Wade

Calculus

CALCULUS

THOMAS L. WADE

The Florida State University

GINN AND COMPANY

Boston · New York · Chicago · Atlanta · Dallas · Columbus
San Francisco · Toronto · London

COPYRIGHT, 1953, BY GINN AND COMPANY

ALL RIGHTS RESERVED

153.1

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE

This book is intended for a first course in the calculus, with the usual prerequisite of analytic geometry. It does not purport to be a comprehensive treatise on the calculus at the intermediate or advanced level. While an attempt is made to recognize and respect points of rigor, the rigorous treatment of a considerable number of situations is regarded as properly being relegated to a calculus course at a higher level.

Chapter 1 is an informal introduction to the calculus. An effort is made to point up the need for the calculus as a means of satisfying man's intellectual curiosity as well as for solving practical problems. The intuitive idea of a limit is exploited. Representative examples and exercises which assist the student to refresh himself in the basic algebraic manipulations of the calculus are given. Some attention is devoted to an historical orientation of the calculus in our present scientific era.

In considering limits in Chapter 2, simple but adequate definitions are given of the limit of a variable and of the limit of a function. To avoid plunging the student into a collection of abstractions, discussion of such topics as infinite discontinuities and indeterminate forms is postponed to later chapters.

Experience indicates that after the student has learned to differentiate the general power function cu^n and has learned that the derivative of the sum is the sum of the derivatives, he is in a position to explore many applications of the differential calculus to geometry, physics, and engineering. Chapter 3 is concerned with these things.

In line with a recent trend the derivative is symbolized by D_xy, $f'(x)$, or y' until differentials are considered in Chapter 4; not until that point is the alternate notation dy/dx for the derivative introduced. Also in agreement with growing practice is the adoption of the notation $\ln x$ for $\log_e x$ in Chapter 8.

It is becoming common practice to introduce integration much earlier in the calculus than formerly. There are two reasons for this: (1) many students take their calculus course concurrently with a physics course which is based upon both the differential and integral processes of the calculus; (2) many teachers of the calculus have found that by interweaving the processes of differentiation and integration numerous of the more interesting and important developments and applications of the calculus can be taken up much earlier, thus making the instruction more interesting and effective. After a student has learned to integrate $cu^n\, du$ and knows that the integral of a sum is the sum of the integrals, he is ready for an over-view of the principles and procedures of the integral calculus. That is given in Chapter 4 for power functions. Differentials are introduced directly, and are not based upon infinitesimals. However, at a later point the term *infinitesimal* is explained. Special care is taken in presenting the definite integral as the limit of a sum in Chapter 5. The author has found that a few illustrative examples which actually show the calculation of the limit of a sum pay dividends in subsequent improved understanding by the student of this often used and all-important process.

The differential calculus of fractional functions, which calls for the derivative of a product and a quotient and their attendant rather complicated computations, is postponed to Chapter 7.

In Chapter 8 we endeavor to give an intelligible and coherent treatment of the exponential and logarithmic functions, covering both differential and integral processes. Chapter 9 deals in like manner with the trigonometric and inverse trigonometric functions. In both instances the discussion is coordinated with the student's earlier study of these functions.

To avoid confronting the student with a long list of formulas for integration at an early stage in the development of integration procedures, the rather extensive work on integration in Chapters 4–8 is based upon the use of only three formulas, namely those for

$$\int u^n\, du,\ \int \frac{du}{u},\ \text{and} \int e^u\, du.$$

The last ten chapters follow somewhat the same plan as that discussed in detail above for the first nine chapters, with emphasis generally being given to the careful exposition in as simple language as possible of the fundamentals for the first-year's course in the calculus.

The author wishes to express his appreciation to six other mathematics teachers and to several hundred students who have enabled him to try out the material of the book at the Florida State University during the past two years.

THOMAS L. WADE

试读结束：需要全本请在线购买： www.ertongbook.com

CONTENTS

INTRODUCTION

The subject which we are now to explore and study is designated variously by the names "the calculus," "the differential and integral calculus," and "the infinitesimal calculus." Later we shall discuss briefly the history of the calculus and the origin of these appellations of the subject.

1 : 1 The calculus as an intellectual achievement and as a tool

First let us raise the question: Why study the calculus? Many of our present-day physical comforts, recreational pleasures, and medical aids are direct consequences of man's development and application of this branch of mathematics. A notable example of a natural phenomenon harnessed by man with the aid of the calculus is electricity. The ancient Egyptians and Greeks knew how to produce electricity on a small scale by rubbing together certain substances of opposite natures, a procedure commonly described in elementary science. But in order to produce, distribute, and utilize electricity on a large scale it was necessary to recognize the essential characteristics of this phenomenon and to state the basic relations as equations connecting these essential characteristics. We are all conscious of the rapid growth during the last sixty years in the use of electricity in lighting, refrigeration, transportation, communication, heating, and the numerous modern devices which use electric motors.

The quantitative theory upon which this large-scale production and use of electricity depends has its foundation in the mathematical formulation given by James Clerk Maxwell (1831–1879) of the experimental results of

试读结束：需要全本请在线购买： www.ertongbook.com

Michael Faraday (1791–1867). These basic mathematical laws of electricity are expressed in the medium of "differential equations," which means that they involve certain processes and concepts of the calculus. On the basis of his mathematical formulation of the laws of electricity in the middle of the nineteenth century, Maxwell predicted theoretically the existence of wireless waves. From that prediction and its experimental verification by Heinrich Hertz in 1888 there followed the development of wireless telegraphy and radio, beginning commercially with Marconi's successful transmission of wireless signals across the English Channel in 1899.

Electricity is typical of many natural phenomena for which man has found the calculus to be the key for their control; some others are sound, light, thermodynamics, work, and pressure. In so far as mere man is privileged to regulate and control certain aspects of his physical environment, he does so largely through a knowledge of the functional relationships (equations) connecting the variables which characterize a given situation. Some of these functional relationships, such as Galileo's law of falling bodies, $s = 16\,t^2$, and Boyle's law of gases, $vp = k$, may be obtained from experimental results organized and expressed mathematically without the aid of the calculus. But in many other situations, as in the case of electricity described above, experimental results lead to relations involving not just the basic variables of the problem, but also the rates of change (derivatives) of these variables with respect to some one of them. From such a "differential equation" one can arrive at the desired functional relation between the variables of the problem only with the aid of processes of the calculus entailing what is commonly called "integration" or "solving the differential equation."

A physicist or an engineer often deals with problems which call for the use of the calculus. Serious students and research workers in physical chemistry, physiological chemistry, econometrics, and biometrics find a knowledge of the calculus necessary. For this reason it is commonly expected that an incipient physicist, chemist, biometrician, engineer, or theoretical economist will study the calculus for a year or more. An increasing number of students in liberal arts colleges study the calculus out of intellectual curiosity and a desire to gain an acquaintanceship with it, for the same reasons they study logic, philosophy, drama, political science, and like subjects.

In summary we may say that *the calculus is worthy of our study because it is at the same time one of the greatest intellectual achievements and one of the most powerful tools known to man.* If we consider contemporary culture as the acquaintanceship with the significant human achievements of the present and recent eras, then the study of the calculus is surely cultural. Too often the study of calculus is considered both dull and difficult. Certainly a subject which has its origin in efforts to solve problems of significant human interest in astronomy, mechanics, hydrodynamics, elasticity, gravitation, electricity, and magnetism cannot be very dull. And the industrious, intelligent student who is imbued with a spirit of adventure for exploring the realm of ideas finds the study of the calculus less difficult than that of most sequential subjects; this is doubtless due to the fact that the calculus unifies practically all the mathematics that the student has learned before with only one main concept added, that of a limit.

The above indication that the calculus gives a procedure for solving differential equations might erroneously leave the impression that such is its only significant contribution. This is not true. For example, the calculus gives immense aids to arithmetic calculations of many sorts. The logarithmic and trigonometric tables with which you have become acquainted in earlier mathematics courses were constructed by means of infinite series, one of the topics of the calculus.

1 : **2** Problems in maximum and minimum values

Among the problems with which the calculus deals extensively are those concerned with maximum and minimum values of a function. We now consider several such problems in a restricted manner.

EXAMPLE 1

Suppose that we have 400 feet of fence, and desire to fence in a rectangular lot which has one side along a high rock wall so that no fence is needed along this side. Find the dimensions of the largest rectangular lot which can be enclosed with the given 400 feet of fence.

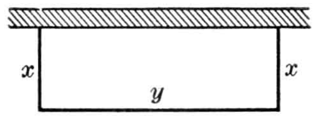

FIG. 1

SOLUTION

Let x represent the width of the rectangle in feet, y the length in feet, and a the area. Then

$$a = xy \quad \text{and} \quad 2x + y = 400.$$

From the second of these relations $y = 400 - 2x$; substituting this value of y in the first relation, we get

$$[1] \qquad a = 400x - 2x^2, \quad \text{or} \quad a = -2(x^2 - 200x).$$

We want to determine the value of x for which a is greatest. Completing the square on x, within the parentheses, and making the proper arithmetic compensation, we have

$$a = -2(x^2 - 200x + 10{,}000) + 2(10{,}000),$$

or

$$[2] \qquad a = -2(x - 100)^2 + 20{,}000.$$

From this expression for a it should be apparent that $-2(x - 100)^2$ is negative or zero for any real value of x, and therefore a attains its greatest value when this term is zero, or when $x = 100$ feet; further, this greatest value of a is 20,000 square feet. The significance of this result is that with 400 feet of fence we may enclose any number of rectangular lots, but there is one of the lots with area greater than that of the others. This lot with greatest area has an area of 20,000 square feet, with width of 100 feet and length of 200 feet. To make the significance of this result more emphatic to the student it may be advantageous to represent some relative values of x, y, and a, as in the table below.

When $x =$	20	40	80	100	125	150
then $y =$	360	320	240	200	150	100
and $a =$	7,200	12,800	19,200	20,000	18,750	15,000

Clearly, with 400 feet of fence we can enclose many rectangular lots of the kind described; of these there is a largest one, with width 100 feet, length 200 feet, and area 20,000 square feet. It is customary to say that relative to the function [*1*] the value $x = 100$ gives the maximum value of a.

This problem we were able to work by a special algebraic procedure and special observation because the relation [*1*] gives a as a quadratic function of x. More complicated problems could not be done in this manner. But the calculus gives a general procedure for handling problems in maxima and minima for all types of elementary functions. To work the present problem by the calculus, we would calculate the *derivative* of a with respect to x, denoted by $D_x a$. We would find $D_x a = 400 - 4x$. Setting this derivative equal to zero, we get $400 - 4x = 0$, which when solved yields the value of x for which a is maximum. The procedure for the construction of the derivative and for its use as indicated here is given in the next section.

EXAMPLE 2

Suppose that we want to construct an open top rectangular box which is to have a volume of 40 cubic feet. The sides are to cost 12 cents per square foot, and the bottom is to cost 15 cents per square foot. Find the dimensions of the box which will make the cost least, the base being a square.

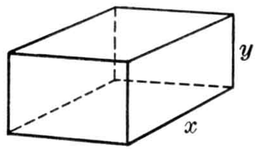

FIG. 2

SOLUTION

Let x be the length of a side of the square base in feet, and y the height of the box in feet. The area of the base is x^2 square feet, and the cost of the base is $15\,x^2$ cents. The area of one of the four equal sides is xy square feet, the area of the four sides is $4xy$ square feet, and the cost of the four sides is $4\,xy(12)=48\,xy$ cents. Let c be the total cost of the box in cents. Then

$$c = 15\,x^2 + 48\,xy \quad \text{and} \quad x^2y = 40.$$

Eliminating y from these relations by substituting from the second in the first, we get

$$[3] \qquad c = 15\,x^2 + \frac{1920}{x}.$$

We want to find the value of x that will make c least, or a minimum. There is no procedure for treating the relation [*3*] comparable to that for the algebraic treatment of [*1*] on page 3. The calculus would tell us to find the *derivative* of c with respect to x, D_xc, set it equal to zero, and solve for x. Here $D_xc = 30\,x - 1920\,x^{-2}$. Setting this equal to zero and solving for x, we get $x^3 = 64$, or $x = 4$. Then $y = 2.5$ and $c = 720$ cents, or \$7.20.

There are many boxes that can be built to satisfy the given conditions. Thus with the same volume one may be constructed with dimensions 2 by 2 by 10 feet with a cost of \$10.20; another can be constructed with dimensions 6 by 6 by $\frac{10}{9}$ feet with a cost of \$8.60. But the one with the dimensions 4 feet by 4 feet by 2.5 feet costs the least, or the minimum, that minimum cost being \$7.20.

EXERCISES

1. For the relation $a = 400\,x - 2\,x^2$ developed in Example 1 construct a table of corresponding values of x and a by assigning to x the values 0, 20, 40, 60, 80, 100, 120, 140, 160, 180, 200 and calculating the corresponding values of a. Regard each pair of corresponding values of x and a as the coordinates of a point, plot these points on squared paper (with the x-axis horizontal and the a-axis vertical), and draw a smooth curve through these points. What kind of curve is this? From the graph what value of x appears to yield the greatest value of a? Does this graphical estimate agree with the result

obtained on page 4 by algebraic means? What is the position of the tangent to the curve at the point for which a is maximum?

2. A sheet of metal 24 inches wide is to be bent upward the same amount on two opposite edges to form a trough. What should the depth x of the trough be in order that its carrying capacity be a maximum; that is, what should the depth be in order that the area of a cross-section perpendicular to the edge shall be a maximum?

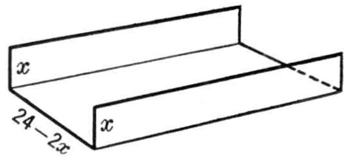

FIG. 3

HINT: Express the area a of the cross section in terms of the depth x, and proceed as in Example 1.

3. By assigning to x appropriate values, construct a table and a graph for the relation of Exercise **2**. Ask yourself questions similar to those of Exercise 1.

4. The power w, in watts, delivered to a load by a current of i amperes through a resistance of 24 ohms when an electromotive force of 110 volts is maintained at one end of the circuit is given by the formula $w = 110\,i - 24\,i^2$. Using the method of Example 1, find the value of i for which w is maximum. What is this maximum value of w?

5. For the relation [*3*] developed in Example 2 construct a table by assigning to x the values 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, 3, $\frac{7}{2}$, 4, $\frac{9}{2}$, 5, $\frac{11}{2}$, 6 and calculating the corresponding values of c. Using these tabular values, draw a graph with the x-axis horizontal and the c-axis vertical. From this graph what value of x appears to yield the least value of c? How does this graphical estimate compare with the value of x obtained in Example 2 with the aid of the anticipated calculus?

1 : 3 The derivative

Consider the algebraic relation

[*4*] $$y = 16\,x^2.$$

As the variable x changes from a given value x_1 to another value x_2, the amount of change in x is $x_2 - x_1$. This change $x_2 - x_1$ in x is called an *increment* of x and is denoted by the symbol Δx (read "delta x"); it may be positive or negative. As the quantity x changes or takes on an increment Δx, the quantity y changes or takes on an increment Δy. Oftentimes, for a given relation $y = f(x)$, we want to obtain an expression for Δy in terms of x and Δx. Among several ways of interpreting the relation [4], one is to consider it as the equation of a curve, a parabola opening upward with the y-axis as the axis of symmetry. Suppose that x_1 and y_1 are corresponding values of x and y for the relation [4], and similarly that $x_1 + \Delta x$, $y_1 + \Delta y$ are another pair of such corresponding values. A portion of the parabola in the first quadrant is represented in Fig. 4, where P is the point (x_1, y_1). Since the point $Q(x_1 + \Delta x, y_1 + \Delta y)$ lies on the parabola, its coordinates satisfy the equation [4] of the parabola, and we have

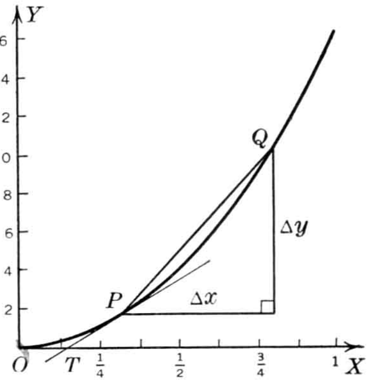

FIG. 4

$$y_1 + \Delta y = 16(x_1 + \Delta x)^2$$

or

[5] $$y_1 + \Delta y = 16\, x_1^2 + 32\, x_1\, \Delta x + 16(\Delta x)^2.$$

Also

[6] $$y_1 = 16\, x_1^2,$$

since the point $P(x_1, y_1)$ is on the parabola. Subtracting [6] from [5] we get

[7] $$\Delta y = 32\, x_1\, \Delta x + 16(\Delta x)^2.$$

Division of both sides of [7] by Δx results in

[8] $$\frac{\Delta y}{\Delta x} = 32\, x_1 + 16\, \Delta x.$$

It is noteworthy that to effect the division by Δx it is necessary to assume that $\Delta x \neq 0$. Therefore, in our subsequent considerations of [8] we must be careful that we do not assign to Δx the value zero. Division by zero is not a permissible operation in conventional mathematics.

In the relation [8] consider x_1 as fixed or constant, and let Δx vary by becoming smaller and smaller (but remember that we have obligated ourselves to refrain from letting Δx take on zero as one of its values). Some representative values of Δx and the corresponding values of $\Delta y/\Delta x$ are given in the following table.

When $\Delta x = \frac{1}{16}$	$\frac{1}{16{,}000}$	$\frac{1}{16{,}000{,}000}$
then $\Delta y/\Delta x = 32\, x_1 + 1$	$32\, x_1 + \frac{1}{1{,}000}$	$32\, x_1 + \frac{1}{1{,}000{,}000}$

试读结束：需要全本请在线购买： www.ertongbook.com