INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS

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PREFACE

THE object of this book is to provide in a compact form an account of the methods of integrating explicitly the commoner types of ordinary differential equation, and in particular those equations that arise from problems in geometry and applied mathematics. It takes the existence of solutions for granted; the reader who desires to look into the theoretical background of the methods here outlined will find what he seeks in my larger treatise *Ordinary Differential Equations* (Longmans, Green & Co., Ltd., 1927). With this qualification, it will be found to contain all the material needed by students in our Universities who do not specialize in differential equations, as well as by students of mathematical physics and technology.

As one of the first things a beginner has to learn is to identify the type to which a given equation belongs, the examples for solution have not been printed after the sections to which they refer, but have been collected at the end of the book. When the contents of the first chapter have been mastered, the reader may test his skill by attacking examples selected at random from Nos. 1 to 122, and similarly for the later chapters. The examples occur roughly in the order of the table of contents, so that working material is always available as reading progresses.

In conclusion, I wish to record my thanks to the General Editors for their encouragement and help during its growth and passage through the press.

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The Second Edition, through the untimely death of the author in March 1941, has not had the advantage of being revised by him. The Editors are indebted to Dr I. M. H. Etherington and Miss N. Walls for some corrections, and to Dr A. Erdélyi for undertaking a scrutiny of the book and recasting parts of Chapter VI.

A. C. A.
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CHAPTER I

EQUATIONS OF THE FIRST ORDER AND DEGREE

1. Definitions. Let x be an independent, and y a dependent variable; let $y', y'', \ldots, y^{(n)}$ represent successive derivatives of y with respect to x. Then any relation of equality which involves at least one of these derivatives is said to be an *ordinary differential equation.* The term *ordinary* distinguishes it from a *partial* differential equation, which would involve two or more independent variables, a dependent variable, and the corresponding partial derivatives. The *order* of any differential equation is the order of the highest derivative involved. Thus any relation of the form

$$F(x, y, y', y'', \ldots, y^{(n)})=0$$

is an ordinary differential equation of order n.

Differential equations, both ordinary and partial, are of frequent occurrence in mechanics and mathematical physics, but the illustrations that best serve to introduce the subject are taken from the geometry of plane curves.

The equation

$$f(x, y, C)=0, \qquad (1.1)$$

in which x and y are rectangular co-ordinates and C is a parameter or arbitrary constant, represents a family of curves, in which one curve corresponds to one value of C, another curve to another value. If, regarding C for the moment as fixed, we differentiate with respect to x, we obtain

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}y'=0. \qquad (1.2)$$

Generally speaking, (1.2) will involve C; if C is eliminated between the two equations, there will result an equation involving x, y, and y', say

$$F(x, y, y')=0, \quad . \quad . \quad . \quad (1.3)$$

that is, an ordinary differential equation *of the first order.* When such an equation is polynomial in y' (but not necessarily in x and y) the index of the highest power of y' involved is said to be the *degree* of the equation.

Geometrically, the differential equation (1.3) implies that at any chosen point of the (x, y)-plane the derivative has a certain value or values, that is to say it symbolizes a property of the gradient of any curve of the family (1.1) that passes through the point (x, y) considered.

Example 1. The equation

$$y=x^2+C$$

represents a family of equal parabolas having the y-axis as their common axis. On differentiating with respect to x we have

$$y'=2x.$$

The arbitrary constant C has disappeared, so that this is actually the differential equation of the family of parabolas. It expresses the fact that all the curves of the family have the same gradient at the points where they are cut by a line parallel to the y-axis, namely a gradient equal to twice the abscissa of the line.

Example 2. The equation

$$y=Cx^2$$

represents a family of similar parabolas having the y-axis as their common axis, and all touching the x-axis at the origin. Differentiating, we obtain

$$y'=2Cx,$$

which involves C; if this constant is eliminated we obtain the differential equation

$$y'=2y/x,$$

which expresses the property that any line $y=mx$ through the origin intersects all curves of the family in points where they have the same gradient $2m$.

2. Integration. The process of elimination by which the differential equation (1.3) was obtained from the *primitive* equation (1.1) is in general not reversible; the action of recovering the primitive, or an equivalent expression, is known as *integration*. More precisely, to integrate or solve a differential equation of the first order is to determine all the relations $f(x, y)=0$ such that the values of y and y' deduced from them in terms of x shall satisfy the differential equation identically.

When an infinite set of such integrals can be grouped in one comprehensive formula, involving an arbitrary constant, say

$$f(x, y, C)=0,$$

it is known as a *general* integral; it is in fact either the primitive or an expression equivalent to it.* Any integral that can be obtained by assigning a definite numerical value to C is a *particular* integral. But there may be integrals other than those that can be obtained by assigning particular values to C; these are *singular* integrals.

As an example, the equation

$$(y')^2-xy'+y=0$$

(which is of the first order and second degree) possesses the general integral $y=Cx-C^2$. This represents a family of straight lines, and any particular integral corresponds to a definite line in the family. But the equation is satisfied also by $y=\frac{1}{4}x^2$, which represents not a straight line but a parabola. This is a singular integral.

* General theory proves that a differential equation of the first order has one and only one distinct general integral. If two integrals exist, each of which involves an arbitrary constant, they can be transformed into one another. See § 3, *Ex.* 2.

The remainder of this chapter will be devoted to equations of the first order and first degree; that is, to equations that may be written in the alternative forms

$$P(x, y) + Q(x, y)y' = 0, \quad . \quad . \quad (2.1)$$

$$P(x, y)dx + Q(x, y)dy = 0, \quad . \quad . \quad (2.2)$$

where P and Q do not involve the derivative y'.

Since from any primitive equation of the form

$$u(x, y) = C, \quad . \quad . \quad . \quad (2.3)$$

where C is an arbitrary constant, we deduce that

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0,$$

it follows that a necessary condition for (2.3) to be an integral of (2.2) is

$$P\frac{\partial u}{\partial y} - Q\frac{\partial u}{\partial x} = 0. \quad . \quad . \quad (2.4)$$

3. Separation of Variables. Among equations of the type $Pdx + Qdy = 0$ the simplest are those in which P is a function of x alone and Q of y alone, say $P = M(x)$, $Q = N(y)$. The general integral is obtained by direct integration, thus:

$$\int M(x)dx + \int N(y)dy = C,$$

where C is the arbitrary constant of integration.

More generally, let P and Q be the products of a term involving x but not y and a term involving y but not x, so that the equation has the form

$$M(x)R(y)dx + N(y)S(x)dy = 0. \quad .$$

The variables are then said to be *separable*, for on division by $R(y)S(x)$ the equation becomes

$$\frac{M(x)}{S(x)}dx + \frac{N(y)}{R(y)}dy = 0$$

and has the general integral

$$\int \frac{M(x)}{S(x)}dx + \int \frac{N(y)}{R(y)}dy = C.$$

When, as in the above cases, the process leads to an expression that involves integral signs the result is said to be an *integration by quadratures*. This implies that the problem has been reduced from one in differential equations to an equivalent in the integral calculus. If it is found impossible to evaluate one or other of the integrals, an explicit solution of the equation is impossible, and the solution by quadratures must be regarded as the best attainable, unless an alternative line of approach can be discovered.

Example 1.

$$x(y^2 - 1)dx - y(x^2 - 1)dy = 0.$$

Separating the variables:

$$\frac{xdx}{x^2 - 1} - \frac{ydy}{y^2 - 1} = 0.$$

Integrating,

$$\log | x^2 - 1 | - \log | y^2 - 1 | = C$$

or

$$\log \left| \frac{x^2 - 1}{y^2 - 1} \right| = -\log c$$

which may be written

$$(y^2 - 1) = c(x^2 - 1).$$

(Note that replacing the arbitrary constant C by another arbitrary form, as in this case $C = -\log c$, may help to simplify the general integral.)

Example 2.

$$\frac{dx}{\sqrt{(1 - x^2)}} + \frac{dy}{\sqrt{(1 - y^2)}} = 0.$$

The variables are separate; direct integration gives

$$\text{arc sin } x + \text{arc sin } y = C$$

which is the general integral. This may be transformed into an equivalent expression by slightly rearranging the terms and taking the sine of both members, thus:

$$\sin\{\text{arc}\sin y\}=\sin\{C-\text{arc}\sin x\}$$

or, since

$$\cos\{\text{arc}\sin x\}=\pm\sqrt{(1-x^2)},$$
$$y=\pm\sqrt{(1-x^2)}\sin C-x\cos C$$

and, rationalising,

$$(y+x\cos C)^2=(1-x^2)\sin^2 C,$$

i.e.

$$x^2+y^2+2xy\cos C=\sin^2 C$$

or, if $c=\cos C$,

$$x^2+y^2+2cxy=1-c^2.$$

Example 3. A change in variables may sometimes succeed in converting an equation into another with separate variables. For instance, in

$$(x+y)dx+dy=0$$

the variables are not separable, but if y is replaced by $v-x$, the equation is transformed into

$$(v-1)dx+dv=0.$$

The variables x and v are separable, thus

$$dx+\frac{dv}{v-1}=0$$

which leads to

$$x+\log|v-1|=\log c \quad \text{or} \quad (v-1)e^x=c$$

so that the original equation has a general integral of the form

$$(x+y-1)e^x=c, \quad \text{or} \quad y=ce^{-x}-x+1.$$

The most important instance of reduction by change of variables occurs in the case of the homogeneous equation which now follows.

4. The Homogeneous Type. The equation

$$P(x, y)dx+Q(x, y)dy=0$$

is said to be of homogeneous type when P and Q are homogeneous functions of x and y *of the same degree.* If the degree is m, the substitution $y = vx$ will reduce P and Q to the forms

$$P(x, vx) = x^m R(v), \qquad Q(x, vx) = x^m S(v),$$

where R and S are independent of x. Thus the factor x^m may be cancelled out of the equation, which becomes

$$R(v)dx + S(v)\{vdx + xdv\} = 0$$

or

$$\{R(v) + vS(v)\}dx + xS(v)dv = 0.$$

Separating the variables and integrating, we have

$$\int \frac{S(v)dv}{R(v) + vS(v)} + \log x = C,$$

and when the integral in v has been evaluated, the substitution $v = y/x$ will give the general integral of the original equation.

Example 1.

$$(x^2 - y^2)dx + 2xydy = 0.$$

The two terms in this equation are homogeneous and of the second degree in x and y; the above process is therefore applicable. Making the substitution mentioned, we have

$$x^2(1 - v^2)dx + 2x^2v(vdx + xdv) = 0;$$

x^2 cancels out, leaving

$$(1 + v^2)dx + 2vxdv = 0.$$

Separating variables and integrating:

$$\int \frac{2vdv}{1 + v^2} + \int \frac{dx}{x} = C$$

or

$$\log (1 + v^2) + \log x = \log c,$$

i.e.

$$(1 + v^2)x = c$$

which leads to the general integral

$$x^2 + y^2 = cx.$$

Example 2.

$$(2ye^{y/x} - x)y' + 2x + y = 0.$$

Here each term is of the first degree in x and y, for $e^{y/x}$ is of degree zero. Writing $y=vx$, $y'=xv'+v$ and cancelling x, we obtain

$$(2ve^{v}-1)xv'+2(v^{2}e^{v}+1)=0.$$

Separating the variables, this becomes

$$\frac{2v-e^{-v}}{v^{2}+e^{-v}}dv+\frac{2dx}{x}=0.$$

Integrating, we have

$$\log(v^{2}+e^{-v})+2\log x=C,$$

whence the general integral

$$y^{2}+x^{2}e^{-y/x}=c.$$

5. The Equation with Linear Coefficients. Although the equation

$$(ax+by+c)dx+(a'x+b'y+c')dy=0 \quad . \quad (5.1)$$

is not of homogeneous type, it may be reduced to that type by a simple substitution. The equations

$$ax+by+c=0, \qquad a'x+b'y+c'=0 \quad . \quad (5.2)$$

represent a pair of straight lines which will intersect unless the condition for parallelism, *i.e.* $a'/a=b'/b$ or $ab'-a'b=0$, is satisfied. Let (h, k) be the point of intersection; transfer the origin to that point by the substitution

$$x=h+X, \qquad y=k+Y$$

and the equation will become

$$(aX+bY)dX+(a'X+b'Y)dY=0.$$

It is now homogeneous; the substitution $Y=vX$ followed by separation of variables leads to the general integral

$$\log CX+\int\frac{(a'+b'v)dv}{a+(a'+b)v+b'v^{2}}=0,$$

whose ultimate form * depends upon whether the roots

* For which see § 17 *infra*.

of the denominator of the integrand are real, coincident or imaginary, *i.e.* according as $(a'+b)^2$ is greater than, equal to, or less than $4ab'$.

In the exceptional case when the lines (5.2) are parallel, that is when $a'/a = b'/b = k$ (say), the equation can be written

$$(ax+by+c)dx + \{k(ax+by)+c'\}dy = 0.$$

Take $z = ax+by$ as a new variable to replace y, then

$$b(z+c)dx + (kz+c')(dz-adx) = 0.$$

Separating the variables and integrating, we have

$$x + \int \frac{(kz+c')dz}{(b-ak)z+bc-ac'} = \text{const.}$$

The above are particular cases of an equation of the type

$$y' = F\left(\frac{ax+by+c}{a'x+b'y+c'}\right)$$

which may be reduced to a form integrable by quadratures by the same routine process.

Example 1.

$$y' = \frac{4x-y+7}{2x+y-1}.$$

The lines $4x-y+7=0$, $2x+y-1=0$ meet in $(-1, 3)$; writing $x = X-1$, $y = Y+3$ we have

$$(2X+Y)dY = (4X-Y)dX.$$

The transformation $Y = vX$ reduces this equation to

$$(2+v)Xdv + (v^2+3v-4)dX = 0$$

which becomes, on separating the variables and taking partial fractions,

$$\left\{\frac{3}{v-1} + \frac{2}{v+4}\right\}dv + \frac{5dX}{X} = 0.$$

Integrating.

$$3\log|v-1| + 2\log|v+4| + 5\log|X| = C,$$

i.e.

$$(v-1)^3(v+4)^2X^5=c$$

or

$$(Y-X)^3(Y+4X)^2=c.$$

Reverting to the variables x, y we have the general integral

$$(y-x-4)^3(y+4x+1)^2=c.$$

Example 2.

$$(2x-4y+5)y'+x-2y+3=0.$$

If $z=x-2y$, $2y'=1-z'$; with this transformation the equation becomes

$$(2z+5)z'=4z+11.$$

Separating the variables, we obtain

$$\left(1-\frac{1}{4z+11}\right)dz=2dx$$

whence

$$4z-\log|4z+11|=8x-C,$$

giving the general integral

$$4x+8y+\log|4x-8y+11|=C.$$

6. Exact Equations. When the primitive of a differential equation involves the arbitrary constant C explicitly, as:

$$u(x, y)=C, \qquad . \quad . \quad . \quad (6.1)$$

the operation of taking the differential eliminates C automatically, thus:

$$du(x, y)=0 \qquad . \quad . \quad . \quad (6.2)$$

or

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=0. \qquad . \quad . \quad . \quad (6.3)$$

Conversely, if a differential equation of the form

$$P(x, y)dx+Q(x, y)dy=0 \qquad . \quad . \quad (6.4)$$

has originated in such a process, and if no variable factor has been cancelled out, it must be equivalent to one of the form (6.3) and thus to (6.2), and therefore it must possess a general integral of the form (6.1). Such an equation is said to be *exact*.

Thus, in order that (6.4) may be exact, there must exist a function $u(x, y)$ such that

$$P(x, y)=\frac{\partial u}{\partial x}, \qquad Q(x, y)=\frac{\partial u}{\partial y} \quad . \quad . \quad (6.5)$$

and therefore

$$\frac{\partial P}{\partial y}=\frac{\partial^2 u}{\partial x \partial y}=\frac{\partial Q}{\partial x}.$$

Hence the theorem: *for the differential equation* (6.4) *to be exact it is necessary that* $P(x, y)$ *and* $Q(x, y)$ *be linked by the relation*

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}. \quad . \quad . \quad . \quad (6.6)$$

This is known as the *condition of integrability*,* for when it is satisfied the primitive $u(x, y)=C$ can be recovered by the following process. Starting from the relation

$$\frac{\partial u(x, y)}{\partial x}=P(x, y),$$

integrating and remembering that since the differentiation with respect to x was partial, the inverse process of integration will introduce, as the arbitrary element, a function of y, we have,

$$\begin{aligned} u(x, y) &= \int P(x, y)dx+\phi(y) \\ &= S(x, y)+\phi(y) \qquad \text{(say).} \end{aligned}$$

But, on the other hand,

$$Q(x, y)=\frac{\partial u}{\partial y}=\frac{\partial S}{\partial y}+\phi'(y),$$

an equation which will give $\phi'(y)$ since Q and S are both known; the final integration to obtain $\phi(y)$ will introduce the arbitrary constant C of the general integral.

* Immediate integrability is implied; when the condition is not satisfied, the equation is still integrable, though not without some preliminary manipulation.

Example 1.

$$\frac{(1+y^2)y\,dx+(1+x^2)x\,dy}{(1+x^2+y^2)^{3/2}}=0.$$

Since, in the above notation,

$$\frac{\partial P}{\partial y}=\frac{\partial}{\partial y}\left\{\frac{y+y^3}{(1+x^2+y^2)^{3/2}}\right\}=\frac{1+x^2+y^2+3x^2y^2}{(1+x^2+y^2)^{5/2}}$$

$$=\frac{\partial}{\partial x}\left\{\frac{x+x^3}{(1+x^2+y^2)^{3/2}}\right\}=\frac{\partial Q}{\partial x}$$

the equation is exact. Hence we are entitled to write

$$u(x,\ y)=\int\frac{(1+y^2)y\,dx}{(1+x^2+y^2)^{3/2}}+\phi(y)$$

$$=\int\frac{y\,dx}{(1+x^2+y^2)^{1/2}}-\int\frac{x^2y\,dx}{(1+x^2+y^2)^{3/2}}+\phi(y)$$

$$=\int\frac{y\,dx}{(1+x^2+y^2)^{1/2}}+\int xy\frac{\partial}{\partial x}\left\{\frac{1}{(1+x^2+y^2)^{1/2}}\right\}dx+\phi(y)$$

$$=\frac{xy}{\sqrt{(1+x^2+y^2)}}+\phi(y),$$

on integrating the second integral by parts. The equation

$$\frac{\partial u}{\partial y}=\frac{x+x^3}{(1+x^2+y^2)^{3/2}}+\phi'(y)$$

shows that $\phi'(y)$ is zero, or $\phi(y)$ is a constant. The general integral is therefore

$$\frac{xy}{\sqrt{(1+x^2+y^2)}}=C.$$

Note that although the given equation is exact as it stands, it would cease to be exact if the denominator of the left-hand member were removed.

Example 2.

$$\log\,(y^2+1)dx+\frac{2y(x-1)}{y^2+1}dy=0.$$

The condition for integrability is satisfied, therefore we write

$$u(x,\ y)=\int\log\,(y^2+1)dx+\phi(y)$$
$$=x\log\,(y^2+1)+\phi(y).$$