1}$ the σ's operate on the first indices as they would on the two-components of a non relativistic spin function, whereas the ρ's operate on the second indices in the same manner. For instance

$$\rho_x \times \sigma_x \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \rho_x \begin{pmatrix} u_2 \\ u_1 \\ u_4 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_4 \\ u_3 \\ u_2 \\ u_1 \end{pmatrix}. \tag{17}$$

The main advantage of the Pauli set is that γ_4 is diagonal, which, at low energy, causes the components u_1 and u_2 to be large (proportional to $\omega+m \approx 2m$) and u_3 and u_4 to be small (proportional to $\omega-m$).

In the Kramers set, the matrix γ_5, which occurs in the coupling of spin $\frac{1}{2}$ particles (e.g., mucleons) with pseudoscalar particles (e.g., π-mesons), is diagonal. The use of this set is convenient in studying space inversions.

Finally, the Majorana matrices $\gamma''_1, \gamma''_2, \gamma''_3$ are real like the space coordinates x, y, z and γ''_4 is imaginary like $x_4 = it$ †. All four matrices are, of course, Hermitian. Thus $\gamma''_1, \gamma''_2, \gamma''_3$ are equal to their transposes, since the Hermiticity condition $a_{ik} = a^*_{ki}$ reduces for them to $a_{ik} = a_{ki}$, whereas γ''_4 is equal to minus its transpose. The formula $\gamma''^{T}_{\mu} = -\gamma''_4 \gamma''_\mu \gamma''_4$, where T denotes transposition, concisely expresses this property. The Majorana set turns out to be the most convenient when defining the operation of particle-antiparticle conjugation.

For completeness we add the expressions for the components of the spin matrices in the different sets:

$$
\begin{aligned}
&\text{(Pauli)} && \mathbf{s} \equiv \tfrac{1}{2}(I\times\sigma_x, && I\times\sigma_y, && I\times\sigma_z);\\
&\text{(Kramers)} && \mathbf{s}' \equiv \tfrac{1}{2}(I\times\sigma_x, && I\times\sigma_y, && I\times\sigma_z);\\
&\text{(Majorana)} && \mathbf{s}'' \equiv \tfrac{1}{2}(\rho_y\times\sigma_z, && I\times\sigma_y, && -\rho_y\times\sigma_x).
\end{aligned}
\tag{15'}
$$

The reader may find it useful to have in full the relations between the three sets of matrices (15). These are

$$
\gamma'_\mu = S_{\mathrm{K,P}}\gamma_\mu S^{-1}_{\mathrm{K,P}}, \qquad \gamma''_\mu = S_{\mathrm{M,K}}\gamma'_\mu S^{-1}_{\mathrm{M,K}}, \qquad \gamma''_\mu = S_{\mathrm{M,P}}\gamma_\mu S^{-1}_{\mathrm{M,P}}, \tag{18}
$$

with ††

$$
\begin{aligned}
S_{\mathrm{K,P}} &= \frac{1}{\sqrt{2}}(1+i\rho_y)\rho_x,\\
S_{\mathrm{M,K}} &= \frac{e^{i\pi/4}}{\sqrt{2}}(1+i\rho_y\sigma_y),\\
S_{\mathrm{M,P}} &= \frac{e^{i\pi/4}}{\sqrt{2}}(1+i\rho_y\sigma_y)(1+i\rho_y)\rho_x,
\end{aligned}
\tag{19}
$$

(the inverse matrices being the Hermitian conjugates of these, e.g., $S^{-1}_{\mathrm{K,P}} = (1/\sqrt{2})\,\rho_x(1-i\rho_y)$). The corresponding transformations for the field operator ψ are

$$
\psi' = S_{\mathrm{K,P}}\psi, \qquad \psi'' = S_{\mathrm{M,K}}\psi', \qquad \psi'' = S_{\mathrm{M,P}}\psi. \tag{20}
$$

3.2. EXPLICIT SOLUTIONS

Tables 1 and 2 show the components of the solutions of given momentum for the three sets of γ-matrices. The symbols used are

† i.e., all the elements of $\gamma''_1, \gamma''_2, \gamma''_3$ are real numbers, while those of γ''_4 are purely imaginary.

†† For simplicity we write $\rho_i\sigma_k$ instead of $\rho_i\times\sigma_k$ in these formulae, and replace unit matrices by 1.

TABLE 1
Particle solutions

	$\sqrt{2}\, u(\mathbf{k}, \uparrow)$	$\sqrt{2}\, u(\mathbf{k}, \downarrow)$	$u'(\mathbf{k}, \uparrow)$	$u'(\mathbf{k}, \downarrow)$	$\sqrt{2}\, e^{-\pi i/4} \cdot u''(\mathbf{k}, \uparrow)$	$\sqrt{2}\, e^{-\pi i/4} \cdot u''(\mathbf{k}, \downarrow)$
(1)	$\alpha(c+s)$	$-\beta^*(c+s)$	$c\alpha$	$-s\beta^*$	$c\alpha-is\beta$	$-s\beta^*-ic\alpha^*$
(2)	$\beta(c+s)$	$\alpha^*(c+s)$	$c\beta$	$s\alpha^*$	$c\beta+is\alpha$	$s\alpha^*-ic\beta^*$
(3)	$\alpha(c-s)$	$\beta^*(c-s)$	$s\alpha$	$-c\beta^*$	$ic\beta+s\alpha$	$is\alpha^*-c\beta^*$
(4)	$\beta(c-s)$	$-\alpha^*(c-s)$	$s\beta$	$c\alpha^*$	$-ic\alpha+s\beta$	$is\beta^*+c\alpha^*$

TABLE 2
Antiparticle solutions

	$\sqrt{2}v(\mathbf{k}, \uparrow)$	$\sqrt{2}v(\mathbf{k}, \downarrow)$	$v'(\mathbf{k}, \uparrow)$	$v'(\mathbf{k}, \downarrow)$	$\sqrt{2}\, e^{-\pi i/4} \cdot v''(\mathbf{k}, \uparrow)$	$\sqrt{2}\, e^{-\pi i/4} \cdot v''(\mathbf{k}, \downarrow)$
(1)	$-\beta^*(c-s)$	$\alpha(c-s)$	$s\beta^*$	$c\alpha$	$s\beta^*-ic\alpha^*$	$c\alpha+is\beta$
(2)	$\alpha^*(c-s)$	$\beta(c-s)$	$-s\alpha^*$	$c\beta$	$-s\alpha^*-ic\beta^*$	$c\beta-is\alpha$
(3)	$\beta^*(c+s)$	$\alpha(c+s)$	$-c\beta^*$	$-s\alpha$	$-is\alpha^*-c\beta^*$	$ic\beta-s\alpha$
(4)	$-\alpha^*(c+s)$	$\beta(c+s)$	$c\alpha^*$	$-s\beta$	$-is\beta^*+c\alpha^*$	$-ic\alpha-s\beta$

$$c = \cos\frac{\chi}{2}, \qquad s = \sin\frac{\chi}{2}, \qquad \operatorname{ctg}\chi = \frac{|\mathbf{k}|}{m}, \qquad k_z = k\cos\theta,$$
$$k_x+ik_y = k\sin\theta\cdot e^{i\phi}, \qquad \alpha = \cos\frac{\theta}{2}\cdot e^{-i\phi/2}, \qquad \beta = \sin\frac{\theta}{2}\cdot e^{i\phi/2}. \tag{21}$$

It is easy to verify that $\uparrow$ and $\downarrow$ correspond to a spin orientation parallel and antiparallel to **k** for the particle solutions, and to a spin orientation parallel and antiparallel to $-\mathbf{k}$ for the antiparticle solutions. For example, for the Kramers set

$$\frac{1}{k}(\mathbf{s}'\cdot\mathbf{k})u'(\mathbf{k},\uparrow) = \tfrac{1}{2}u'(\mathbf{k},\uparrow),$$
$$\frac{1}{k}(\mathbf{s}'\cdot\mathbf{k})u'(\mathbf{k},\downarrow) = -\tfrac{1}{2}u'(\mathbf{k},\downarrow), \tag{22}$$

and

$$\frac{1}{k}(\mathbf{s}'\cdot-\mathbf{k})v'(\mathbf{k},\uparrow) = \tfrac{1}{2}v'(\mathbf{k},\uparrow),$$
$$\frac{1}{k}(\mathbf{s}'\cdot-\mathbf{k})v'(\mathbf{k},\downarrow) = -\tfrac{1}{2}v'(\mathbf{k},\downarrow). \tag{22'}$$

Another important point is that, for the Majorana set, the particle and antiparticle solutions are the complex conjugate of one another,

$$(u''_\alpha(\mathbf{k}, \uparrow))^* = v''_\alpha(\mathbf{k}, \uparrow),$$
$$(u''_\alpha(\mathbf{k}, \downarrow))^* = v''_\alpha(\mathbf{k}, \downarrow) \qquad (\alpha = 1, 2, 3, 4). \tag{23}$$

That this must be so can be seen by writing the Dirac equation for the two kinds of solutions,

$$(i\gamma'' \cdot \mathbf{k} - \omega\gamma''_4 + m)u''(\mathbf{k}, \lambda) = 0,$$
$$(-i\gamma'' \cdot \mathbf{k} + \omega\gamma''_4 + m)v''(\mathbf{k}, \lambda) = 0, \tag{24}$$

and by noticing that, γ'' being real and γ''_4 purely imaginary, the second equation results from the first by complex conjugation (not to be confused with Hermitian conjugation).

The relationship between particle and antiparticle solutions is more complicated for the other two sets. Taking

$$\psi^* \gamma, \tag{25}$$

where γ is a matrix and ψ is a four-component quantity, to mean a quantity with components $\alpha = 1, 2, 3, 4$ given by

$$\psi_\beta^* \gamma_{\beta\alpha} \tag{26}$$

we see that

$$v(\mathbf{k}, \lambda) = u^*(\mathbf{k}, \lambda)\gamma_2, \qquad v'(\mathbf{k}, \lambda) = u'^*(\mathbf{k}, \lambda)\gamma'_2 \tag{27}$$

and conversely

$$u(\mathbf{k}, \lambda) = v^*(\mathbf{k}, \lambda)\gamma_2, \qquad u'(\mathbf{k}, \lambda) = v'^*(\mathbf{k}, \lambda)\gamma'_2. \tag{27'}$$

3.3. MAJORANA'S NEUTRINO THEORY

Some neutral particles of spin ½ (e.g. neutrons) are distinguishable from their antiparticles by virtue of their electromagnetic properties, such as their magnetic moment.

The Majorana theory of neutrino [6, 9]) concerns itself with particles which are identical with their antiparticles. The transition from the ordinary theory to the Majorana theory is effected by writing

$$\mathsf{a} \to \mathsf{c}, \qquad \mathsf{a}^* \to \mathsf{c}^*, \qquad \mathsf{b} \to \mathsf{c}, \qquad \mathsf{b}^* \to \mathsf{c}^* \tag{28}$$

in eq. (12) i.e. by identifying the creation and destruction operators for particles and antiparticles. The consequences of doing so are particularly evident if Majorana's own set of γ-matrices is employed. In fact, from (12), (28) and tables 1 and 2, we find

$$\begin{pmatrix} \psi''_1 \\ \psi''_2 \\ \psi''_3 \\ \psi''_4 \end{pmatrix} = \begin{pmatrix} \psi''^*_1 \\ \psi''^*_2 \\ \psi''^*_3 \\ \psi''^*_4 \end{pmatrix}, \tag{29}$$

so that the components of the field operator for Majorana particles are

Hermitian for the Majorana set. For the Pauli and Kramers sets we have

$$\psi_\alpha = \psi_\beta^* \gamma_{2,\beta\alpha}, \qquad \psi'_\alpha = \psi_\beta'^* \gamma'_{2,\beta\alpha} \tag{30}$$

so that

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} -\psi_4^* \\ \psi_3^* \\ \psi_2^* \\ -\psi_1^* \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_2^* \\ -\psi_1^* \end{pmatrix} \tag{31}$$

in the former, and

$$\begin{pmatrix} \psi'_1 \\ \psi'_2 \\ \psi'_3 \\ \psi'_4 \end{pmatrix} = \begin{pmatrix} \psi'_1 \\ \psi'_2 \\ -\psi_2'^* \\ \psi_1'^* \end{pmatrix} \tag{31'}$$

in the latter. This shows that the Majorana theory uses only two independent non-Hermitian components, or four independent Hermitian components in the Majorana set.

3.4. ANGULAR MOMENTUM REPRESENTATION

In the angular momentum representation, particles of spin $\frac{1}{2}$ are described by the eigenvalues $j(j+1)$ of the square of the total angular momentum $\mathbf{j} = \mathbf{l}+\mathbf{s}$ $(j = \frac{1}{2}, \frac{3}{2}, \ldots)$ and by the component $j_z = m (m = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots)$. For each pair of values j and m and a given sign of the energy, the Dirac equation is known to have two independent solutions (according as to whether $j = l \mp \frac{1}{2}$, l being the orbital angular momentum, they will be denoted by $j^{(\pm)}$). For positive energy and for the Pauli set of γ-matrices these are of the form [3])

$$u(k, j^{(\pm)}, m; \mathbf{r}, t) = \begin{pmatrix} a^{(\pm)} Y_{j\pm\frac{1}{2}}^{(m-\frac{1}{2})} \\ b^{(\pm)} Y_{j\pm\frac{1}{2}}^{(m+\frac{1}{2})} \\ ic^{(\pm)} Y_{j\mp\frac{1}{2}}^{(m-\frac{1}{2})} \\ id^{(\pm)} Y_{j\mp\frac{1}{2}}^{(m+\frac{1}{2})} \end{pmatrix} \cdot e^{-i\omega t}. \tag{32}$$

Where two signs occur, only the upper signs or the lower signs must be taken.

The (real) coefficients $a^{(\pm)}, \ldots, d^{(\pm)}$ need not be specified. For what follows it is, however, necessary to notice that

$$a^{(\pm)}(-m) = \mp b^{(\pm)}(m), \qquad c^{(\pm)}(-m) = \pm d^{(\pm)}(m). \tag{33}$$

It follows immediately from $Y_l^{(m)}(-\mathbf{r}) = (-1)^l Y_l^{(m)}(\mathbf{r})$ that

$$u(k, j^{(\pm)}, m; -\mathbf{r}, t) = (-1)^{j\pm\frac{1}{2}} \gamma_4 \, u(k, j^{(\pm)}, m; \mathbf{r}, t), \tag{34}$$

from which it is apparent that, j being a half integer, the solutions with $j^{(+)}$ and $j^{(-)}$ are of different parity.

Antiparticle solutions for $j_z = m$ (which are negative energy solutions for $j_z = -m$) are given by

$$v(k, j^{(\pm)}, m; \mathbf{r}, t) = (-1)^{m+\frac{1}{2}} \begin{pmatrix} id^{(\pm)} Y_{j\mp\frac{1}{2}}^{(-m-\frac{1}{2})} \\ ic^{(\pm)} Y_{j\mp\frac{1}{2}}^{(-m+\frac{1}{2})} \\ b^{(\pm)} Y_{j\pm\frac{1}{2}}^{(-m-\frac{1}{2})} \\ a^{(\pm)} Y_{j\pm\frac{1}{2}}^{(-m+\frac{1}{2})} \end{pmatrix} \cdot e^{-i\omega t} \tag{35}$$

and are related to the particle solutions by the equation

$$v(k, j^{(\pm)}, m; \mathbf{r}, t) = u^*(k, j^{(\pm)}, m; \mathbf{r}, t)\gamma_2, \tag{36}$$

as is easy to show by using the properties of the spherical harmonics. Eq. (36) is analogous to (27) for the momentum representation.

The analogue of (34) is

$$v(k, j^{(\pm)}, m; -\mathbf{r}, t) = (-1)^{j\mp\frac{1}{2}} \gamma_4 v(k, j^{(\pm)}, m; \mathbf{r}, t). \tag{37}$$

The expansion of the field operators is

$$\psi = \sum [\mathfrak{a}(k, j^{(\pm)}, m)u(k, j^{(\pm)}, m; \mathbf{r}, t) + \mathfrak{b}^*(k, j^{(\pm)}, m)v(k, j^{(\pm)}, m; \mathbf{r}, t)], \tag{38}$$

which defines creation and destruction operators for particles and anti-particles characterized by the quantum numbers $k, j^{(\pm)}, m$.

3.5. TWO-PARTICLE STATES

To construct state vectors for two particles of spin $\frac{1}{2}$, we proceed as in the case of spinless particles. For a particle and an antiparticle in the centre-of-momentum frame, the most general state of definite energy is

$$\int d\Omega_{\mathbf{k}}\, \phi(\mathbf{k}; s_a s_b)\mathfrak{a}^*(\mathbf{k}, s_a)\mathfrak{b}^*(-\mathbf{k}, s_b) \quad |0), \tag{39}$$

where now it is convenient to use as one-particle states those of given momentum and spin parallel and antiparallel to the z-axis ($s = \pm 1$). The corresponding creation operators are related to the creation operators for spin parallel and antiparallel to the momentum by the formulae

$$\mathfrak{a}^*(\mathbf{k}, 1) = \alpha^* \mathfrak{a}^*(\mathbf{k}, \uparrow) - \beta \mathfrak{a}^*(\mathbf{k}, \downarrow), \tag{40}$$

$$\mathfrak{a}^*(\mathbf{k}, 2) = \beta^* \mathfrak{a}^*(\mathbf{k}, \uparrow) + \alpha \mathfrak{a}^*(\mathbf{k}, \downarrow) \tag{41}$$

(non-relativistic).

The magnitude of the total spin $\mathbf{S} = \mathbf{s}_a + \mathbf{s}_b$ of the particle-antiparticle system is a good quantum number, so that we have triplet and singlet states.

For the singlet states, the orbital angular momentum L is also a good quantum number. For the triplet, orbital angular momenta are mixed to form $^3P_0$, $^3S_1 + {}^3D_1$, $^3P_1$, $^3P_2 + {}^3F_2$, $^3D_2$ etc. states. (The subscript denotes the total angular momentum J.) In both cases, the momentum wave function has the property

$$\phi(-\mathbf{k}, s_a s_b) = (-1)^L \phi(\mathbf{k}, s_a a_b), \tag{42}$$

which holds also for mixtures such as $^3S_1 + {}^3D_1$, since the orbital angular momenta of the two components differ by 2. On the other hand, for interchange of s_a and s_b, $\phi(\mathbf{k}, s_a s_b)$ is symmetrical for the triplet, antisymmetrical for the singlet. The non-relativistic expressions

$$Y_L^{(M)} \left[\delta_{s_a,1}\,\delta_{s_b,1},\ \frac{\delta_{s_a,1}\,\delta_{s_b,-1}+\delta_{s_a,-1}\,\delta_{s_b,1}}{\sqrt{2}},\ \delta_{s_a,-1}\,\delta_{s_b,-1}\right] \tag{43}$$

$$Y_L^{(M)} \frac{\delta_{s_a,1}\,\delta_{s_b,-1}-\delta_{s_a,-1}\,\delta_{s_b,1}}{\sqrt{2}} \tag{44}$$

for the triplet (no mixture) and the singlet satisfy these general conditions.

4. Photons [11]

4.1. POLARIZATION STATES

For a photon of momentum $\mathbf{k}$ two independent polarization states are possible, from which states with any other polarization can be constructed by linear combination. They are represented by the unit vectors $\mathbf{e}(\mathbf{k}, \lambda)$ $(\lambda = 1, 2)$ perpendicular to $\mathbf{k}$ and to each other. Creation and destruction operators $\mathfrak{a}^*(\mathbf{k}, \lambda)$ and $\mathfrak{a}(\mathbf{k}, \lambda)$ can be defined, so that

$$\mathfrak{a}^*(\mathbf{k}, \lambda)\,|0) \tag{45}$$

is a state with a photon of momentum $\mathbf{k}$ and polarization λ.

The field operator is the vector potential, which can be represented as

$$\mathbf{A}(x) = \sum \mathbf{e}(\mathbf{k}, \lambda)\,[\mathfrak{a}(\mathbf{k}, \lambda)e^{i(k,x)}+\mathfrak{a}^*(\mathbf{k}, \lambda)e^{-i(k,x)}], \tag{46}$$

where

$$\sum \equiv \sum_{\mathbf{k}} \sum_{\lambda} \frac{1}{\sqrt{2Vk_0}} \text{ and } k_0 = |\mathbf{k}|.$$

Consider now the simple case of two classical waves both circularly polarized. The first,

$$\begin{aligned} A_x^{(R)} &= A_0 \cos\,(kz-k_0t+\delta^R), \\ A_y^{(R)} &= A_0 \sin\,(kz-k_0t+\delta^R) \end{aligned} \tag{47}$$

propagates with wave length $2\pi k$ in the z-direction and has right-circular polarization, and

$$\begin{aligned} A_x^{(L)} &= A_0 \cos\,(kz-k_0t+\delta^L), \\ A_y^{(L)} &= -A_0 \sin\,(kz-k_0t+\delta^L) \end{aligned} \tag{48}$$

has left-circular polarization and propagates in the same direction as the first.

We compare (47) and (48) with the vector potentials for two waves both propagating in the z-direction, one

$$\mathbf{A}^{(1)} = A_0\mathbf{e}(1)e^{i(kz-k_0t)}+\text{c.c.} \tag{49}$$

with $\mathbf{e}(1) \equiv (1, 0, 0)$, polarized in the x-direction, the other,

$$\mathbf{A}^{(2)} = A_0 \mathbf{e}(2) e^{i(kz-k_0 t)} + \text{c.c.} \tag{50}$$

with $\mathbf{e}(2) \equiv (0, 1, 0)$, polarized in the y-direction. We can now write

$$\mathbf{A}^{(R)} = \frac{A_0}{\sqrt{2}} \mathbf{e}(R) e^{i(kz-k_0 t+\delta^R)} + \text{c.c.} \tag{51}$$

and

$$\mathbf{A}^{(L)} = \frac{A_0}{\sqrt{2}} \mathbf{e}(L) e^{i(kz-k_0 t+\delta^L)} + \text{c.c.} \tag{52}$$

Here

$$\mathbf{e}(R) = \frac{\mathbf{e}(1) - i\mathbf{e}(2)}{\sqrt{2}}, \qquad \mathbf{e}(L) = \frac{\mathbf{e}(1) + i\mathbf{e}(2)}{\sqrt{2}} \tag{53}$$

and

$$\mathbf{e}^*(R) \cdot \mathbf{e}(R) = \mathbf{e}^*(L) \cdot \mathbf{e}(L) = 1.$$

We can now proceed to define circularly polarized photons. We introduce

$$a(\mathbf{k}, R) = \frac{1}{\sqrt{2}} [a(\mathbf{k}, 1) + i a(\mathbf{k}, 2)] \tag{54}$$

as the destruction operator for a right-circular photon of momentum $\mathbf{k}$ and polarization vector

$$\mathbf{e}(\mathbf{k}, R) = \frac{1}{\sqrt{2}} [\mathbf{e}(\mathbf{k}, 1) - i\mathbf{e}(\mathbf{k}, 2)], \tag{55}$$

and

$$a(\mathbf{k}, L) = \frac{1}{\sqrt{2}} [a(\mathbf{k}, 1) - i a(\mathbf{k}, 2)] \tag{56}$$

and

$$\mathbf{e}(\mathbf{k}, L) = \frac{1}{\sqrt{2}} [\mathbf{e}(\mathbf{k}, 1) + i\mathbf{e}(\mathbf{k}, 2)] \tag{57}$$

for left-circular photons. Then the right member in the expression (46) for the vector potential can be rearranged to read

$$\mathbf{A}(x) = \sum \{ [\mathbf{e}(\mathbf{k}, R) a(\mathbf{k}, R) + \mathbf{e}(\mathbf{k}, L) a(\mathbf{k}, L)] e^{i(k,x)} + [\mathbf{e}^*(\mathbf{k}, R) a^*(\mathbf{k}, R) + \mathbf{e}^*(\mathbf{k}, L) a^*(\mathbf{k}, L)] e^{-i(k,x)} \} \tag{58}$$

States with one circularly polarized photon of momentum $\mathbf{k}$ will be given by

$$a^*(\mathbf{k}, R)\,|0) \quad \text{and} \quad a^*(\mathbf{k}, L)\,|0)$$

according to whether the polarization is right or left. By eqs. (54) and (56) they can be expressed as superpositions of states with linear polarization.

4.2. TWO-PHOTON STATES

Later on, in connection with the selection rules for positronium decay, we shall have to consider states with two photons of equal and opposite momenta. For simplicity we shall assume that the momenta are along the z-axis, and denote by $a^*(+, R)$, $a^*(-, R)$, $a^*(+, L)$, $a^*(-, L)$ the creation operators for photons with right- and left-circular polarization, travelling in the positive (+) and negative (−) z-direction.

By applying to the vacuum state the product of two of the above operators we can construct the four states $|+R, -R)$, $|+L, -L)$, $|+R, -L)$, $|+L, -R)$. It is convenient, however, to consider mixtures of the first two:

$$\begin{aligned}|+R, -R)+|+L, -L) &= [a^*(+, R)a^*(-, R)+a^*(+, L)a^*(-, L)]\,|0) \\ &= [a^*(+, 1)a^*(-, 1)-a^*(+, 2)a^*(-, 2)]\,|0), \qquad (59)\\ |+R, -R)-|+L, -L) &= -i[a^*(+, 1)a^*(-, 2)+a^*(+, 2)a^*(-, 1)]\,|0).\end{aligned}$$

The first represents two photons travelling in opposite directions with planes of polarization always parallel. There is an equal chance that the planes of polarization are either both in the x-direction or both in the y-direction. The second represents the state of two photons having planes of polarization always perpendicular to one another. On the other hand

$$\begin{aligned}|+R, -L) &= \tfrac{1}{2}[a^*(+, 1)a^*(-, 1)+a^*(+, 2)a^*(-, 2) \\ &\quad +ia^*(+, 1)a^*(-, 2)-ia^*(+, 2)a^*(-, 1)]\,|0), \\ |+L, -R) &= \tfrac{1}{2}[a^*(+, 1)a^*(-, 1)+a^*(+, 2)a^*(-, 2) \\ &\quad -ia^*(+, 1)a^*(-, 2)+ia^*(+, 2)a^*(-, 1)]\,|0)\end{aligned} \qquad (60)$$

are states where there is an equal chance that the planes of polarization are parallel or perpendicular to one another.

4.21. TRANSFORMATION PROPERTIES OF PHOTON STATES

We cannot dispense with a few remarks about transformation properties of photons under space rotations, later to be used for deriving selection rules for the decay of particles into two photons.

A rotation by the angle ϕ around the z-axis

$$\begin{aligned}x' &= x\cos\phi + y\sin\phi, \\ y' &= -x\sin\phi + y\cos\phi, \qquad (61)\\ z' &= z\end{aligned}$$

transforms the classical vector potential as

$$\begin{aligned}A'_x &= A_x\cos\phi + A_y\sin\phi, \\ A'_y &= -A_x\sin\phi + A_y\cos\phi, \qquad (62)\\ A'_z &= A_z.\end{aligned}$$

In the quantized theory the rotation is associated with a unitary operator R_ϕ such that

$$\mathsf{R}_\phi \mathsf{A} \mathsf{R}_\phi^{-1} = \mathsf{A}' \qquad (63)$$

is the transformation law for the vector potential operator, $\mathbf{A}'$ being expressed in terms of $\mathbf{A}$ through (62). In order to determine R_ϕ we consider one typical term in the expansion (58). According to (61) and (62) the polarization vectors $\mathbf{e}(\mathbf{k}, R)$ and $\mathbf{e}(\mathbf{k}, L)$ transform as †

$$\begin{aligned} \mathbf{e}'(\mathbf{k}, R) &= e^{-i\phi} \cdot \mathbf{e}(\mathbf{k}, R), \\ \mathbf{e}'(\mathbf{k}, L) &= e^{i\phi} \cdot \mathbf{e}(\mathbf{k}, L), \end{aligned} \tag{64}$$

if $\mathbf{k}$ is in the z-direction. For (63) to be realized we must have

$$R_\phi[\mathbf{e}(\mathbf{k}, R)a(\mathbf{k}, R)]R_\phi^{-1} = \mathbf{e}'(\mathbf{k}, R)a(\mathbf{k}, R) \tag{65}$$

and similarly for left polarization, and therefore

$$\begin{aligned} R_\phi\, a(\mathbf{k}, R)R_\phi^{-1} &= e^{-i\phi} \cdot a(\mathbf{k}, R), \\ R_\phi\, a(\mathbf{k}, L)R_\phi^{-1} &= e^{i\phi} \cdot a(\mathbf{k}, L). \end{aligned} \tag{66}$$

For the creation operators we have

$$\begin{aligned} R_\phi\, a^*(\mathbf{k}, R)R_\phi^{-1} &= e^{i\phi} \cdot a^*(\mathbf{k}, R) \\ R_\phi\, a^*(\mathbf{k}, L)R_\phi^{-1} &= e^{-i\phi} \cdot a^*(\mathbf{k}, L). \end{aligned} \tag{67}$$

We now apply R_ϕ to states with one and two photons. For instance

$$R_\phi\, a^*(\mathbf{k}, R)\,|0) = R_\phi\, a^*(\mathbf{k}, R)R_\phi^{-1} \cdot R_\phi\,|0). \tag{68}$$

If we assume that the vacuum is invariant under rotations,

$$R_\phi\,|0) = |0), \tag{69}$$

comparing (68) with (67) we find

$$R_\phi\, a^*(\mathbf{k}, R)\,|0) = e^{i\phi} \cdot a^*(\mathbf{k}, R)\,|0) \tag{70}$$

and similarly

$$R_\phi\, a^*(\mathbf{k}, L)\,|0) = e^{-i\phi} \cdot a^*(\mathbf{k}, L)\,|0). \tag{71}$$

Thus states with one right or left circularly polarized photon moving along the z-axis are eigenstates of R_ϕ with eigenvalues $e^{-i\phi}$ and $e^{+i\phi}$ respectively. It is useful to remind the reader of the connection between the angular momentum component in a certain direction, say z, and the operator R_ϕ for rotations around that direction,

$$J_z = \frac{1}{i}\left(\frac{dR_\phi}{d\phi}\right)_{\phi=0} \tag{72}$$

It is apparent from this formula and from (70), (71) that $a^*(\mathbf{k}, R)\,|0)$ and $a^*(\mathbf{k}, L)\,|0)$ are photons with $J_z = \pm 1$ respectively.

The discussion of two photon states follows similar lines. We have

$$R_\phi\{|+R, -R)\pm|+L, -L)\} = |+R, -R)\pm|+L, -L) \tag{73}$$

$$R_\phi|+R, -L) = e^{2i\phi}|+R, -L), \tag{73'}$$

$$R_\phi|+L, -R) = e^{-2i\phi}|+L, -R). \tag{73''}$$

† The exponentials $e^{\pm i(k,x)}$ are invariant under rotations.