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Alexandru Buium

Differential Algebraic Groups of Finite Dimension

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INTRODUCTION

1. Scope

Differential algebraic groups are defined roughly speaking as "groups of solutions of algebraic differential equations" in the same way in which algebraic groups are defined as "groups of solutions of algebraic equations". They were introduced in modern literature by Cassidy [C_1] and Kolchin [K_2]; their pre-history goes back however to classical work of S. Lie, E. Cartan and J.F. Ritt.

Let's contemplate a few examples before giving the formal definition (for which we send to Section 3 of this Introduction). Start with any linear differential equation:

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0 \tag{1}$$

where the unknown y and the coefficients a_i are, say, meromorphic functions of the complex variable t. The difference of any two solutions of (1) is again a solution of (1); so the solutions of (1) form a group with respect to addition. This provides a first example of "differential algebraic group".

Similarily, consider the system

$$\begin{cases} xy - 1 = 0 \\ yy'' - (y')^2 + ayy' = 0 \end{cases} \tag{2}$$

where the unknowns x,y and the coefficient a are once again meromorphic in t. This system (extracted from a paper of Cassidy [C_4]) has the property that the quotient $(x_1/x_2, y_1/y_2)$ of any two solutions (x_1,y_1), (x_2,y_2) is again a solution; so the solutions of (2) form a group with respect to the multiplicative group of the hyperbola $xy - 1 = 0$ and we are led to another example of "differential algebraic group".

Examples of a more subtle nature are provided by the systems

$$\begin{cases} y^2 - x(x-1)(x-c) = 0 \\ x''y - x'y' + ax'y = 0 \end{cases} \tag{3a}$$

$$\begin{cases} y^2 - x(x-1)(x-t) = 0 \\ -y^3 - 2(2t-1)(x-t)^2x'y + 2t(t-1)(x-t)^2(x''y - 2x'y') = 0 \end{cases} \tag{3b}$$

where the unknowns x, y are still meromorphic functions in t, the coefficient a is meromorphic in t and $c \neq 0,1$ is a constant in $\mathbb{C}$. These systems (of which (3b) is extracted from a paper of Manin [Ma]) have the property that if (x_1,y_1) and (x_2,y_2) are distinct solutions of one of the systems, then their difference, in the group law of the elliptic curve A defined by the first equation of that system, is again a solution; so the solutions of (3a), (3b) together with the point at infinity (0 : 1 : 0) of A form groups with respect to the group law of A and lead to other examples of "differential algebraic groups".

A differential algebraic group will be called of finite dimension if roughly speaking its elements "depend on finitely many integration constants" rather than on "arbitrary functions". This is the case with the "differential algebraic groups" derived from (1), (2), (3); on the contrary, for instance, the group, say, of all matrices

$$\begin{pmatrix} x & 0 & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \quad x \neq 0$$

satisfying the system

$$(4) \qquad \begin{cases} z' = 0 \\ x' = zx \end{cases}$$

has its elements depending on the "arbitrary function" y so it is an "infinite dimensional differential algebraic group". Differential algebraic groups are known today but to a few mathematicians; and this is because Kolchin's language $[K_1][K_2]$ through which they are studied is known but to a few mathematicians. Yet differential algebraic groups certainly deserve a much broader audience, especially among algebraic geometers. The scope of the present research monograph is twofold namely: 1) to provide an algebraic geometer's introduction to differential algebraic groups of finite dimension and 2) to develop a structure and classification theory for these groups. Unless otherwise stated, all results appearing in this book are due to the author and were never published before.

2. Motivation

The original motivation for the study of differential algebraic groups in $[C_1]$ and $[K_2]$ was undoubtedly their intrinsic beauty and variety. Admittedly no such group appeared so far to play a role, say, in mathematical physics; but as we hope to demonstrate in the present work, differential algebraic groups play a role in algebraic geometry. This is already suggested for instance by the implicit occurence of the differential algebraic group (3b) in Manin's paper [Ma] on the geometric Mordell conjecture. Moreover as shown in the author's papers $[B_6]$, $[B_7]$ differential algebraic groups may be used along a line quite different from Manin's to prove the geometric and infinitesimal analogues of a diophantine conjecture of S. Lang (about intersections of subvarieties of abelian varieties with finite rank subgroups). The latter application will be presented without proof in an appendix. Other applications and interesting links with other topics of algebraic geometry (such as deformations of algebraic groups and their automorphisms, moduli spaces of abelian varieties, the Grothendieck-Mazur-Messing crystalline theory of universal extensions of abelian varieties [MM], a.s.o.) will appear in the body of the text (and will also be touched in this Introduction).

3. Formalization

Before explaining our strategy and results in some detail it will be convenient to provide the formal definition of differential algebraic groups with which we will operate in this book.

The frame in which this definition will be given is that of "differential algebraic geometry" by which we mean here the analogue (due to Ritt and Kolchin) of algebraic geometry in which algebraic equations are replaced by algebraic differential equations. Let's quickly review the basic concepts of this geometry, cf. [R], [K_1], [C_1]; for details we send to the first section of Chapter 5 of this book.

One starts with a field of characteristic zero $\mathcal{U}$ equiped with a derivation $\delta : \mathcal{U} \to \mathcal{U}$ (i.e. with an additive map satisfying $\delta(xy) = (\delta x)y + x(\delta y)$ for all $x,y \in \mathcal{U}$) and set $\mathcal{K} = \{x \in \mathcal{U}; \delta x = 0\}$ the field of constants. We assume that $\mathcal{U}$ is "sufficiently big" as to "contain" all "solutions of algebraic differential equations with coefficients in it"; for the reader familiar with [K_1] what we assume is that $\mathcal{U}$ is "universal". Such a field $\mathcal{U}$ is a quite artificial being; but since all problems related to algebraic differential equations can be "embedded" into "problems over $\mathcal{U}$" the use of this field appears to be extremely useful and in any case it greatly simplifies language. Next one considers the ring of differential polynomials (shortly Δ - polynomials) $\mathcal{U}\{y_1, \dots y_n\}$ which by definition is the ring of polynomials with coefficients in $\mathcal{U}$ in the indeterminates

$$y_1, \dots y_n, y'_1, \dots, y'_n, \dots, y_1^{(k)}, \dots, y_n^{(k)}, \dots$$

E.g. the expressions appearing in the left hand side of the equations (1), (2), (3) (where we assume now $a_i, a, t \in \mathcal{U}$, $t' = 1$, $c \in \mathcal{K}$) are Δ - polynomials. Now for any finite set $F_1, \dots, F_m$ of Δ - polynomials we may consider their set Σ of common zeroes in the affine space $\mathbf{A}^n(\mathcal{U}) = \mathcal{U}^n$; such a set will be called a Δ - closed subset of $\mathcal{U}^n$. Δ - closed sets form a Noetherian topology on $\mathcal{U}^n$ called the Δ - topology which is stronger than the Zariski topology. For instance the equation (1) defines an irreducible Δ - closed subset of the affine line $\mathbf{A}^1(\mathcal{U}) = \mathcal{U}$ while the systems (2) and (3) define irreducible Δ - closed subset of the affine plane $\mathbf{A}^2(\mathcal{U}) = \mathcal{U}^2$.

Any irreducible Δ - closed subset Σ in $\mathcal{U}^n$ has a natural sheaf (for the induced Δ - topology) of $\mathcal{U}$- valued functions on it called the sheaf of Δ - regular functions; roughly speaking a function on a Δ - open set of Σ is called Δ - regular if, locally in the Δ - topology, it is given by a quotient of Δ - polynomials in the coordinates. Σ together with this sheaf will be called an affine differential algebraic manifold (shortly, an affine Δ - manifold).

An irreducible ringed space locally isomorphic to an affine Δ - manifold will be called a differential algebraic manifold (or simply a Δ - manifold). Given a Δ - manifold Σ, the direct limit, over all Δ - open sets Ω of Σ, of the rings of Δ - regular functions on Ω is a field denoted by $\mathcal{U}\langle \Sigma \rangle$. The transcendence degree of $\mathcal{U}\langle \Sigma \rangle$ over $\mathcal{U}$ will be called the dimension of Σ and intuitively represents the number of "integration constants" of which the points of Σ depend. E.g. the Δ - closed sets given by equations (1), (2), (3) have dimensions n, 2, 2 respectively.

A map between two Δ - manifolds will be called Δ - regular if it is continuous and pulls back Δ - regular functions into Δ - regular functions. We are provided thus with a category which has direct products called the category of Δ - manifolds. Note that any algebraic $\mathcal{U}$- variety X (respectively any $\mathcal{K}$- variety X_o) has a natural structure of Δ - manifold which we denote by $\hat{X}$ (respectively by $\check{X}_o$); we have $\dim \check{X}_o = \dim X_o$ and $\dim \hat{X} = \infty$ provided $\dim X > 0$.

Finally we define differential algebraic groups (or simply Δ - group) as being group objects in the category of Δ - manifolds, i.e. Δ - manifolds Γ whose set of points is given a group law such that the multiplication $\Gamma \times \Gamma \to \Gamma$ and the inverse $\Gamma \chi \Gamma$ are Δ - regular maps.

Differential algebraic groups of finite dimension will be simply called Δ_o - groups.

The equations (1), (2) clearly provide examples of Δ_o - groups. As for equations (3) the Δ_o - groups which can be derived are not the Δ - closed subsets of $\mathbf{A}^2(\mathcal{U})$ given by (3) but their Δ - closures in $(\mathbf{P}^2)^\wedge$, which contain an additional point $(0 : 1 : 0)$, the origin of these groups.

We leave open the question whether our Δ - groups "are" the same with Kolchin's [K_2]; one can show that any Δ - group in our sense "is" a Δ - group in Kolchin's sense. Moreover one can show that our Δ_o - groups "are" precisely Kolchin's irreducible "Δ - groups of Δ - type zero". Finally note that our concept of "dimension" corresponds to Ritt's concept of "order" [R] and also (in case we have "finite dimension") with Kolchin's concept of "typical Δ - dimension" [K_1][K_2].

4. Strategy

Cassidy and Kolchin developed their theory of differential algebraic groups in analogy with the theory of algebraic group. Our viewpoint will be quite different: we will base our approach on investigating the relations, not the analogies, between the two theories.

Our strategy has two steps. Step 1 will consist in developing a theory of what we call "algebraic D-groups"; this will be done in Chapters 1-4 of the book. Then Step 2 will consist in applying the latter theory to the study of Δ_o - groups; this will be done in Chapter 5.

Let's explain the concept of algebraic D-group; for convenience we will give in this Introduction a rather restricted definition of it (which will be "enlarged" in the body of the text). Let $\mathcal{U}$, δ, $\mathcal{K}$ be as in Section 3 above and let

$$D = \mathcal{U}[\delta] = \sum_{i \geq 0} \mathcal{U}\delta^i \qquad \text{(direct sum)}$$

be the $\mathcal{K}$ - algebra of linear differential operators generated by $\mathcal{U}$ and δ. By an algebraic D-group we will understand an irreducible algebraic $\mathcal{U}$- group G whose structure sheaf $\mathcal{O}_G$ of regular functions is given a structure of sheaf of D - modules such that the multiplication, comultiplication, antipode and co-unit are D-module maps; in other words if $\mu : G \times G \to G$, $S : G \to G$ are the group multiplication and inverse and if $e \in G$ is the unit then for any regular functions ϕ, ψ defined on some open set of G we have the formulae

$$(5)\quad \begin{aligned} &\delta(\phi\psi) = (\delta\phi)\psi + \phi(\delta\psi) \\ &(\delta\phi)\circ\mu = (\delta\otimes 1 + 1\otimes\delta)(\phi\circ\mu) \\ &\delta(\phi\circ S) = (\delta\phi)\circ S \\ &\delta(\phi(e)) = (\delta\phi)(e) \end{aligned}$$

Some people might like to call such a structure "an algebraic group with connection along δ"; we were inspired in our terminology by the paper of Nichols and Weisfeiler [NW]. Note however that unlike in [NW] we do not assume G is affine, imposing instead that G is of finite type over $\mathcal{U}$! Algebraic D-groups entirely belong to "algebraic geometry" (rather than to the Ritt-Kolchin "differential algebraic geometry") hence Step 1 will inevitably be performed in the field of algebraic geometry; the task of classifying algebraic D-groups will be sometimes quite technical but in the end rewarding.

Step 2 will be based on the result that "the category of Δ_o - groups" is equivalent to "the category of algebraic D - groups"; for any Δ_o - group Γ we shall denote by $G(\Gamma)$ the corresponding algebraic D-group. Note that (the underlying group of) Γ appears as the group of all points $\alpha \in G(\Gamma)(\mathcal{U})$ for which the evaluation map $\mathcal{O}_{G(\Gamma),\alpha} \to \mathcal{U}$ is a D-module map; actually Γ appears as a Δ - closed subgroup of $G(\Gamma)^{\wedge}$. Note that the function field $\mathcal{U}(G(\Gamma))$ identifies with $\mathcal{U}\langle \Gamma \rangle$. Then Step 2 will deal with the (sometimes not so obvious) translation of properties of algebraic D-groups into properties of Δ_o - groups. In order to get a feeling about the correspondence $\Gamma \mapsto G(\Gamma)$ let's look at the examples we began with (equations (1), (2), (3)).

If Γ is derived from equation (1) then $G(\Gamma)$ is the algebraic vector group $G_a^n = \operatorname{Spec}\mathcal{U}[\xi_o,\xi_1,\ldots\xi_{n-1}]$ equiped with the D-module structure of its coordinate algebra defined by

$$\begin{aligned} &\delta\xi_o = \xi_1 \\ &\delta\xi_1 = \xi_2 \\ &\cdots\cdots \\ &\delta\xi_{n-1} = -a_1\xi_{n-1} - a_2\xi_{n-2} - \cdots - a_n\xi_o \end{aligned}$$

(and by the condition that δ is a derivation of $\mathcal{U}[\xi_o,\ldots,\xi_{n-1}]$).

If Γ is derived from equation (2) then $G(\Gamma)$ is $G_m \times G_a = \operatorname{Spec}\mathcal{U}[\chi,\chi^{-1},\xi]$, the product of the multiplicative and additive groups, equiped with the D-structure defined by

$$\begin{aligned} &\delta\chi = \chi\xi \\ &\delta\xi = -a\xi \end{aligned}$$

In case Γ is derived from (3a), $G(\Gamma)$ can be proved to be the product $A \times G_a$ while in case Γ is derived from (3b), $G(\Gamma)$ is a non-trivial extension of the elliptic curve A by G_a. Note that in the latter case $G(\Gamma)$ does not descend to $\mathcal{K}$ (because A doesn't).

5. Results

According to the preceeding section the classification problem for Δ_o - groups reduces to answering the following question: given an algebraic $\mathcal{U}$- group G describe all structures of algebraic D-groups on G; call their set $P(G/\delta)$. Note that $P(G/\delta)$ is a principal homogeneous space for the $\mathcal{U}$- linear space $P(G/\mathcal{U})$ of all $\mathcal{U}$- linear maps $D : \mathcal{O}_G \to \mathcal{O}_G$ satisfying equations (5) with D instead of δ. So we are faced with two problems here namely:

1) What irreducible algebraic $\mathcal{U}$- groups G admit at least one structure of algebraic D - group ?

2) Describe $P(G/\mathcal{U})$ for any such G.

Both these problems have a deformation - theoretic flavour: the first is related to deformations of the algebraic groups themselves while the second is related to infinitesimal deformations of automorphisms of our groups.

Let's consider these problems separately and start by stating our results on the first of them.

THEOREM 1. Let G be an affine algebraic $\mathcal{U}$- group. Then G admits a structure of algebraic D - group if and only if G descends to $\mathcal{K}$.

The proof of Theorem 1 will be done in Chapter 2 and will involve analytic arguments, specifically results of Mostow-Hochschildt [HM_1] and Hamm [Ha]. We already noted that Theorem 1 may fail in the non-affine case; we shall describe in what follows a complete answer to problem 1) in the commutative (non-affine) case. The formulation of the next results requires some familiarity with [Se], [KO]. Their proofs will be done in Chapter 3.

So assume G is an irreducible commutative algebraic $\mathcal{U}$- group and make the following notations:

$L(G)$ = Lie algebra of G

$X_m(G) = Hom(G,G_m)$ = group of multiplicative characters of G

$X_a(G) = Hom(G,G_a)$ = group of additive characters of G.

Recall [KO] that we dispose of the Gauss-Manin connection:

$$\nabla : Der_{\mathcal{K}} \mathcal{U} \to Hom_{\mathcal{K}}(H^1_{DR}(A), H^1_{DR}(A)), \qquad p \mapsto \nabla_p$$

on the de Rham cohomology space $H^1_{DR}(A)$ of the abelian part A of G (where A = G/B, B = linear part of G = maximum linear connected subgroup of G).

We will introduce in Chapter 3 a "multiplicative analogue" of the Gauss-Manin connection which is a $\mathcal{U}$- linear map

$$\ell\nabla : Der_{\mathcal{K}} \mathcal{U} \to Hom_{gr}(H^1_{DR}(A)_m, H^1_{DR}(A)), \qquad p \mapsto \ell\nabla_p$$

where $H^1_{DR}(A)_m$ is the first hypercohomology group of the complex [MM]:

$$1 \to \mathcal{O}^*_A \xrightarrow{dlog} \Omega^1_{A/\mathcal{U}} \xrightarrow{d} \Omega^2_{A/\mathcal{U}} \xrightarrow{d} \cdots$$

View now G as an extension of A by B, let $S(G)_a$ and $S(G)_m$ be the images of the natural maps $X_a(B) \to H^1(\mathcal{O}_A)$ and $X_m(B) \to Pic^o(A)$ [Se] and let $S_{DR}(G)_a$ and $S_{DR}(G)_m$ be the inverse images of $S(G)_a$ and $S(G)_m$ via the "edge morphisms" $H^1_{DR}(A) \to H^1(\mathcal{O}_A)$ and $H^1_{DR}(A)_m \to Pic^o(A)$.

THEOREM 2. Let G be an irreducible commutative algebraic $\mathcal{U}$-group. Then G admits a structure of algebraic D - group if and only if

$$\ell\nabla(S_{DR}(G)_m) \subset S_{DR}(G)_a \qquad \text{and}$$

$$\nabla(S_{DR}(G)_a) \subset S_{DR}(G)_a$$

In particular if G is the universal extension E(A) of an abelian $\mathcal{U}$- variety A by a vector group then G has a structure of algebraic D-group (recall that E(A) is an extension of A by G_a^g, $g = \dim A$, having no affine quotient); this consequence of Theorem 2 is also a consequence of the Grothendieck-Messing-Mazur crystalline theory [MM]. Note that the algebraic D - group $G(\Gamma)$ associated to the Δ_o - group derived from the equation (3b) above is a special case of this construction! Note also that Theorem 2 says more generally that any extension of E(A) by a torus G_m^N has a structure of algebraic D - group !

Theorem 2 will be deduced from a general duality theorem relating the Gauss-Manin connection and its multiplicative analogue to the "adjoint connection" on the Lie algebra of commutative algebraic D - groups. Our duality theorem generalizes certain aspects of the theory in [MM] and [BBM] and our proof is quite "elementary" (although computational !).

Let's pass to discussing "problem 2)" of "describing $P(G/\mathcal{U})$.

It is "well known" that the automorphism functor

$$\underline{Aut}\, G : \{\mathcal{U}\text{- schemes}\} \to \{\text{groups}\}$$

$$(\underline{Aut}\, G)(S) = Aut_{S\text{-grsch.}}(G \times S)$$

of an algebraic $\mathcal{U}$- group G is not representable in general [BS] (by the way we will prove in Chapter 4 that the restriction of $\underline{Aut}\, G$ to {reduced $\mathcal{U}$- schemes} is representable by a locally algebraic $\mathcal{U}$- group Aut G this providing a positive answer to a question of Borel and Serre [BS] p. 152). Then $P(G/\mathcal{U})$ is obviously identified with the Lie algebra $L(\underline{Aut}\, G)$ of the functor $\underline{Aut}\, G$, which contains the Lie algebra L(Aut G), but in general exceeds it (due to "nonrepresentability of $\underline{Aut}$ over non-reduced schemes"). If G is commutative $P(G/\mathcal{U})$ is easily identified with $X_a(G) \otimes L(G)$ (where $Der_{\mathcal{U}} \mathcal{O}_G$ is identified with $\mathcal{O}(G) \otimes L(G)$). For noncommutative G the analysis of $P(G/\mathcal{U})$ will be quite technical; we will perform it in some detail for G affine and get complete results in some special cases (e.g. in case the radical of G is nilpotent or if the unipotent radical of G is commutative), cf. Chapter 2.

Let's discuss in what follows the "splitting" problem for algebraic D-groups. If G_o is any irreducible algebraic $\mathcal{K}$- group one can construct an algebraic D - group $G = G_o \otimes_{\mathcal{K}} \mathcal{U}$ by letting D act on $\mathcal{O}_G$ via $1 \otimes \delta$; any algebraic D - group isomorphic to one obtained in this way will be called split. A Δ_o - group will be called split if it is isomorphic to $\check{G}_o$ for some algebraic

$\mathcal{K}$- group G_o; this is equivalent to saying that $G(\Gamma)$ is split. For instance the Δ_o - group Γ defined by equation (1) above is split; for if $\gamma_1, \dots, \gamma_n \in \mathcal{U}$ is a fundamental system of solutions of (1) then we have an isomorphism of Δ - groups

$$(G^n_{a,\mathcal{K}})^{\vee} = \mathcal{K}^n \to \Gamma$$

$$(c_1, \dots, c_n) \to \Sigma c_i \gamma_i$$

On the contrary the Δ_o - groups derived from equations (2) and (3) are not split; for (3b) this is clear because $G(\Gamma)$ does not descend to $\mathcal{K}$ while for (2) this follows because the D-submodule of $\mathcal{U}[\chi, \chi^{-1}, x]$ generated by χ is infinite dimensional. Now for any algebraic $\mathcal{U}$- group the set of split algebraic D - group structures on G can be proved to be a principal homogenous space for L(Aut G) so, at least in case G descends to $\mathcal{K}$, the problem of determining what algebraic D - group structures on G are split is equivalent to determining what elements of L(<u>Aut</u> G) actually lie in L(Aut G). In particular if G descends to $\mathcal{K}$ and <u>Aut</u> G is representable then any algebraic D - group structure on G is split; this is the case if G is linear reductive or unipotent. More generally we will prove in Chapter 4 the following:

THEOREM 3. For an algebraic D- group G the following are equivalent:

1) G is split.

2) δ preserves the (ideal sheaf of the) unipotent radical U of the linear part of G.

3) δ preserves the (ideal sheaf of) $U/U \cap [G,G]$ in $G/[G,G]$.

4) δ preserves the maximum semiabelian subfield of the function field $\mathcal{U}(G)$ (which by definition is the function field of the maximum semiabelian quotient of G; recall that "semiabelian" means "extension of an abelian variety by a torus").

Consequently a Δ_o - group Γ is split iff $\Gamma/[\Gamma,\Gamma]$ is so, iff δ preserves the maximum semiabelian subfield of $\mathcal{U}<\Gamma>$; if in addition Γ is a Δ - closed subgroup of $\widehat{GL}_N$ for some $N \geq 1$ and we put $I := \{$ideal of all functions in $\mathcal{U}\{y_{ij}\}_d$, $d = \det(y_{ij})$, vanising on $\Gamma\}$ and $\mathcal{U}\{\Gamma\} :=$ $:= \mathcal{U}\{y_{ij}\}_d/I$ then Γ is split iff all group-like elements of the Hopf algebra $\mathcal{U}\{\Gamma\}$ are killed by δ.

We would like to close this presentation of our main results with a theorem about Δ - subgroups of abelian varieties; we need one more definition. We say that a Δ_o - group Γ has no non-trivial linear representation if any Δ - regular homomorphism $\Gamma \to \widehat{GL}_N$ is trivial.

THEOREM 4. Let A be an abelian $\mathcal{U}$-variety of dimension g. Then there is a unique Δ_o - subgroup $A^{\#}$ with the following properties:

1) it is Zariski dense in $\hat{A}$,

2) it has no non-trivial linear representation.

Moreover as A varies in the modului space $\mathcal{A}_{g,n}$ of principally polarized abelian $\mathcal{U}$- varieties with level n - structure, $n \geq 3$, the function $A \mapsto \dim A^{\#}$ varies lower semicontinuously with respect to the Δ - topology of $\hat{\mathcal{A}}_{g,n}$ and assumes all values between g and 2g.

In case $g = 1$ one may say more namely:

a) either A descends to $\mathcal{K}$, say $A = A_o \otimes_{\mathcal{K}} \mathcal{U}$, and then $A^{\#} = A_o(\mathcal{K})$ so $\dim A^{\#} = 1$.

b) or A does not descend to $\mathcal{K}$ and in this case $\dim A^{\#} = 2$.

Note that the Δ - group Γ defined by (3b) is nothing but the group $A^{\#}$ above. On the other hand the Δ - group Γ defined by (3a) is an extension of $(G_{a,\mathcal{K}})^{\vee} \simeq \mathcal{K}$ by $A^{\#}$ above.

6. Amplifications

Most definitions given so far in our Introduction will be "enlarged" in the body of the text and most results will be proved in a generalized form. For instance the definitions related to Δ - manifolds and Δ - groups will be given for "partial" rather than "ordinary" differential fields, i.e. for fields $\mathcal{U}$ equiped with several commuting derivations. Algebraic D - groups will be allowed to be reducible and will be defined for $D = K[P]$ any "k-algebra of linear differential operators on a field extension K of k" which is "built on a Lie K/k-algebra P" (cf. [NW] or Section 0 below); this degree of generality and abstractness might seem excessive, but it is motivated in many ways:

1) our wish to deal with Δ - groups over partial differential universal fields

2) our wish to "compute", for a given algebraic K-group G (K algebraically closed of characteristic zero, containing a field k) the smallest algebraically closed field of definition K_G of G between k and K; the existence of K_G was proved in [B_5] but we will reprove this in a different way in Chapter 4.

3) our wish to relate our topic to Deligne's "regular D-modules".

4) our wish to take a (shy) look at "algebraic D-groups in positive characteristic".

The reader will appreciate the usefulness of this more general concept of algebraic D-group in the light of the arguments 1)-4) above.

7. Plan

The book opens with a preliminary section 0 which fixes terminology and conventions. In Chapter 1 we present the main concepts related to algebraic D-groups. Chapters 2 and 3 are devoted to affine, respectively to commutative algebraic D-groups. Chapter 4 deals with algebraic D-groups which are not necessarily affine or commutative. Chapter 5 opens with an introduction to Δ - manifolds and Δ - groups and then deals with the "structure" and "moduli" of Δ_o - groups.

Internal references to facts not contained in section 0 will be given in the form (X,y,z) or simply (X,y) where X is the number of the chapter and y is the number of the section. Within the same chapter X we write (y,z) instead of (X,y,z). References to section 0 will be given in the form (0,z). Each chapter will begin with a brief account of its contents and of its specific conventions.

8. Prerequisites

The reader is assumed to have only some basic knowledge of algebraic geometry [Har] and of algebraic groups [H], [Ro], [Se]. The non-experts in these fields might still appreciate the results of this book via our comments and various examples.

No knowledge of the Kolchin-Cassidy theory [K_i][C_j] is assumed; the elements of this theory which are relevant for our approach will be quickly reviewed in the book.

Finally one should note that the present book is ideologically a continuation of our previous book [B_1]; but it is logically independent of it.

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CONTENTS

0. TERMINOLOGY AND CONVENTIONS

(0.1) Unless otherwise stated rings, fields and algebras are generally assumed associative commutative with unit. This will not apply however to Lie algebras, universal enveloping algebras or to Hopf algebras (the latter are understood in the sense of Sweedler [Sw]).

(0.2) Terminology of algebraic geometry is the standard one (cf. for instance [Har][DG]). Nevertheless we make the convention that all schemes appearing are separated. For any morphism of schemes $f : X \to Y$ we generally denote the defining sheaf morphism $\mathcal{O}_Y \to f_* \mathcal{O}_X$ by the same letter f. If both X and Y are schemes over some field K then by a K- f-derivation we mean a K-derivation of $\mathcal{O}_Y$ into the $\mathcal{O}_Y$-module $f_* \mathcal{O}_X$. By a K-variety (K a field) we will always mean a (separated!) geometrically integral K-scheme of finite type. By a (locally) algebraic K-group we will mean a geometrically reduced (locally) algebraic K-group scheme. For any integral K-algebra A (respectively for any integral K-scheme X) we let K(A) (respectively K(X)) denote the quotient field of A (respectively of X). For any K-variety X and any field extension K_1/K we denote by $X(K_1)$ the set $\mathrm{Hom}_{K\text{-sch}}(\mathrm{Spec}\, K_1, X)$ of K_1-points of X; more generally we write X(Y) instead of $\mathrm{Hom}_{K\text{-sch}}(Y,X)$ for any K-schemes X,Y. A morphism $X \to Y$ of K-varieties will be called surjective if the map $X(K) \to Y(K)$ is surjective. For any K-scheme X, T_X will denote the sheaf $\mathrm{Der}_K \mathcal{O}_X$. If $X = \mathrm{Spec}\, A$ for some K-algebra A we sometimes write Der(A/K) instead of $\mathrm{Der}_K A$.

(0.3) All affine K-group schemes G are tacitly assumed to be such that the ring $\mathcal{O}(G)$ of global regular functions is at most countably generated as a K-algebra. This is a harmless assumption in view of our applications in Chapter 5.

The unit in any K-group scheme G will be denoted by e; if G is commutative we will sometimes write 0 instead of e.

For any locally algebraic K-group G, G^o will denote the identity component; Z(G) will denote the center and we put $Z^o(G) := (Z(G))^o$.

(0.4) For any algebraic $\mathbb{C}$ - variety X (respectively algebraic $\mathbb{C}$ - group G) we denote by X^{an} (resp. G^{an}) the associated analytic space (respectively the analytic Lie group). For any analytic manifold $\mathfrak{X}$ we denote by $T_{\mathfrak{X}}$ the analytic tangent bundle.

(0.5) Our terminology of differential algebra is a combination of terminologies from $[C_i]$ $[K_i]$ [NW] and $[B_i]$. In what follows we shall review it in some detail and also introduce some new concepts.

Let K/k be a field extension. By a Lie K/k - algebra [NW] we mean a K-vector space P which is also a Lie k-algebra, equipped with a K-linear map $\partial : P \to \mathrm{Der}_k K$ of Lie k-algebras such that

$$[p_1,\lambda p_2] = \partial(p_1)(\lambda)p_2 + \lambda[p_1,p_2] \quad \text{for} \quad \lambda \in K,\ p_1,p_2 \in P$$

Let's give some basic examples of Lie K/k - algebras.

EXAMPLE 1. Start with a derivation $d \in Der_k K$; one can associate to it a Lie K/k - algebra P of dimension 1 over K by letting

$$P = Kp$$

$$\partial(\lambda p) = \lambda d, \quad \lambda \in K$$

$$[\lambda_1 p, \lambda_2 p] = (\lambda_1 d\lambda_2 - \lambda_2 d\lambda_1)p, \qquad \lambda_1, \lambda_2 \in K$$

So we call the attention on the fact that this P is far from being commutative!

EXAMPLE 2. Start now with a family of derivations $(d_i)_{i\in I}$, $d_i \in Der_k K$. Then one can associate to it the "free Lie K/k-algebra" P built on this family: by definition P has the property that it contains a family of elements $(p_i)_{i\in I}$ with $\partial(p_i) = d_i$ such that for any K/k - algebra P' and any family $(p'_i)_{i\in I}$, $p'_i \in P'$ with $\partial'(p'_i) = d_i$ there is a unique Lie K/k - algebra map $f : P \to P'$ with $f(p_i) = p'_i$. We leave to the reader the task of constructing this P. Note that if I consists of one element this P is the same with the one in Example 1.

EXAMPLE 3. Assume we are given a family $(d_i)_{i\in I}$, $d_i \in Der_k K$ such that $[d_i, d_j] = 0$ for all $i,j \in I$. Then one can construct a new Lie K/k-algebra P as follows: we let P have a K-basis $(p_i)_{i\in I}$ we let $\partial(p_i) = d_i$ and define the bracket [,] by the formula

$$[\lambda p_i, \mu p_j] = \lambda(d_i \mu)p_j - \mu(d_j \lambda)p_i, \qquad \lambda, \mu \in K, \ i,j \in I$$

This P can be called the "free integrable Lie K/k-algebra" built on our family of derivations. Once again if I consists of one element, this P coincides with the one in Example 1.

EXAMPLE 4. Let $k \subset E \subset K$ be an intermediate field. Then $P = Der_E K$ together with its inclusion $\partial : Der_E K \to Der_k K$ is an example of Lie K/k - algebra.

Other remarkable examples of Lie K/k - algebras will appear in (I.1).

Given such a P one associates to it [NW] a k-algebra of differential operators $D = K[P]$: by definition K[P] is the associative k-algebra generated by K and P subject to the relations

$\lambda p = v(\lambda, p)$ for $\lambda \in K$, $p \in P$ where $v : K \times P \to P$ is the vector space structure map

$p\lambda - \lambda p = \partial(p)(\lambda)1$ for $\lambda \in K$, $p \in P$

$p_1 p_2 - p_2 p_1 = [p_1, p_2]$ for $p_1, p_2 \in P$

Recall [NW] that $D = K[P]$ has a K-basis consisting of all the monomials of the form $p_{i_1}^{e_1} \dots p_{i_n}^{e_n}$ where $e_j \geq 0$ and $i_1 \leq \dots \leq i_n$ in some total order on a basis $(p_i)_i$ of P. For instance if P is 1-dimensional, $P = Kp$, as in Example 1 above then

$$D = \sum_{i \geq 0} Kp^i \qquad \text{(direct sum)}$$

is the ring of "linear ordinary differential operators" on K "generated by K and p"; multiplication in D corresponds to "composition of linear differential operators". This example is quite familiar: indeed, assume $k = \mathbb{C}$, $K = \mathbb{C}(t)$ is the field of rational functions and $\partial(p) = d/dt$; then $D = \mathbb{C}(t)[d/dt]$ is nothing but the "rational" Weyl algebra. Similarity one can obtain the "rational" Weyl algebra in several variables $D = \mathbb{C}(t_1, \ldots, t_n)[d/dt_1, \ldots, d/dt_n]$ starting with P as in Example 3 above. Since D is a ring we may speak about D-modules (always assumed to be left modules). If V is a D-module we define its set of P-constants $V^D = \{x \in V;\ px = 0$ for all $p \in P\}$; it is a vector space over the field $K^D = \{\lambda \in K;\ px = 0$ for all $p \in P\}$. We sometimes write also V^P and K^P instead of V^D and K^D and speak about P-modules instead of D-modules. Recall from [NW] that if V,W are D- modules then $V \oplus W$, $V \otimes_K W$ and $Hom_K(V,W)$, hence in particular $V^o = Hom_K(V,K)$, have natural structures of D - modules and the following formulae hold

$$p(x,y) = (px,py) \quad \text{for } p \in P,\ (x,y) \in V \oplus W$$

$$p(x \otimes y) = px \otimes y + x \otimes py \quad \text{for } p \in P,\ x \otimes y \in V \otimes_K W$$

$$(pf)(x) = p(f(x)) - f(px) \quad \text{for } p \in P,\ x \in V,\ f \in Hom_K(V,W)$$

By a D - algebra (respectively associative D - algebra, Lie D - algebra, Hopf D - algebra) we understand a D - module A which is also a K - algebra (respectively an associative K - algebra, Lie K - algebra, Hopf K - algebra) in such a way that all structure maps are D - module maps; here by "structure maps" we understand multiplication $A \otimes_K A \to A$, unit $K \to A$, comultiplication $A \to A \otimes_K A$ and co-unit $A \to K$($A \otimes_K A$ and K are viewed with their natural D - module structure). A D - algebra is called D - finitely generated if it is generated as a K - algebra by some finitely generated D - submodule of it.

Note that if P is as in Example 1 (respectively 2, 3) a D - algebra is simply a K - algebra A together with a lifting of the derivation $d \in Der_k K$ to a derivation $\tilde{d} \in Der_k A$ (respectively with liftings of d_i to $\tilde{d}_i \in Der_k A$ in case of Example 2 and with pairwise commuting liftings $\tilde{d}_i \in Der_k A$ in case of Example 3).

A D - algebra which is a field will be called a D - field. Clearly K is a D - field. K will be called D - algebraically closed if for any D - finitely generated D - algebra A there exists a D - algebra map $A \to K$.

Of course one would like to "see" an example of D - algebraically closed field. Unfortunately we can't "show" any (although one can prove the existence of enough of them, see (0.12) below). But one should not forget that with a few exceptions a similar situation occurs with algebraically closed fields.

Note that if A_1, A_2 are D - algebras then $A_1 \otimes_K A_2$ becomes a D - algebra.

Clearly if A is a D - algebra then $Im(P \to End_k A) \subset Der_k A$. If no confusion arrises we denote the image of any $p \in P$ in $Der_k A$ by the same letter p.

(0.6) Following [B_1] we can define D - schemes: these are K - schemes X whose structure sheaf $\mathcal{O}_X$ is given a structure of sheaf of D - algebras (i.e. $\mathcal{O}_X(U)$ is a D - algebra for all open sets U and restriction maps $\mathcal{O}_X(U_1) \to \mathcal{O}_X(U_2)$ for $U_2 \subset U_1$ are D - algebra maps). For any