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Chain conditions in topology

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Introduction

Our monograph passed through several stages before acquiring its present form. It began, several years ago, as material gathered for our earlier work on ultrafilters but eventually discarded as too peripheral to the principal subject. Lying inert for some time, and slowly gaining some unity, it appeared later in our minds as a systematization of existing applications of the Erdős–Rado principle on quasi-disjoint sets to topological situations (mainly, in product spaces). Finally, though, through the contributions of researchers such as Gaifman, Laver, Galvin, Hajnal, Kunen, Argyros, Tsarpalias, and Shelah, it became something more fascinating and delightful: a study of the fine structure of the (countable) chain condition; and, a study of topological spaces (and also of partially ordered sets, and of Boolean algebras, and even of Banach spaces) as a function of their Souslin number.

The tools for the most part are the classical, yet constantly developing and inexhaustibly fertile, principles of infinitary combinatorics (given in Chapter 1). Early in the development of topology, especially in the Moscow School of Alexandroff and Urysohn, in the work of Lusin and Souslin, in the Polish School, and in the work of Hausdorff, informal set-theoretic and infinitary combinatorial considerations were prominent. The subsequent systematic development of infinitary combinatorics by the Hungarian School, led by Erdős, based on Dedekind's box (pigeonhole) principle and inspired by Ramsey's theorem, provided concrete techniques through which topological questions could be examined. Combinatorial tools returned to a central position in the work of Shanin, who studied fundamental questions on the intersection properties of families of open sets (the chain conditions, defined in detail in Chapter 2) in product spaces, using quasi-disjoint sets. More recently in the same spirit Arhangel'skiĭ, Hajnal, Juhász, Šapirovskiĭ and others have produced significant results on cardinal invariants associated with topological spaces.

A simple but quite useful extension to singular cardinals of the Erdős–Rado theorem on quasi-disjoint sets, noted independently by Shelah and Argyros, allows for positive statements concerning the conservation of chain conditions in cartesian products or powers (in fact, in various stronger box topologies) in Shanin's spirit. However, methods involving quasi-disjoint sets, for both regular and singular cardinals, have their limitations; their usefulness lies with spaces that are products, or have a product-like structure. This is the case with the results of Chapter 3, where we study some classes of chain conditions (calibres, compact-calibres, and pseudo-compactness numbers); with Shelah's result in Chapter 4, which systematically exploits calibres of Σ-products of dyadic powers to define (non-compact) spaces whose calibre gaps are created more or less at will; and with the results in Chapter 10, where the pseudo-compactness properties given in Chapter 3 are applied to determine the dependence of continuous functions defined on 'large' subsets of products (with the cartesian or various stronger box topologies) on a 'small' set of coordinates. Furthermore some of the results in Chapters 6 and 7, concerning which we say more below, where a limited use of quasi-disjoint sets can be observed, also concern the dyadic powers $\{0,1\}^I$ with topologies quite different from the cartesian product topology.

Results to the effect that a class of cardinals satisfying certain obvious restrictions is realized as the set of non-compact-calibres of a space, analogous to the results of Shelah for calibres, are obtained in Chapter 8; here we use spaces of non-uniform ultrafilters rather than Σ-products of dyadic powers. The corresponding statements for pseudo-compactness numbers are not yet available and indeed it is not clear in this case whether there are conditions analogous to the 'obvious restrictions' dealing with calibre and compact-calibre. The difficulty, as described in Chapter 9 using (permutation) types of ultrafilters, derives from the fact that properties of pseudo-compactness type are not finitely productive.

In Chapter 5 we study (arbitrary) topological spaces as a function of their Souslin number, enlarging greatly Shanin's original program.

Combinatorial concepts more powerful than quasi-disjoint sets are needed to deal with general spaces, where there is no explicit or implicit product structure. Such principles have been formed by Argyros and Tsarpalias, and used to determine a large class of regular and singular calibres of compact spaces. In fact, as is mentioned below, assuming the generalized continuum hypothesis, most calibres of compact spaces are determined by these methods (see the second chart in section 7.18). It is worth noting that the proof of the regular cardinal case uses a combinatorial kernel sufficiently strong that it yields in Chapter 1 several related and classical results (quasi-disjoint families and the early 'arrow relations' of Erdős and Rado); we indicate in Chapter 1 that these results can be proved also by use of a simple form of the pressing-down lemma.

These techniques were in fact inspired in part by questions in functional analysis not directly concerned with chain conditions and described only informally in this monograph. These questions concern the existence of large independent families, and the resulting isomorphic embedding of the Banach space l_α^1 into α-dimensional subspaces of the space $C(X)$ with X a compact, Hausdorff space whose Souslin number is 'small' relative to α.

We noted above that Shelah's result in Chapter 4 allows the creation of completely regular, Hausdorff spaces whose classes of calibres are assigned in advance. In contrast, the class of non-calibres of a compact space is more restricted. In fact, assuming the generalized continuum hypothesis, a regular cardinal α either is not a calibre of a compact space for the trivial reason that α is smaller than the Souslin number of the space, or α is indeed a calibre of the space (with some possible 'boundary' exceptions of the form $\alpha = \beta^+$ with the cofinality of β smaller than the Souslin number of the space, in which case the space still has calibre (α, β)).

The deeper results describing the fine structure of the countable chain condition, especially the examples of Chapters 6 and 7, rely on infinitary combinatorial techniques that in addition contain dialectical (mostly diagonal) arguments. The success of a large part of this undertaking depends on the continuum hypothesis: although some remarkable fragments hold without any special hypotheses (essentially, the statements concerning the existence of spaces with no strictly positive measure), other parts definitely collapse. These examples lie, in the countable case, between the very strong property of separability and the very general countable chain condition (here abbreviated c.c.c.). This spectrum of chain conditions – including calibre ω^+, the existence of a strictly positive measure and the related properties (*) and (**), Knaster's property (K) and the related properties K_n for natural numbers $n \geq 2$, and the productive countable chain condition – grew up around the problem posed in 1920 by Souslin. The examples given in Chapter 7 by Laver, Hajnal, Galvin and Kunen serve to differentiate some of these properties. Thus, assuming the continuum hypothesis, there is a c.c.c. space whose square is not a c.c.c. space, and there is a productively c.c.c. space that does not have Knaster's property. The celebrated example in Chapter 6 of Gaifman, a c.c.c. space with no strictly positive measure (and, assuming CH, with no calibre), was for many years the only known example of its kind. Then came the Galvin–Hajnal example of a compact space, obtained without any special set-theoretic assumptions (and solving as well a problem of Horn and Tarski), with no strictly positive measure and for which every regular uncountable cardinal is a calibre. Recently for every natural number $n \geq 2$ Argyros found a space with no strictly positive measure, with property K_n and, assuming CH, without property K_{n+1}; motivated by a problem in model theory, Rubin and Shelah found, assuming CH, another example of a space with K_n and without K_{n+1}. Argyros found also, given an infinite cardinal α, a c.c.c. space X such that for every set $\{\mu_i : i < \alpha\}$ of regular, Borel measures on X there is a non-empty, open subset U of X with $\mu_i(U) = 0$ for all $i < \alpha$; subsequently Galvin observed that an appropriate modification of the Galvin–Hajnal example mentioned above produces the same phenomenon. In the opposite direction it is shown, usually assuming appropriate segments of the generalized continuum hypothesis, that the Stone spaces of the homogeneous algebras are – indeed, they are the only examples known so far – compact spaces with strictly positive measures but without various (arbitrarily large) calibres. This class of examples, due to Erdős for the separable homogeneous measure algebra, together with a further example of Argyros (given in Chapter 5) related to his examples on strictly positive measure, serves as well to delineate the class of allowed calibres of compact spaces.

In the Notes for Chapter 7, we describe the study of the chain conditions on certain classes of spaces (for example, the Eberlein-compact and the Corson-compact spaces) arising in the theory of Banach spaces. As M. Wage has remarked, 'the fields of analysis, general topology and set theory have another happy reunion in the study of weakly compact subsets of Banch spaces'.

After all is said, not everything is settled and complete. We are at a point where fascinating problems still abound while beyond, far-reaching connections can only be imagined. Our work we hope will find its rest in changing: *μεταβάλλον ἀναπαύεται.*

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Middletown, Connecticut, USA and Athens, Greece
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Contents

1

Some Infinitary Combinatorics

This introductory chapter describes the basic combinatorial tools used in the proofs of most of the results contained in the present monograph.

Our treatment of this classical material is based on two (alternative) principles for regular cardinals: Argyros' ramification lemma (1.1) and the pressing-down lemma (1.3). The main combinatorial results established for regular cardinals – the Erdős–Rado theorem for quasi-disjoint families (1.4) and the Erdős–Rado arrow relations (1.5, 1.7) – are consequences of each of these principles. We need also a simple extension to singular cardinals of the Erdős–Rado theorem for quasi-disjoint families.

We note that sometimes, especially in Chapter 5, we obtain information on calibres of spaces directly from Argyros' ramification lemma (and a technique for singular cardinals due to Tsarpalias), rather than from some derived combinatorial result.

1.1 *Lemma.* (Argyros' ramification lemma.) Let $\omega \leq \beta \leq \alpha$ with β and α regular, let κ be a cardinal such that $0 < \kappa \ll \beta$, and for every $A \subset \alpha$ with $|A| = \alpha$ let $\mathscr{P}_A$ be a partition of A such that $|\mathscr{P}_A| < \beta$. Then there is a family $\{A_\eta : \eta < \kappa\}$ of subsets of α such that

$$|A_\eta| = \alpha \text{ for } \eta < \kappa,$$
$$A_{\eta+1} \in \mathscr{P}_{A_\eta} \text{ for } \eta < \kappa,$$
$$A_{\eta'} \subset A_\eta \text{ for } \eta < \eta' < \kappa, \text{ and}$$
$$\cap_{\eta<\kappa} A_\eta \neq \varnothing.$$

Proof. We define families $\{\mathscr{A}_\eta : \eta < \kappa\}$ such that

(i) $\mathscr{A}_0 = \{\alpha\}$;

(ii) $0 < |\mathscr{A}_\eta| < \beta$ for $\eta < \kappa$;

(iii) if $A, B \in \mathscr{A}_\eta$ and $A \neq B$, then $A \cap B = \varnothing$ for $\eta < \kappa$;

(iv) if $A \in \mathscr{A}_\eta$ then $|A| = \alpha$ for $\eta < \kappa$;

(v) $\mathscr{A}_{\eta+1} \subset \cup\{\mathscr{P}_A : A \in \mathscr{A}_\eta\}$ for $\eta < \kappa$; and
(vi) $|\alpha \backslash \cup \mathscr{A}_\eta| < \alpha$ for $\eta < \kappa$.

We proceed by recursion.

We define $\mathscr{A}_0$ by (i).

Next for $\eta < \kappa$ we define $\mathscr{A}_{\eta+1}$. We set

$$\mathscr{A}'_{\eta+1} = \cup\{\mathscr{P}_A : A \in \mathscr{A}_\eta\}, \text{ and}$$
$$\mathscr{A}_{\eta+1} = \{A \in \mathscr{A}'_{\eta+1} : |A| = \alpha\}.$$

We verify conditions (ii), (iii), (iv), (v), and (vi) for $\mathscr{A}_{\eta+1}$.

(ii) Let $A \in \mathscr{A}_\eta$. Since $|A| = \alpha$ and $|\mathscr{P}_A| < \beta \leq \alpha$ and $A = \cup \mathscr{P}_A$ and α is regular, there is $B \in \mathscr{P}_A$ such that $|B| = \alpha$; we have $B \in \mathscr{A}_{\eta+1}$ and hence $\mathscr{A}_{\eta+1} \neq \varnothing$. Further, since $|\mathscr{A}_\eta| < \beta$ and $|\mathscr{P}_A| < \beta$ for $A \in \mathscr{A}_\eta$ and β is regular, we have $|\mathscr{A}'_{\eta+1}| < \beta$ and hence $|\mathscr{A}_{\eta+1}| < \beta$.

(iii), (iv) and (v) for $\mathscr{A}_{\eta+1}$ are clear from the definitions.

(vi) For $A \in \mathscr{A}_\eta$ we set $S_A = \cup\{B \in \mathscr{P}_A : |B| < \alpha\}$, and we set $S = \cup\{S_A : A \in \mathscr{A}_\eta\}$. Since α is regular and $|\mathscr{P}_A| < \alpha$ we have $|S_A| < \alpha$ for $A \in \mathscr{A}_\eta$; hence $|S| < \alpha$. Since

$$\alpha \backslash \cup \mathscr{A}_{\eta+1} = (\alpha \backslash \cup \mathscr{A}_\eta) \cup S,$$

we have $|\alpha \backslash \cup \mathscr{A}_{\eta+1}| < \alpha$, as required.

Now we assume that η is a limit ordinal such that $0 < \eta < \kappa$, and that $\mathscr{A}_\xi$ has been defined for $\xi < \eta$, and we define $\mathscr{A}_\eta$. We set

$$\mathscr{A}'_\eta = \{\bigcap_{\xi<\eta} A_\xi : A_\xi \in \mathscr{A}_\xi \text{ and } \bigcap_{\xi<\eta} A_\xi \neq \varnothing\}, \text{ and}$$
$$\mathscr{A}_\eta = \{A \in \mathscr{A}'_\eta : |A| = \alpha\}.$$

We verify conditions (ii), (iii), (iv), (v) and (vi) for $\mathscr{A}_\eta$.

(ii) We define $\varphi : \mathscr{A}'_\eta \to \prod_{\xi<\eta} \mathscr{A}_\xi$ by the rule

$$\varphi(\bigcap_{\xi<\eta} A_\xi) = \langle A_\xi : \xi < \eta \rangle.$$

It is clear that φ is a one-to-one function. Since $|\mathscr{A}_\xi| < \beta$ for $\xi < \eta$ and $|\eta| < \kappa$ and $\kappa \ll \beta$, we have $|\prod_{\xi<\eta} \mathscr{A}_\xi| < \beta$ by A.5(a); it follows that

$$|\mathscr{A}_\eta| \leq |\mathscr{A}'_\eta| < \beta.$$

We set $S = \cup_{\xi<\eta}(\alpha \backslash \cup \mathscr{A}_\xi)$. Since $|\alpha \backslash \cup \mathscr{A}_\xi| < \alpha$ for $\xi < \eta$ and $|\eta| < \kappa < \alpha$ and α is regular, we have $|S| < \alpha$. Since $\alpha \backslash S = \cup \mathscr{A}'_\eta$ and $|\mathscr{A}'_\eta| < \alpha$, there is $A \in \mathscr{A}'_\eta$ such that $|A| = \alpha$; we have $A \in \mathscr{A}_\eta$ and hence $\mathscr{A}_\eta \neq \varnothing$.

(iii) and (iv) are clear for $\mathscr{A}_\eta$.

(vi) We have

$$\alpha\backslash\cup\mathscr{A}_\eta=\cup\{A\in\mathscr{A}'_\eta:|A|<\alpha\}\cup S.$$

Since $|\mathscr{A}'_\eta|<\beta\leq\alpha$ and $|S|<\alpha$, we have $|\alpha\backslash\cup\mathscr{A}_\eta|<\alpha$, as required.

The definition of the family $\{\mathscr{A}_\eta:\eta<\kappa\}$ is complete.

Since $|\alpha\backslash\cup\mathscr{A}_\eta|<\alpha$ for $\eta<\kappa$ and $\kappa<\alpha$ and α is regular, we have $|\cup_{\eta<\kappa}(\alpha\backslash\cup\mathscr{A}_\eta)|<\alpha$, and hence there is

$$\zeta\in\alpha\backslash\bigcup_{\eta<\kappa}(\alpha\backslash\cup\mathscr{A}_\eta).$$

For $\eta<\kappa$ there is $A_\eta\in\mathscr{A}_\eta$ such that $\zeta\in A_\eta$. It is clear that the family $\{A_\eta:\eta<\kappa\}$ satisfies the required conditions.

We apply Lemma 1.1 most frequently with $\beta=\alpha$. In Chapter 5 we will need the case

$$\beta=(\kappa^\kappa)^+\leq\alpha \text{ with } \kappa \text{ and } \alpha \text{ regular.}$$

The definition of a weakly compact cardinal is given in Appendix A.

1.2 *Lemma.* Let α be a weakly compact cardinal, and for every $A\subset\alpha$ with $|A|=\alpha$ let $\mathscr{P}_A$ be a partition of A such that $|\mathscr{P}_A|<\alpha$. Then there is a family $\{A_\eta:\eta<\alpha\}$ of subsets of α such that

$$|A_\eta|=\alpha \text{ for } \eta<\alpha,$$
$$A_{\eta+1}\in\mathscr{P}_{A_\eta} \text{ for } \eta<\alpha, \text{ and}$$
$$A_{\eta'}\subset A_\eta \text{ for } \eta<\eta'<\alpha.$$

Proof. We define families $\{\mathscr{A}_\eta:\eta<\alpha\}$ such that

(i) $\mathscr{A}_0=\{\alpha\}$;

(ii) $0<|\mathscr{A}_\eta|<\alpha$ for $\eta<\alpha$;

(iii) if $A,B\in\mathscr{A}_\eta$ and $A\neq B$, then $A\cap B=\varnothing$ for $\eta<\alpha$;

(iv) if $A\in\mathscr{A}_\eta$ then $|A|=\alpha$ for $\eta<\alpha$; and

(v) $\mathscr{A}_{\eta+1}\subset\cup\{\mathscr{P}_A:A\in\mathscr{A}_\eta\}$ for $\eta<\alpha$.

(The argument is essentially that of Lemma 1.1. To verify (ii) for limit ordinals η such that $0<\eta<\alpha$ we set $\gamma=\sup\{|\mathscr{A}_\xi|:\xi<\eta\}$ and we note that since $\gamma<\alpha$ we have

$$|\mathscr{A}_\eta|\leq|\mathscr{A}'_\eta|\leq\prod_{\xi<\eta}|\mathscr{A}_\xi|\leq\gamma^{|\eta|}\leq 2^\gamma\cdot 2^{|\eta|}<\alpha.)$$

Now we set $\mathscr{A}=\cup_{\eta<\alpha}\mathscr{A}_\eta$ and we define a partial order $\preccurlyeq$ on $\mathscr{A}$

by $A \preccurlyeq B$ if $A \supset B$. Then $\langle \mathscr{A}, \preccurlyeq \rangle$ is a tree of height α with $|\mathscr{A}| < \alpha$ for $\eta < \alpha$; hence there is a branch

$$\Sigma = \{A_\eta : \eta < \alpha\}$$

of $\mathscr{A}$ with $A_\eta \in \mathscr{A}_\eta$ for $\eta < \alpha$. It is clear that the family $\{A_\eta : \eta < \alpha\}$ is as required.

1.3 *Lemma.* (The pressing-down lemma.) Let $\omega \leq \kappa < \alpha$ with α and κ regular, let

$$S = \{\xi < \alpha : \mathrm{cf}(\xi) \geq \kappa\},$$

and let f be a function from S to α such that $f(\xi) < \xi$ for $\xi \in S$. Then there are $T \subset S$ with $|T| = \alpha$ and $\bar{\zeta} < \alpha$ such that $f(\xi) < \bar{\zeta}$ for all $\xi \in T$.

Proof. We suppose the lemma fails. Then for $\zeta < \alpha$ we have $|f^{-1}(\zeta)| < \alpha$ and hence there is $g(\zeta)$ such that

$$\zeta < g(\zeta) < \alpha, \text{ and}$$

$$f(\xi) \geq \zeta \quad \text{for } \xi \in S, \xi \geq g(\zeta).$$

We define $\{\zeta(\eta) : \eta < \kappa\}$ by the rule

$\zeta(0) = 0$,
$\zeta(\eta) = \sup\{\zeta(\eta') : \eta' < \eta\}$ for non-zero limit ordinals $\eta < \kappa$, and
$\zeta(\eta + 1) = g(\zeta(\eta))$ for $\eta < \kappa$;

and we set $\bar{\xi} = \sup_{\eta < \kappa} \zeta(\eta)$. The function $\eta \to \zeta(\eta)$ is an ordered-set isomorphism of κ into $\bar{\xi}$, and since κ is a regular cardinal we have $\mathrm{cf}(\bar{\xi}) = \kappa$ and hence $\bar{\xi} \in S$. For $\eta < \kappa$ we have

$$g(\zeta(\eta)) = \zeta(\eta + 1) < \bar{\xi},$$

and hence $f(\bar{\xi}) \geq \zeta(\eta)$. From $\bar{\xi} = \sup_{\eta < \kappa} \zeta(\eta)$ it then follows that $f(\bar{\xi}) \geq \bar{\xi}$, a contradiction.

The proof is complete.

We remark, retaining the notation of Lemma 1.3, that since $T = \cup_{\zeta < \bar{\zeta}} (f^{-1}(\{\zeta\}) \cap T)$ and α is regular, there are $T' \subset T$ with $|T'| = \alpha$ and $\zeta < \bar{\zeta}$ such that $f(\xi) = \zeta$ for all $\xi \in T'$.

Definition. An indexed family $\{S_i : i \in I\}$ of sets is a *quasi-disjoint* family if

$$\bigcap_{i \in I} S_i = S_j \cap S_{j'} \text{ whenever } j, j' \in I, j \neq j'.$$

It is clear that a family $\{S_i : i \in I\}$ is quasi-disjoint if and only if there is a set S such that

$$S = S_j \cap S_{j'} \text{ whenever } j, j' \in I, j \neq j'.$$

1.4 *Theorem.* Let $\omega \leq \kappa \ll \alpha$ with α regular and let $\{S_\xi : \xi < \alpha\}$ be a family of sets such that $|S_\xi| < \kappa$ for $\xi < \alpha$. Then there are $A \subset \alpha$ with $|A| = \alpha$ and a set J such that

$$S_\xi \cap S_{\xi'} = J \text{ for } \xi, \xi' \in A, \xi \neq \xi'.$$

Proof. (In the terminology of the definition above we are to show that there is $A \in [\alpha]^\alpha$ such that $\{S_\xi : \xi \in A\}$ is a quasi-disjoint family. We give two proofs.)

First Proof (using Lemma 1.1). We suppose that if $A \subset \alpha$ and $\{S_\xi : \xi \in A\}$ is quasi-disjoint then $|A| < \alpha$; and for every $A \subset \alpha$ with $|A| = \alpha$ we set

$$J_A = \bigcap_{\xi \in A} S_\xi, \text{ and}$$
$$\mathscr{B}_A = \{B \subset A : \text{if } \xi, \xi' \in B, \xi \neq \xi', \text{ then } S_\xi \cap S_{\xi'} = J_A\}.$$

Since the set $\mathscr{B}_A$ partially ordered by inclusion is inductive, there is a maximal element $B_A \in \mathscr{B}_A$. We have $|B_A| < \alpha$, and since $\{\xi\} \in \mathscr{B}_A$ for all $\xi \in A$ we have $B_A \neq \emptyset$.

For $\xi \in A \backslash B_A$ it follows from the maximality of B_A that there is $\zeta(\xi) \in B_A$ such that

$$S_\xi \cap S_{\zeta(\xi)} \supsetneqq J_A.$$

We define

$$\varphi_A : A \backslash B_A \to \cup \{\mathscr{P}(S_\zeta) : \zeta \in B_A\}$$

by the rule

$$\varphi_A(\xi) = S_\xi \cap S_{\zeta(\xi)},$$

and we set

$$\mathscr{P}_A = \{B_A\} \cup \{\varphi_A^{-1}(\{S\}) : S \in \cup \{\mathscr{P}(S_\zeta) : \zeta \in B_A\}\}.$$

Then $\mathscr{P}_A$ is a partition of A, and since $|S_\zeta| < \kappa$ for $\zeta \in B_A$ and $\kappa \ll \alpha$, we have $|\mathscr{P}(S_\zeta)| < \alpha$ and hence (since $|B_A| < \alpha$ and α is regular) we have $|\mathscr{P}_A| < \alpha$. It follows from (the case $\beta = \alpha$ of) Lemma 1.1 that there is a family $\{A_\eta : \eta < \kappa\}$ of subsets of α such that

$$|A_\eta| = \alpha \quad \text{for } \eta < \kappa,$$
$$A_{\eta+1} \in \mathscr{P}_{A_\eta} \quad \text{for } \eta < \kappa,$$
$$A_{\eta'} \subset A_\eta \quad \text{for } \eta < \eta' < \kappa, \text{ and}$$
$$\bigcap_{\eta<\kappa} A_\eta \neq \varnothing.$$

For $\eta < \kappa$ there is $S(\eta) \in \cup\{\mathscr{P}(S_\zeta) : \zeta \in B_{A_\eta}\}$ such that

$$A_{\eta+1} = \varphi_{A_\eta}^{-1}(\{S(\eta)\}) \in \mathscr{P}_{A_\eta} \quad \text{and}$$
$$\varphi_{A_\eta}(\xi) = S_\xi \cap S_{\zeta(\xi)} = S(\eta) \supsetneqq J_{A_\eta} \quad \text{for } \xi \in A_{\eta+1}.$$

Since $S(\eta) = \varphi_{A_\eta}(\xi) \subsetneqq S_\xi$ for $\xi \in A_{\eta+1}$, we have

$$J_{A_\eta} \subsetneqq S(\eta) \subset \bigcap_{\xi \in A_{\eta+1}} S_\xi = J_{A_{\eta+1}}$$

and hence $J_{A_{\eta+1}} \backslash J_{A_\eta} \neq \varnothing$ for $\eta < \kappa$; it follows that

$$\left| \bigcup_{\eta<\kappa} J_{A_\eta} \right| \geq \sum_{\eta<\kappa} |J_{A_{\eta+1}} \backslash J_{A_\eta}| \geq \kappa.$$

Now let $\xi \in \bigcap_{\eta<\kappa} A_\eta$. Then $\xi \in A_{\eta+1}$ for $\eta < \kappa$, and from $S_\xi \supset J_{A_\eta}$ we have

$$S_\xi \supset \bigcup_{\eta<\kappa} J_{A_\eta}$$

and hence $|S_\xi| \geq \kappa$, a contradiction.

Second Proof (using Lemma 1.3). We assume without loss of generality that $S_\xi \subset \alpha$ for $\xi < \alpha$. We set

$$S = \{\xi < \alpha : \mathrm{cf}(\xi) \geq \kappa\}$$

and we define $f : S \to \alpha$ by the rule

$$f(\xi) = \sup(S_\xi \cap \xi) \quad \text{for } \xi \in S.$$

For $\xi \in S$ we have $|S_\xi| < \kappa \leq \mathrm{cf}(\xi)$ and hence $f(\xi) < \xi$. It follows from Lemma 1.3 that there are $T \subset S$ with $|T| = \alpha$ and $\bar{\zeta} < \alpha$ such that $f[T] \subset \bar{\zeta}$.

For $\xi \in T$ we have $S_\xi \cap \bar{\zeta} \in \mathscr{P}_\kappa(\bar{\zeta})$, and since $\kappa \ll \alpha$ and α is regular we have $|\mathscr{P}_\kappa(\bar{\zeta})| < \alpha$; it follows that there are $T' \subset T$ with $|T'| = \alpha$ and $J \in \mathscr{P}_\kappa(\bar{\zeta})$ such that $S_\xi \cap \bar{\zeta} = J$ for $\xi \in T'$.

We define a function $\varphi : \alpha \to T'$ as follows. We set $\varphi(0) = \min T'$, and if $\xi < \alpha$ and $\varphi(\xi')$ has been defined for all $\xi' < \xi$ we choose $\varphi(\xi) \in T'$ such that

$$\sup\{\varphi(\xi') : \xi' < \xi\} < \varphi(\xi), \text{ and}$$
$$\sup\big(\bigcup_{\xi' < \xi} S_{\varphi(\xi')}\big) < \varphi(\xi);$$

such a choice is possible because

$$\Big|\bigcup_{\xi' < \xi} S_{\varphi(\xi')}\Big| \le \sum_{\xi' < \xi} |S_{\varphi(\xi')}| \le |\xi| \cdot \kappa < \alpha$$

and T' is cofinal in α.

We set $A = \{\varphi(\xi) : \xi < \alpha\}$ and we claim that if $\xi' < \xi < \alpha$ then

$$S_{\varphi(\xi')} \cap S_{\varphi(\xi)} = J.$$

Indeed since $\varphi(\xi')$, $\varphi(\xi) \in T'$ we have

$$S_{\varphi(\xi')} \cap S_{\varphi(\xi)} \cap \bar{\zeta} = J \cap J = J;$$

and if $\eta \ge \bar{\zeta}$ and $\eta \in S_{\varphi(\xi)}$, then since

$$\sup(S_{\varphi(\xi)} \cap \varphi(\xi)) = f(\varphi(\xi)) < \bar{\zeta},$$

we have $\eta \ge \varphi(\xi) > \sup S_{\varphi(\xi')}$ and hence $\eta \notin S_{\varphi(\xi')}$.

The extension to singular cardinals of the Erdős–Rado theorem for quasi-disjoint families is deferred to Theorem 1.9 below.

Though it is not needed later in this work, we note a strong converse to Theorem 1.4.

Theorem. Let κ and α be cardinals with $\kappa < \alpha$ and $\alpha \ge \omega$. If for every family $\{S_\xi : \xi < \alpha\}$ of sets with $|S_\xi| < \kappa$ for $\xi < \alpha$ there is $A \subset \alpha$ such that $|A| = \alpha$ and $\{S_\xi : \xi \in A\}$ is quasi-disjoint, then $\kappa \ll \alpha$.

Proof. Suppose there are $\beta < \alpha$, $\lambda < \kappa$ such that $\beta^\lambda \ge \alpha$, and let $\{f_\xi : \xi < \alpha\}$ be a subset of β^λ with $f_\xi \ne f_{\xi'}$ for $\xi' < \xi < \alpha$. Then f_ξ is a function from λ to β, and we set

$$S_\xi = \text{graph } f_\xi = \{\langle \eta, f_\xi(\eta)\rangle : \eta < \lambda\} \quad \text{for } \xi < \alpha.$$

Since $|S_\xi| = \lambda < \kappa$, there is $A \subset \alpha$ such that $|A| = \alpha$ and $\{S_\xi : \xi \in A\}$ is quasi-disjoint. Since $|A| = \alpha > \beta$, for $\eta < \lambda$ the function from A to β defined by $\xi \to f_\xi(\eta)$ is not one-to-one and hence there are distinct elements ξ, ξ' of A and $\varphi(\eta) < \beta$ such that

$$f_\xi(\eta) = f_{\xi'}(\eta) = \varphi(\eta);$$