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Preface

These papers are the proceedings of the Twelfth George H. Hudson Symposium: Advances in Non-Commutative Ring Theory which was held by the Department of Mathematics of the State University College of Arts and Science at Plattsburgh, New York, and which took place on April 23-25, 1981.

The conference consisted of talks by five invited speakers and thirteen other speakers who contributed papers, and in this volume we have collected papers by two of the invited speakers and seven of the contributors. While not all of the papers given at the Symposium appear in this volume, some of the contributors have taken the opportunity to elaborate on their contributions.

At this time, the organizers of the Symposium would like to express their thanks to the following:

The National Science Foundation and, especially, Dr. Alvin Thaler for support under NSF Grant MC580-1655.

Dean Charles O. Warren and Mr. Robert G. Moll of the Dean's Office for expert administrative support.

The Mathematics Department at PSUC and its chairman, Dr. Robert Hofer, for moral support and a great deal of hard work.

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Finally, to Dr. Paul Roman, Dean of Graduate Studies and Research at PSUC who supplied excellent advice, unstinting support, vast amounts of time, and a great deal of encouragement, we can only give a very inadequate "Thank you."

P. Fleury

Plattsburgh, N.Y.

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TORSION MODULES AND THE FACTORIZATION OF MATRICES

P. M. Cohn
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1. For firs (and even semifirs) there is a fairly complete factorization theory for elements and more generally for square matrices. In terms of modules this leads to the category of torsion modules, and two questions arise naturally at this point:

1. Do these or similar results hold for more general rings?

2. What can be said about the factorization of rectangular matrices?

Below is a progress report. It turns out that torsion modules can be defined over very general rings (weakly finite rings), but as soon as we ask for more precise information we are hampered by the lack of a good factorization theory, which so far is missing even for the semifirs' nearest neighbours, the Sylvester domains. The basic results on the factorization of rectangular matrices are stated here, but some shortcomings will be pointed out, which will need to be overcome in a definitive treatment.

2. If R is a principal ideal domain, it is well known that any submodule of R^n has the form R^m with $m \leqslant n$. So any finitely generated R-module M has a resolution

(1) $$0 \to R^m \to R^n \to M \to 0,$$

and $n - m$ is an invariant, the <u>characteristic</u> of M, written $\chi(M)$. By what has been said, $\chi(M) \geqslant 0$ always; the modules M with $\chi(M) = 0$ are just the <u>torsion</u> <u>modules</u>.

An obvious generalization is to take rings in which each submodule of a free module is free, of unique rank. These are just the firs (= free ideal rings, cf. [2], Ch. 1), e.g. the free algebra k<X> on a set X over a field k. But there is an important difference, in that we can now have $\chi(M) < 0$; e.g. when $R = k\langle X\rangle$,

$M = R/(Rx + Ry)$ has $\chi(M) = -1$. To find an analog to the PID case we need the notion of a positive module. This is a module M such that $\chi(M') \geqslant 0$ for all submodules M' of M. If M is positive and $\chi(M) = 0$, we call M a torsion module. As the presentation (1) shows, M is then defined by a square matrix A, and the positivity of M means that A is full, i.e. we cannot write A = PQ, where P has fewer columns than A.

For completeness we define a negative module as a module M such that $\chi(M'') \leqslant 0$ for all quotients M" of M. It is not hard to see that there is a duality (= anti-equivalence) between the category of all negative left R-modules and the category of all positive right R-modules such that $\mathrm{Hom}_R(M,R) = 0$ (the bound modules), for any fir, or more generally, any semifir (cf. [3]). A module is said to be prime if M is either positive and $\chi(M') > 0$ for any non-zero submodule M', or negative and $\chi(M'') < 0$ for any non-zero quotient M". As an example of a prime module of characteristic 1 we can take the semifir R itself. Now we have

Proposition 1 (cf. [4]). *If R is a semifir and M, N are prime R-modules of characteristic 1, then any non-zero homomorphism f: M → N is injective.*

Proof. We have the exact sequence

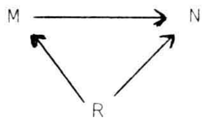
$$0 \to \ker f \to M \xrightarrow{f} N \to \operatorname{coker} f \to 0.$$

If $\ker f \neq 0$, then $\chi(\ker f) > 0$, so $\chi(\operatorname{coker} f) > 0$, $\chi(\operatorname{im} f) = 1 - \chi(\operatorname{coker} f) \leqslant 0$, hence $\operatorname{im} f = 0$.

From the Proposition we easily obtain the

Corollary. *If M is a prime module of characteristic 1 over a semifir, then $\mathrm{End}_R(M)$ is an integral domain.*

Let me outline, following G.M. Bergman [1], how Prop. 1 can be used to show the existence of a field of fractions for a fir. Consider all the prime left R-modules of characteristic 1 extending R. They form a category which is a partial ordering: two homomorphisms M → N agreeing on R must be equal, by Prop. 1. The category



is directed since we can form pushouts (it is at this point that one needs firs rather than semifirs). Let L be the direct limit, then $End_R(L)$ contains R (via right multiplications), and it is a skew field, because the set of all endomorphisms is transitive on non-zero points.

3. We now examine what assumptions on the ring are really needed in the preceding development. To begin with, let R be any ring, ${}_R P$ the class of all finitely generated projective left R-modules and $K_0(R)$ the projective module group, with generators [P], for $P \in {}_R P$, and defining relations $[P \oplus Q] = [P] + [Q]$. As is well known (and easily seen), each element of $K_0(R)$ has the form $[P] - [Q]$ and $[P] - [Q] = [P'] - [Q']$ if and only if

(2) $$P \oplus Q' \oplus T \cong P' \oplus Q \oplus T \text{ for some } T \in {}_R P.$$

Here we may of course replace T by R^n.

We define a partial preorder, the <u>natural preorder</u> on $K_0(R)$ by putting

(3) $$[P] - [Q] > 0 \text{ whenever } [P] = [Q] + [S] \text{ for some } S \in {}_R P.$$

Our first concern is to know when this is a partial order:

Proposition 2. <u>The natural preorder on</u> $K_0(R)$ <u>is a partial order if and only if</u>

(4) $$S \oplus T \oplus R^n \cong R^n \Rightarrow S \oplus R^m \cong R^m.$$

For we have a partial order if and only if $[P] \geq [Q] \geq [P]$ implies $[P] = [Q]$, i.e. $[S] \leq 0 \Rightarrow [S] = 0$, and this is just (4).

We recall that a ring R is said to be <u>weakly finite</u> if for any square matrices of the same size, $AB = I \Rightarrow BA = I$, or equivalently, $P \oplus R^n = R^n \Rightarrow P = 0$ (other names: R_n for all n, is v. Neumann finite, directly finite, inverse symmetric). It is clear that in a weakly finite ring (4) holds, so we have

Theorem 1. In any weakly finite ring R the natural preorder on $K_0(R)$ is a partial order and $[P] = 0 \Rightarrow P = 0$.

To define torsion modules we have to limit ourselves to modules with a finite resolution. Let us call a module M finitely resolvable if it has a finite resolution by finitely generated projective R-modules:

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0, \qquad (P_i \in {}_R P). \tag{5}$$

Write ${}_R\overline{P}$ for the class of all such M. Given two finite resolutions of M, say (5) and

$$0 \to Q_n \to \cdots \to Q_1 \to Q_0 \to M \to 0, \tag{6}$$

(without loss of generality both are of the same length, we have by the extended Schanuel-lemma (cf. [6], p. 137)

$$P_0 \oplus Q_1 \oplus P_2 \oplus \cdots \cong Q_0 \oplus P_1 \oplus Q_2 \oplus \cdots$$

Hence the alternating sums for the sequences (5) and (6) define the same element of $K_0(R)$ and we can define the characteristic of M by the formula

$$\chi(M) = \Sigma(-1)^i[P_i]. \tag{7}$$

Starting from any resolution (5) of M, we can modify $P_1, \cdots, P_{n-1}$ so that they become free of finite rank. If in this case the last module P_n is also free, M is said to have a finite free resolution. Clearly when this is so, we have $\chi(M) = n[R]$ for some $n \in Z$ (this holds more generally whenever the last term P_n in the above resolution is stably free).

It is easily seen (and well known) that $\chi(M)$ is additive on short exact sequences: Given a short exact sequence

$$0 \to M' \to M \to M'' \to 0,$$

if two of M, M', M'' are in $_R\overline{P}$, then so is the third, and we have $\chi(M) = \chi(M') + \chi(M'')$.

We can now define for any $M \in {}_R\overline{P}$:

1. M is *positive* if $\chi(M') \geqslant 0$ for all submodules M' of M in $_R\overline{P}$.
2. M is *negative* if $\chi(M'') \leqslant 0$ for all quotients M'' of M in $_R\overline{P}$.
3. M is a *torsion module* if it is both positive and negative.
4. M is *prime* if either M is positive and $\chi(M') > 0$ for non zero submodules M' or M is negative and $\chi(M'') < 0$ for non-zero quotients M''.

Now it is clear that Prop. 1 holds for any projective free ring (i.e. a ring over which every finitely generated projective module is free, of unique rank). More generally, a similar result will hold for any ring with a minimal positive projective module.

As in the case of semifirs (cf. [2], Th. 5.3.3, p. 185) one now has

Theorem 2. *For any weakly finite ring* R *the torsion modules form an abelian category* T *which is a full subcategory of R-Mod*.

The proof follows closely the semifir case, using the natural ordering in $K_0(R)$, and the following criterion (cf. [2], Prop. A.3, p. 321. I am obliged to C.M. Ringel for drawing my attention to an omission in the enunciation, which is rectified below).

Let A *be an abelian category and* B *a full subcategory; then* B *is abelian if and only if it has finite direct sums and the kernel and cokernel (taken in* A) *of any map in* B *lie again in* B.

4. Over a commutative Noetherian ring every torsion module is annihilated by a non-zero divisor (Auslander-Buchsbaum theorem, cf. [6], p. 140). This is certainly no longer true in general, e.g. R/xR, where $R = k\langle x,y\rangle$, is a torsion module whose annihilator is 0, but it may well extend to non-commutative Noetherian

domains.

When we come to look at general (weakly finite) rings, one difficulty is the paucity of prime modules. We saw that for a semifir R, R itself is prime. Below we examine another, wider, class of rings for which this is true.

We recall that for any matrix A (over any ring) the <u>inner rank</u> of A, rk A, is defined as the least r such that A = PQ, where P has r columns. Now Dicks and Sontag [5] have defined a <u>Sylvester domain</u> as a ring R such that

(8) $$A \; m \times r,\; B \; r \times n,\; AB = 0 \Rightarrow \mathrm{rk}A + \mathrm{rk}B \leqslant r.$$

The reason for the name is that (8) is a special case of Sylvester's law of nullity:

(9) $$\mathrm{rk}\, A + \mathrm{rk}\, B \leqslant r + \mathrm{rk}\, AB,$$

for A m × r, B r × n. Conversely, we can deduce Sylvester's law (9) from (8). For if AB in (8) has inner rank s, say AB = CD, where D has s rows, then $(A,C)\begin{pmatrix}-B\\ D\end{pmatrix} = 0$, hence rk A + rk B $\leqslant$ r + s, i.e. (9). Any Sylvester domain has a universal field of fractions inverting all full matrices, in fact this property can be used to characterize them; thus Sylvester domains include semifirs. Further, any Sylvester domain is projective free, of weak global dimension at most 2. For an Ore domain the converse holds: any projective free Ore domain of weak global dimension at most 2 is a Sylvester domain. E.g. k[x,y] is a Sylvester domain, but not k[x,y,z].

Proposition 3. <u>For any coherent Sylvester domain</u> R, R <u>is a prime module</u>.

Proof. We must show that for every finitely presented non-zero left ideal a of R, X(a) > 0.

Let a be generated by $c_1,\cdots,c_n$ and take a resolution

(10) $$0 \rightarrow F \xrightarrow{\alpha} R^n \rightarrow a \rightarrow 0.$$

We note that w.dim(R/a) $\leqslant$ 2, hence w.dim(a) $\leqslant$ 1, so the first term F in (10) is flat; by coherence it is finitely generated, hence finitely presented, therefore projective, and so free (because R is projective free). If α has a matrix $A = (a_{ij})$, then $Ac = 0$, where $c = (c_1, \cdots, c_n)^T = 0$, hence rk A + rk c $\leqslant$ n, but rk c $\geqslant$ 1, so rk A $\leqslant$ n - 1. Thus A = PQ, where P is m × p, Q is p × n and p $\leqslant$ n - 1. It follows that PQc = 0 and rk P = p = rk Q, and rk P + rk Qc $\leqslant$ p, hence Qc = 0. Moreover, Qx = 0 implies Ax = 0, hence we have a presentation of a by Q instead of A and X(a) = n - p $\geqslant$ 1.

5. It looks at first sight as if much of the theory of semifirs carries over to Sylvester domains, but we run into difficulties as soon as we consider the factorization (of elements or matrices) over Sylvester domains. To make a beginning let us see how the factorization theory of semifirs treated in Ch. 5 of [2] extends to rectangular matrices. In Ch. 5 of [2] there is a factorization theory for square matrices over semifirs, but nothing beyond a few remarks (on p. 202f.) about rectangular matrices.

Let R be any ring, then any matrix $A \in {}^mR^n$ defines a module M:

$$R^m \rightarrow R^n \rightarrow M \rightarrow 0, \tag{11}$$

where the map α has matrix A, and A is determined by M if and only if xA = 0 => x = 0, i.e. A is a right non-zerodivisor. We remark that A is a left non-zerodivisor if and only if $M^* = \mathrm{Hom}_R(M,R) = 0$, i.e. M is a bound module. We also note that M = 0 if and only if A has a left inverse.

When R is a semifir, every finitely presented module M is defined by a right non-zerodivisor matrix A, i.e. α in (11) is then injective. In that case X(M) = n - m; we shall also call n - m the characteristic of the matrix A: char A = n - m.

Two matrices A, A' define isomorphic modules if and only if they are <u>stably associated</u>, i.e. $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} = U \begin{pmatrix} A' & 0 \\ 0 & 1 \end{pmatrix} V$ for invertible matrices U, V (where the unit matrices need not be of the same size.) Conversely, every matrix A which is a right non-zerodivisor defines a left module M, and a matrix product C = AB corresponds to a short exact sequence

$$0 \to M' \to M \to M'' \to 0,$$

where A, B, C define M', M", M respectively. More generally, if we consider all factorizations of a full matrix C, there is a correspondence between the left and right factors, the factorial duality (cf. [2], p. 119), which means for example, that an integral domain which satisfies the ascending chain condition on principal right ideals (right ACC_1 for short), also satisfies the descending chain condition on principal left ideals containing a given non-zero left ideal. We recall that a fir satisfies right ACC_n, i.e. ACC on n-generator right ideals, for any n ([2], p. 49). by an atom in a ring we mean a non-unit which cannot be written as a product of two non-units. Now the factorization theorem for firs may be stated as follows: Theorem (cf. [2], p. 201). In an n x n matrix ring over a fir every full matrix can be written as a product of atoms, and given two factorizations into atoms:

$$C = A_1 \cdots A_r = B_1 \cdots B_s, \tag{12}$$

we have r = s and there is a permutation $i \to i'$ such that A_i is stably associated to $B_{i'}$.

Here all the matrices are n x n over the ground ring, for some fixed n. We are interested in the generalization to the case where the A's and B's are not necessarily square and even C need not be square. For this purpose we have to examine more closely the steps by which one passes from one factorization of C in (12) to another.

We recall that a relation between matrices

$$AB' = BA' \tag{13}$$

is called comaximal if (A,B) has a right inverse and $\begin{pmatrix} B' \\ A' \end{pmatrix}$ a left inverse. Let A be $r \times m$, B $r \times n$, A' $n \times s$ and B' $m \times s$, then by the law of nullity in semifirs,

(14) $$r + s \leq m + n.$$

If equality holds in (14), so that char A = char A' = m - r, char B = char B' = n - r, we call (13) a proper comaximal relation. Thus for any comaximal relation C = AB' = BA' over a semifir we have

$$\text{char } C \leq \text{char } A + \text{char } B,$$

with equality if and only if the relation is proper. Now one has

Lemma 1 ([4], Prop. 2.2). Two matrices A, A' over a weakly finite ring R are stably associated if and only if there is a proper comaximal relation (13) for A, A'.

If in some factorization a product AB' is replaced by BA', where AB' = BA' is a (proper) comaximal relation, we shall call the change a (proper) comaximal transposition. This extends the usage in [2], p. 134.

Now we have

Theorem 3 (Refinement theorem). Let R be a semifir and $C \in {}^{m}R^{n}$, then any two factorizations of C have refinements which can be obtained from each other by comaximal transpositions.

The proof, which is quite straightforward, is analogous to the corresponding refinement theorem for factorizations into square matrices. However, this theorem does not seem to be in the best possible form in that we cannot always choose the comaximal transpositions to be proper. This happens (roughly speaking) when C is too narrow in shape, i.e. of large positive or negative characteristic. If we translate this into module language we find that comaximal relations correspond to sums and intersections, but when the relation is improper, the intersection contains a free summand.

In order to state a factorization theorem we need to find an analog of atoms for rectangular matrices. Let us call a matrix factorization $C = AB$ proper if A has no right inverse and B no left inverse. If C is neither a unit nor a zero-divisor and has no proper factorizations, then we call it unfactorable.

It is easily seen that a matrix C has a proper factorization if and only if the module M defined by it has a proper non-zero bound submodule. This leads to the following description of the modules defined by unfactorable matrices:

Proposition 4. Let R be a semifir, then a finitely presented R-module M has an unfactorable matrix if and only if every proper finitely generated submodule of M is free.

Proof. Suppose that M has a proper bound submodule $M' \neq 0$, then M' is clearly not free. Conversely, if M' is a non-free proper submodule of M, either M' is bound and we have a proper factorization, or $M'^* \neq 0$, so there is a non-zero homomorphism $F: M' \to R$. Its image (finitely generated as image of M') is free, as submodule of R, and hence splits off M': $M' = F \oplus M'_1$. By induction on the number of generators M'_1 has a bound non-zero submodule and the result follows.

Sometimes a module M is called almost free if M is not free, but every proper finitely generated submodule is free. Thus unfactorable matrices correspond to almost free modules. However, we shall not pursue the module aspect here further.

To prove the factorization theorem we isolate the essential step in the follow g basic lemma:

Lemma 2. Let R be a semifir and C any matrix over R. Given

$$C = AB' = BA',$$

where A is unfactorable and BA' is a proper factorization, either there exists a matrix U such that $B = AU$, $B' = UA'$, or there is a comaximal relation $AB_1 = BA_1$ such that $A' = A_1Q$, $B' = B_1Q$, for some matrix Q.

The proof is quite similar to the corresponding result for elements

([2], p. 124f.). With the help of this lemma we obtain

Theorem 4 (Factorization theorem). Let R be a fir, then every matrix C over R which is a non-zerodivisor has a proper factorization into unfactorables, and given any two such factorizations of C, we can pass from one to the other by a series of comaximal transpositions.

The existence of factorizations was proved in [2], Th. 5.6.5, p. 202, and the uniqueness follows by repeated application of Lemma 2.

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