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Topics in Locally Convex Spaces

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Topics in Locally Convex Spaces

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PREFACE

The choice of topics considered here are dictated by the author's own interest in the field and concentrated heavily on his own research work done in the last years. No claim for completeness is made for the bibliography at the end of the notes. Numbers in square brackets refer to it.

The notes are aimed to persons who already have an acquaintance with the general theory of locally convex spaces. Since the proofs are presented with detail and since some efforts have been made to give a number of simple arguments replacing some rather cumbersome constructions, most of the notes should be readable for graduate students but they can also serve as a reference for the more advanced mathematician.

These notes consist of three Chapters. Each chapter splits into several paragraphs and each paragraph in sections which are enumerated in consecuteve fashion. Gross references are usually u. v. w. z meaning that reference is made to statement z of section w in paragraph v in chapter u. Gross references within the chapter are v. w. z and within the paragraph w. z.

Nine paragraphs constitue the first chapter. Paragraphs 1, 2, 3, 5 are dedicated to the study of classes of locally convex spaces which are used as domain class for the closed graph theorem. Paragraph 4 is devoted to the closed graph theorem when the range class is the quasi - Suslin, K - Suslin, Suslin or semi - Suslin spaces. Paragraph 6 studies the incidence of the duality theory on the closed graph theorem. A characterization of the locally convex spaces which are weakly realcompact is included as well as a discussion on generalized countable inductive limits. Some properties on bounded sets in (LN) - spaces are given.

The second chapter is concerned with sequence spaces which are studied along six paragraphs. A general study of the Kothe perfect spaces and echelon and co - echelon spaces is included. A characterization of echelon quasi - normable spaces is given as well as a discussion on echelon and co - echelon spaces of order p, $1 < p < \infty$, and of order zero. Paragraph 5 contains examples of sequences spaces which answer several questions on aspects of the general theory of locally convex spaces. An example of a Banach space which is an hyperplane of its strong bidual due to R. C. JAMES inspires the end of the chapter where a construction of some vector - valued sequence spaces can be found.

Chapter three has three paragraphs: the first includes easy representations of the more interesting spaces of infinitely differentiable functions and distributions. In the second paragraph representations of spaces of C^m - differentiable functions can be found. The last paragraph is a detailed exposition of Milutin's representation theorem: if X and Y are non - countable compact metric spaces, then the Banach spaces $C(X)$ and $C(Y)$ are isomorphic.

I wish to acknowledge the help I have received from colleagues who have read parts of the manuscript: P. Pérez Carreras, J. Bonet, M. López Pellicer, M. Maestre and R. Crespo. I thank also my wife for her help in typing the manuscript.

Manuel Valdivia

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TABLE OF CONTENTS

CHAPTER THREE

SPACES OF CONTINUOUS FUNCTIONS

CHAPTER ONE
SOME CLASSES OF LOCALLY CONVEX SPACES

Certain classes of locally convex spaces are studied: Baire, convex-Baire, ordered convex-Baire, suprabarrelled, realcompact, Γ_r *,* Γ*- and* (LB) *spaces. Two paragraph are dedicated to the closed graph and open mapping theorems.*

§ 1. BAIRE SPACES

1. TOPOLOGICAL SPACES OF SECOND CATEGORY. The topological spaces we shall use in this paragraph are supposed distinc from the void set. Let B be a subset of a topological space X. B is nowhere dense or rare if and only its closure has void interior. It is obvious that, if B is rare, every subset of B is also rare. B is of first category or meager if and only if it is the countable union of rare sets of X. Clearly, if B is of first category, every subset of B is also of first category. B is of second category if and only if it is not of first category. If B is of second category every subset of X containing B is of second category. If the subset X of X is of second category we say that X is a space of second category. If every non-void open subset of X is of second category, X is said to be a Baire space. It is immediate that if X is a Baire space, it is a space of second category.

In what follows R denotes the field of the real numbers. If we set

$$A = \{(x,0) : x \in R\} ,$$

$$B = \{(0,y) : y \text{ rational number}, y \neq 0\}$$

and if $Y = A \cup B$ is endowed with the topology induced by the euclidian space R^2, it is easy to show that Y is a space of second category which is not Baire, since B is an open subset of Y which is countable union of rare

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subsets which have only one element.

Given a subset M of a topological space X we set $\overline{M}$ to denote the closure of M and $\mathring{M}$ for its interior. A open set M is regular and if only if $M = \mathring{\overline{M}}$.

(1) *A topological space is Baire if and only if given any sequence* (A_m) *of dense open subsets of* X, *then* $\cap\{A_m : m = 1,2, \ldots\}$ *is dense in* X

Proof. Suppose that X is a Baire space. Let A be a non-void open set of X. For every positive integer m, $X \sim A_m$ is a closed subset of X without interior point and therefore $(X \sim A_m) \cap A$ is rare. Since A is of second category we have that

$$A \neq \cup\{(X \sim A_m) \cap A : m = 1,2, \ldots\}$$

and therefore there is a point X in A. which is not in $X \sim A_m$, m = 1,2,.., thus x belongs to $A_m \cap A$, m = 1,2, ..., and therefore $\cap\{A_m : m = 1,2,\ldots\}$ is dense in X.

Reciprocally, let A be a non-void open set of X. Let (M_m) be a sequence of rare subsets of X contained in A. For every positive integer m, we set A_m for $X \sim \overline{M}_m$. Then (A_m) is a sequence of dense open sets of X and therefore $\cap\{A_m : m = 1,2, \ldots\}$ is dense in X. Then

$$A \cap \{A_m : m = 1,2, \ldots\} \neq \emptyset$$

and thus A is not contained in

$$U \cup \{M_m : m = 1,2, \ldots\}.$$

Consequently, A is of second category. The proof is complete.

Result (1) can be stated as

(2) *The topological space* X *is Baire if and only if, given any sequence* (A_m) *of dense open subset of* X *and given a non-void open subset* A *of* X

$$A \cap (\cap \{A_m : m = 1,2,\ldots\})$$

is non-void.

(3) *Let* B *be a subset of a topological space* X. *Let* $A = \{A_i : i \in I\}$ *the family of all open sets of* X *such that* $A_i \cap B$ *is of first category,* $i \in I$ *Then* $A = \cup\{A_i : i \in I\}$ *is an open regular subset of* X *which intersects* B *in a set of first category.*

Proof. Let

(4) $\{P_j : j \in J\}$

be the collection of all subfamilies of A such that if j belongs to J and P and Q are different elements of P_j, then P and Q are disjoint.

We order the collection (4) by inclusion. We apply Zorn's lemma to obtain a maximal element $P = \{M_h : h \in H\}$ in (4). Set $M = \bigcup\{M_h : h \in H\}$. For every h in H there is a sequence (M_h^n) of rare subsets of X such that

$$M_h \cap B = \bigcup \{M_h^n : m = 1,2,\ldots\}.$$

For every positive integer n, we set

$$M^n = \bigcup\{M_h^n : h \in H\},\ m = 1,2,\ldots$$

Suppose that the interior P of $\overline{M^n}$ is non-void. Then $P \cap M^n$ is non-void and therefore there is k in H such that $P \cap M_k^n$ is non-void. Since the elements of P are pairwise disjoint and since M_k is open we have that the closure Q of

$$\bigcup \{M_h^n : h \in H,\ h \neq k\}$$

is disjoint from M_k. Therefore

$$P \cap M_k \subset \overline{M^n} \cap M_k = \overline{(M_k^n \cup Q)} \cap M_k = \overline{M_k^n} \cap M_k \subset \overline{M_k^n}$$

and that is a contradiction. Thus M^n is a rare subset of X and since

$$M \cap B = \bigcup\{M_h \cap B : h \in H\} = \bigcup\{M_h^n : h \in H,\ n = 1,2,\ldots\}$$

$$= \bigcup\{M^n : n = 1,2,\ldots\}$$

it follows that $M \cap B$ is of first category. Since $\overline{M} \sim M$ is rare, it follows that $U \sim M$ is rare, U being the interior of $\overline{M}$ and from

$$U \cap B = ((U \sim M) \cap B) \cup (M \cap B)$$

we obtain that $U \cap B$ is of first category. We shall see now that U coincides with A. Let L be an element of A. If L is not contained in U, then $L \sim \overline{M}$ is a non-void open set which is disjoint from each of the elements of P and intersects B in a set of first category, contradicting the maximality of P. Now the conclusion follows.

Using the same notations as in (3), we denote by D(B) the set of all points x of X such that every neighbourhood of x meets B in a set of second category. Then D(B) coincides with $X \sim A$. We set O(B) to denote the interior of D(B). We conclude from (3) the so called Banach's condesation theorem:

(5) *The set* $(X \sim D(B)) \cap B$ *is of first category and* D(B) *coincides with* $\overline{O(B)}$.

(6) *For every subset* B *of a topological space* X, $B \sim O(B)$ *is of first category.*

Proof. Let A be the open set defined in (3). Then $A \cap B$ is of first category and $D(B) \sim O(B)$ is a rare set. Consequently,

$$B \cap (A \cup (D(B) \sim O(B)))$$

is of first category. Finally

$$B \sim O(B) = B \cap (X \sim O(B)) = B \cap (A \cup (D(B) \sim O(B)))$$

and the conclusion follows.

(7) *Let* (B_m) *be a sequence of subsets of a topological space* X *whose union is* B. *Then*

(8) $$D(B) \sim \bigcup \{O(B_m) : m = 1,2,\ldots\}$$

is rare.

Proof. Suppose that the closed set (8) has non-void interior S. Then $S \cap B$ is of second category and therefore there is a positive integer p such that $S \cap B_p$ is of second category. Consequently $S \cap D(B_p)$ is non-void and therefore $S \cap O(B_p)$ is non-void and that is a contradiction.

A subset B in a topological space X has the Baire property if there exists an open set U such that $U \sim B$ and $B \sim U$ are of first category.

(9) *A subset* B *in a topological space* X *has the Baire property if and only if* $O(B) \sim B$ *is of first category.*

Proof. According to (6), $B \sim O(B)$ is of first category. Therefore if $O(B) \sim B$ is of first category B has the Baire property.

Now suppose that B has the Baire property. Let U be an open subset of X such that $U \sim B$ and $B \sim U$ are of first category. Then $X \sim \bar{U}$ meets B in the set of first category $B \sim \bar{U}$. Therefore D(B) is contained in $\bar{U}$. Since $\bar{U} \sim U$ is rare we have that $\bar{U} \sim B$ is of first category. On the other hand,

$$O(B) \sim B \subset D(B) \sim B \subset \bar{U} \sim B$$

and therefore $O(B) \sim B$ is of first category. The proof is complete.

2. PRODUCTS OF BAIRE SPACES. In what follows N denotes the set of the positive integers. Let d be a metric on a topological space X. We say that d is compatible with the topology of X if this topology coincides with the topology of the metric space (X,d).

Let $\{X_i : i \in I\}$ be a family of topological spaces. For every i of I let d_i a metric on X_i compatible with the topology of X_i such that (X_i, d_i) is a complete metric space. Then we have the following result due to BOURBAKI:

(1) *The topological product* $X = \Pi\{X_i : i \in I\}$ *is a Baire space.*

Proof. Let A be a non-void open set of X. Let (A_m) be a sequence of dense open sets of X. Since $A \cap A_1$ is non-void there is a finite subset I_1 of I and a closed ball A_i^1 in (X_i, d_i), of radius less than $\frac{1}{2}$, $i \in I_1$, such that

$$\Pi\{A_i^1 : i \in I_1\} \times \Pi\{X_i : i \in I \sim I_1\} \subset A \cap A_1$$

Proceeding by recurrence suppose that, for a positive integer n, we have selected a finite subset I_n of I and a closed ball A_i^n in (X_i, d_i) of positive radius less than $\frac{1}{2^n}$, $i \in I_n$. Since

$$M_n = \Pi\{A_i^n : i \in I_n\} \times \Pi\{X_i : i \in I \sim I_n\}$$

has non-void interior we can find a finite subset I_{n+1} in I, $I_n \subset I_{n+1}$, and a closed ball A_i^{n+1} in (X_i, d_i) of positive radius less than $\frac{1}{2^{n+1}}$, $i \in I_{n+1}$, such that

$$\Pi\{A_i^{n+1} : i \in I_{n+1}\} \times \Pi\{X_i : i \in I \sim I_{n+1}\}$$

is contained in the interior of $M_n \cap A_{n+1}$. We set $J = \bigcup\{I_n : n = 1,2,\ldots\}$. For every i in $I \sim J$ take a point x_i in X_i and set $x_i^n = x_i$, $n = 1,2,\ldots$

If i belongs to J and n to N we take x_i^n in A_i^n. The sequence (x_i^n) is obviously a Cauchy sequence in (X_i, d_i) and therefore converges in this space to a point x_i belonging to

$$\cap \{A_i^n : n = 1,2,\ldots\}.$$

Consequently the sequence $((x_i^n : i \in I))$ of X converges in this space to $(x_i : i \in I)$ and

$$(x_i : i \in I) \in \cap \{M_n : n = 1,2,\ldots\} \subset A \cap (\cap \{A_n : n = 1,2,\ldots\})$$

and the conclusion follows.

If we suppose that the former index set I has only one element we obtain from (1) the classical theorem of Baire:

(2) *If there is a metric* d *in a topological space* X *compatible with its topology and such that* (X,d) *is complete, then* X *is a Baire space.*

Now suppose that Y_i is a topological space, $i \in I$. Denote by Y the topological product $\Pi\{Y_i : i \in I\}$. A cylinder in Y is a subset of Y of the form

$$\Pi \{A_i : i \in I\}$$

where $A_i = Y_i$ save a finite number of indices i.

(3) *If the cardinal of* I *is less or equal than the cardinal of* R *and if* Y_i *is separable,* i C I, *then is separable.*

Proof. For every i in I let (x_{im}) be a sequence in Y_i whose elements form a dense subspace Z_i of Y_i. The topological space

$$Z = \Pi\{Z_i : i \in I\}$$

is dense in Y and therefore it is enough to show that Z is separable. Now suppose that N has the discrete topology. The mapping T from the topological space N^I onto Z such that

$$T(n_i : i \in I) = (x_{in_i} : i \in I)$$

is obviously continuous and therefore it is enough to show that N^I is separable. Let J be a non-void set, $J \cap I = \emptyset$, such that $I \cup J$ has the cardinality of R. Then $N^I \times N^J$ is homeomorphic to N^R. On the other hand, the projection of $N^I \times N^J$ onto N^I is continuous and therefore it is enough to show that N^R is separable.

Let P be the set of all the functions defined on R which are characteristic functions on intervals of rational ends. If k is the element of N^R which takes the value one in every point of R we set

$$H = \{k + \sum_{j=1}^{p} (n_j - 1) f_j : n_j, p \in N, f_j \in P, j = 1,2,\ldots,p\}.$$

H is a countable subset of N^R and we shall show that it is dense in N^R. Let U be a neighborhood of an element f of N^R. We find pairwise distinct real numbers $x_1, x_2,\ldots,x_q$ such that

$$\{g \in N^R : g(x_j) = f(x_j),\ j = 1,2,\ldots,q\} \subset U.$$

Take pairwise disjoint intervals $A_1, A_2,\ldots,A_q$ of rational ends such that x_j is in A_j and set h_j to denote the characteristic function of A_j, j=1,2, ...q. Then

$$k + \sum_{j=1}^{q} (f(x_j)-1)h_j \in H \cap U$$

and the conclusion follows.

(4) *Let*

(5) $\{A_j : j \in J\}$

be a family of pairwise disjoint non-void open sets of Y. *If for every* i *in* I, Y_i *is separable then* J *is a countable set.*

Proof. Suppose the property is not true. Take a subfamily of (5), which we denote by (5) again, such that the cardinality of J is less or equal than the cardinality of R, and J is not countable.

For every j in J we find a finite subset I_j in I and a non-void open subset B_i of Y_i, $i \in I_j$, such that

$$\Pi\{B_i : i \in I_j\} \times \Pi\{Y_i : i \in I \sim I_j\} \subset A_j$$

If we set $L = \cup \{I_j : j \in J\}$ we have that the cardinality of L is less or equal than the cardinality of R. We write

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