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Precise Spectral Asymptotics for Elliptic Operators Acting in Fiberings over Manifolds with Boundary

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INTRODUCTION

This book is devoted to the determination of the precise asymptotics for eigenvalues of certain elliptic selfadjoint operators acting in fiberings over compact manifolds with boundary and for more general elliptic selfadjoint spectral problems. The precise asymptotics for restriction to the diagonal of the Schwartz kernels of the corresponding spectral projectors is derived too. These asymptotics for closed manifolds were determined in author's paper [57] using the same method; therefore one can consider [57] as a simple and short introduction to the methods and ideas of this book. More general results were obtained, for example, in [32 - 35], but with much weaker remainder estimates.

In Part I we derive these asymptotics for the second-order elliptic selfadjoint differential operators acting in fiberings over manifolds with boundary on which an elliptic boundary condition is given; this condition must be either the Dirichlet condition or the generalized Neumann condition; the latter means that the boundary value of the derivative of the function with respect to the direction transversal to the boundary is expressed through the boundary value of the function by means of first-order pseudo-differential operator on the boundary; examples show that for more general elliptic boundary conditions our results may not be valid.

We shall derive the following asymptotics for the eigenvalue distribution function:

$$N(k) = \varkappa_0 k^d + O(k^{d-1}),$$

$$N(k) = \varkappa_1 k^{d-1} + O(k^{d-2}) \quad \text{or}$$

$$N(k) = O(1)$$

where d is the dimension; the second asymptotics may occur only if the boundary is not empty. It is impossible to derive stronger estimates for remainder without some condition of global nature. For the Laplace - Beltrami operator this condition is:

The set of all points periodic with respect to the geodesic flow with reflection at the boundary has measure zero.

In a general case if certain strong conditions involving some global condition are satisfied then the following asymptotics hold:

$$N(k) = \varkappa_0 k^d + \varkappa_1 k^{d-1} + o(k^{d-1}) \quad \text{or} \quad N(k) = \varkappa_1 k^{d-1} + \varkappa_2 k^{d-2} + o(k^{d-2}).$$

Under some conditions the asymptotics for the restriction to the diagonal of the Schwartz kernels of the spectral projectors are derived too. Inside these asymptotics have the same nature: $\alpha_0(x)k^d + O(k^{d-1})$ or $O(1)$ but near the boundary the leading part of the asymptotics contains the term of boundary-layer type of the order k^d. If and only if this term exhausts the leading part of the asymptotics then we have the asymptotics $N(k) = \varkappa_1 k^{d-1} + O(k^{d-2})$ for the eigenvalue distribution functions.

In Part I the methods and ideas of [56, 57] are generalized and improved upon and in Part II these methods and ideas are applied to certain new situations and the results of Part I are extended to these situations. In §6 the asymptotics for the elliptic selfadjoint first-order operators are derived. In §7 we derive the asymptotics for the spectral problem

$$(*) \qquad (-\mu \mathcal{J} + A)u = 0$$

where A is the elliptic selfadjoint second-order positive definite operator acting in the fibering over a manifold with boundary and $\mathcal{J}$ is non-degenerate Hermitian matrix, acting in the fibers of this fibering. The opposite case when $\mathcal{J}$ is positive definite A is not necessarily positive or negative definite may be reduced to the case $\mathcal{J} = I$ investigated in Part I. In §8 we study the problem (*), when A and $\mathcal{J}$ differ from $(-\Delta)^p$, $(-\Delta)^q$ respectively only by lower order terms where $p > q \geqslant 0$ and Δ is the Laplace - Beltrami operator. We must note that the latter case is the most complicated and in this case we need some improvements and modifications of our methods.

We intend to write a few papers in future; in these papers the spectral asymptotics for global and partially global operators [62] and the quasiclassical spectral asymptotics for h-(pseudo)-differential operators [63] will be derived and applications of these results to the problem (*) where $\mathcal{J}$ may degenerate in a definite manner, will be given.

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TABLE OF CONTENTS

Part I. THE ASYMPTOTICS FOR SECOND-ORDER OPERATORS

§ 0. Main theorems

0.1. Let X be a compact d-dimensional C^∞-manifold with the boundary $Y \in C^\infty$, $d \geqslant 2$, dx a C^∞-density on X, E a Hermitian D-dimensional C^∞-fibering over X. Let $\mathcal{A}: C^\infty(X, E) \to C^\infty(X, E)$ be a second-order elliptic differential operator, formally selfadjoint with respect to inner product in $L_2(X, E)$; if $Y = \varnothing$, then $\mathcal{A}$ may be a classical pseudo-differential operator. Let ν be a C^∞ vector field transversal to Y at every point, τ the operator of restriction to Y, $B = \tau$ or $B = \tau\nu + B_1\tau$ a boundary operator, where $B_1: C^\infty(Y, E) \to C^\infty(Y, E)$ is a first-order classical pseudo-differential operator on Y; certainly, only B but not its components is an invariant object. We suppose that (E). Operator $\{\mathcal{A}, B\}: C^\infty(X, E) \to C^\infty(X, E) \oplus C^\infty(Y, E)$ satisfies the Šapiro - Lopatinskii condition; (S) $\mathcal{A}_B$ - the restriction of $\mathcal{A}$ to the $\operatorname{Ker} B$ (i.e. operator with the domain $D(\mathcal{A}_B) = C^\infty(X, E) \cap \cap \operatorname{Ker} B$) is a symmetric operator in $L_2(X, E)$.

Let $A_B: L_2(X, E) \to L_2(X, E)$ be a closure of $\mathcal{A}_B$ in $L_2(X, E)$; then A_B is selfadjoint, its spectrum is discrete, with finite multiplicity and tends either to $\pm\infty$, or to $+\infty$, or to $-\infty$. Without the loss of generality one can suppose that 0 is not an eigenvalue of A_B. Then, if X is a closed manifold, then $\Pi^\pm$ - selfadjoint projectors to positive(negative) invariante subspaces of A_B are zero-order classical pseudo-differential operators on X. If X is a manifold with boundary, then $\Pi^\pm$ belong to Boutet de Monvel algebra, i.e. $\Pi^\pm = \Pi^{0\pm} + \Pi'^{\pm}$ where $\Pi^{0\pm}$ are zero-order classical pseudo-differential operators with the transmission property on X and $\Pi'^{\pm}$ are zero-order classical singular Green operators (see [14] or § 1 of this book). $\pi^\pm$ - the principal symbols of $\Pi^\pm$ - are selfadjoint projectors to positive and negative invariant subspaces of a; a is the principal symbol of $\mathcal{A}$. If $\pi^\pm = 0$ then $\Pi^\pm$ is a zero-order classical singular Green operator. It should be pointed out that for a more general boundary operator B even satisfying the Šapiro - Lopatinskii condition, these statements may not be valid: $\Pi^\pm$ may be operators of a more general nature;

are likely to belong to Rempel - Schulze algebra [83].

Let H be a closed subspace in $L_2(X,E)$ such that Π -selfadjoint projector to H - belongs to Boutet de Monvel algebra (if X is a closed manifold then Π is a classical zero-order pseudo-differential operator). We suppose that $\Pi A_B \subset A_B \Pi$; this means that H is an invariant subspace of A_B. Then $A_{B,H} : H \to H$, $\mathfrak{D}(A_{B,H}) = \mathfrak{D}(A_B) \cap H$ - the restriction of A_B to H - is a selfadjoint operator; its spectrum is discrete, with finite multiplicity and tends either to $\pm\infty$, or to $+\infty$, or to $-\infty$.

Let us introduce the eigenvalue distribution functions for $A_{B,H}$: $N^{\pm}(k) = N^{\pm}_H(k)$ is the number of eigenvalues of $A_{B,H}$ lying between 0 and $\pm k^2$.

Let $E(s)$ be spectral selfadjoint projectors of A_B, $E_H(s) = \Pi E(s)$ and $e^{\pm}_H(s) = \pm(E_H(\pm k^2) - E_H(0))$; let $e^{\pm}_H(x,y,k)$ be Schwartz kernels of $e^{\pm}_H(k)$; then

$$N^{\pm}_H(k) = \operatorname{tr} e^{\pm}_H(k) = \int_X e^{\pm}_H(x,x,k)\,dx .$$

We are interested in the asymptotics of $N^{\pm}_H(k)$ and $e^{\pm}_H(x,x,k)$ as k tends to ∞ assuming, of course, that $\pm A_{B,H}$ is not semibounded from above.

0.2. Let a and a^s be the principal and subprincipal symbols of $\mathcal{A}$, $\pi^{\pm}$ and π the principal symbols of $\Pi^{0\pm}$ and Π^0, $\varepsilon(s)$ spectral selfadjoint projectors of a, $\varepsilon^{\pm} = \pm(\varepsilon(\pm 1) - \varepsilon(0))$.

0.3. We start from the results of [57] concerning asymptotics on closed manifolds. One can assume that $\pi\pi^{\pm} \neq 0$ on some connective component of $T^*X \setminus 0$; otherwise $\Pi\Pi^{\pm}$ is of order (-1), therefore $\Pi\Pi^{\pm}$ is finite-dimensional projector and $\pm A_{B,H}$ is semibounded from above.

THEOREM 0.1. If X is closed manifold then the following asymptotics hold:

$$N^{\pm}_H(k) = \varkappa^{\pm}_0 k^d + O(k^{d-1}) \quad \text{as} \quad k \to \infty , \tag{0.1}$$

$$e^{\pm}_H(x,x,k) = \alpha^{\pm}_0(x) k^d + O(k^{d-1}) \tag{0.2}$$

as $k \to \infty$ uniformly with respect to $x \in X$,

where

$$\alpha^{\pm}_0(x) = \int_{T^*_x X} \varepsilon^{\pm}\, d\xi , \tag{0.3}$$

$\bar{d}\xi = (2\pi)^{-d} d\xi$, $d\xi$ is the measure on $T^*_x X$, generated by the natural measure $dx\,d\xi$ on T^*X and by fixed measure dx on X; $\alpha^{\pm}_0(x) > 0$ on Ω -connective component of X , if $T^*\Omega \cap$ cone $\operatorname{supp} \Pi\Pi^{\pm} \neq \emptyset$; otherwise

(0.2) $$e^{\pm}_H(x, x, k) = O(1) \quad \text{as } k \to \infty$$

uniformly with respect to $x \in \Omega$;

(0.4) $$\varkappa^{\pm}_0 = \int_X \operatorname{tr} \alpha^{\pm}_0(x)\, dx = \int_{T^*X} \operatorname{tr} \varepsilon^{\pm}\, dx\, \bar{d}\xi.$$

REMARK. If X is a manifold with the boundary Y then the asymptotics (0.2) holds for $x \in X \setminus Y$ uniformly on every compact subset of $X \setminus Y$.

REMARK. It is obvious that $\alpha^{\pm}_0 \in C^{\infty}$; all coefficients $\alpha^{\pm}_{\cdot}(x)$ which will be introduced in future will also be smooth.

To obtain the second term of the asymptotics for the eigenvalue distribution function we need some condition of a global character. Let $0 < \sigma^2_1 \leqslant \sigma^2_2 \leqslant \ldots \leqslant \sigma^2_{D_\pm}$ be all the positive eigenvalues of $\pm a$; $D_\pm$ may depend on connective component of $T^*X \setminus 0$. Let $\Sigma^{\pm}_j$ be a conical with respect to ξ closed nowhere dense subsets of $T^*X \setminus 0$ such that for every j σ^2_j has the constant multiplicity outside of $\Sigma^{\pm}_j$; this multiplicity may depend on the connective component of $T^*X \setminus 0 \setminus \Sigma^{\pm}_j$. These subsets exist without question.

Consider the <u>bicharacteristics</u> of σ_j - the curves along which

$$\frac{d\rho}{dt} = H_{\sigma_j}(\rho), \qquad \rho = \rho(t) \in T^*X \setminus 0 \setminus \Sigma^{\pm}_j ,$$

where $H_f = \left\langle \frac{\partial f}{\partial \xi}, \frac{\partial}{\partial x} \right\rangle - \left\langle \frac{\partial f}{\partial x}, \frac{\partial}{\partial \xi} \right\rangle$ is the Hamiltonian field generated by f . a bicharacteristic is called periodic if there exists $T \neq 0$ such that $\rho(t+T) = \rho(t)$; T is called the <u>period</u> of bicharacteristic.

THEOREM 0.2. Let X be a closed manifold and the following condition hold:

(H1). There exists a set $\Lambda^{\pm}$ of measure zero, $\bigcup_j \Sigma_j \subset \Lambda^{\pm} \subset T^*X \setminus 0$, such that through each point of $T^*X \setminus 0 \setminus \Lambda^{\pm}$ for every j there passes a nonperiodic bicharacteristic of σ_j lying in $T^*X \setminus 0 \setminus \Sigma^{\pm}_j$, which is infinite in both directions.

Then the following asymptotics holds:

(0.5) $$N^{\pm}_H(k) = \varkappa^{\pm}_0 k^d + \varkappa^{\pm}_1 k^{d-1} + o(k^{d-1}) \qquad k \to \infty ,$$

where

$$(0.6)\qquad x_1^{\pm} = \mp \frac{d}{2} \int\limits_{T_x^* X} tr\, \varepsilon^{\pm} (\pm a)^{-1/2}\, \Pi\Pi^{\pm} a^{s}\, dx\, d\xi + \bar{\varkappa}^{\pm}.$$

REMARK. If A is a differential operator and $\Pi(x,-\xi) = \Pi(x,\xi)$, then $\varkappa_1^{\pm} = 0$ where $\bar{\varkappa}_1^{\pm}$ depends only on a, Π (see Appendices D, F_∞)

0.4. Now let X be a manifold with the boundary. Let $x_1 \in C^\infty$, $x_1 = 0$ and $dx_1 \neq 0$ at Y, $x_1 > 0$ in $X \setminus Y$. Consider the characteristic symbol $g_{\pm}(\rho, \tau) = \det(\tau^2 \mp a(\rho))$.

DEFINITION. The point $\rho \in T^* X \setminus Y$ has the multiplicity $l = l_{\pm}(\rho)$ if

$$\left(\frac{\partial}{\partial \tau}\right)^{i} g_{\pm}(\rho,\tau)\Big|_{\tau=1} = 0 \qquad \forall i < l,$$

$$\theta_{\pm}(\rho) = \left(\frac{\partial}{\partial \tau}\right)^{l} g_{\pm}(\rho,\tau)\Big|_{\tau=1} \neq 0.$$

DEFINITION. We shall say that the point $\rho \in T^* X|_Y \setminus 0$ with $l = l_{\pm}(\rho) \geq 1$

(i) is positive if

$$\theta_{\pm}^{-1}(\rho) \left(\frac{\partial}{\partial \tau}\right)^{i} \left(\frac{\partial}{\partial \xi_1}\right)^{l-i} g_{\pm}(\rho,\tau)\Big|_{\tau=1} > 0$$

$$\forall i = 0, \dots, l-1;$$

(ii is negative if

$$\theta_{\pm}^{-1}(\rho) \left(\frac{\partial}{\partial \tau}\right)^{i} \left(-\frac{\partial}{\partial \xi_1}\right)^{l-i} g_{\pm}(\rho,\tau)\Big|_{\tau=1} > 0$$

$$\forall i = 0, \dots, l-1;$$

(iii) is tangential if

$$\left(\frac{\partial}{\partial \tau}\right)^{i} \left(\frac{\partial}{\partial \xi_1}\right)^{l-i} g_{\pm}(\rho,\tau)\Big|_{\tau=1} = 0$$

$$\forall i = 0, \dots, l-1;$$

(iv) is indefinite in the remaining cases.

Here ξ_1 is the dual to x_1 variable, $\frac{\partial}{\partial \xi_1} = -H_{x_1}$.

Let $j : T^* X|_Y \longrightarrow T^* Y$ be a natural mapping.

THEOREM 0.3. Let $\Pi\Pi^{\pm}$ be not a singular Green operator (i.e.

$\Pi\Pi^{\pm} \neq 0$).
Then the asymptotics (0.1) holds with the same coefficient $\mathscr{x}_0^{\pm}$.

THEOREM 0.4. Let $\Pi\Pi^{\pm}$ be a singular Green operator. Then the following asymptotics holds:

$$(0.7) \quad N_H^{\pm}(k) = \mathscr{x}_1'^{\pm} k^{d-1} + O(k^{d-2})^{*)} \quad \text{as} \quad k \to \infty \qquad {}^{*)}$$

where $0 < \mathscr{x}_1'^{\pm}$ depends only on $a|_Y$ and on $(\Pi\Pi^{\pm})_0'$ - the principal symbol of singular Green operator $(\Pi\Pi^{\pm})'$; b is the principal symbol of B.

0.5. To obtain the second term of the asymptotics for the eigenvalue distribution function we need some condition of a global nature.

Suppose first that

$$(H.2) \quad g_{\pm}(\rho,\tau) = (\tau^2 - \mu(\rho))^{\ell} h(\rho,\tau),$$

where $\mu(\rho) = \mu(x,\xi)$ is a positive definite quadratic form in ξ (i.e. a Riemannian metric on X) and $h(\rho,\tau)$ does not vanish.

Then on S^*X -tangent spheres bundle of the Riemannian manifold X (more precisely on its subset of the complete measure) one can define a continious measure - preserving <u>geodesic flow with reflection at the boundary</u>. More particulary: consider S^*X with identified points ρ and ρ' such that $j\rho = j\rho'$.

By a <u>geodesic</u> we mean a curve lying in S^*X , along which

$$\frac{d\rho}{dt} = \frac{1}{2} H_{\mu}(\rho)$$

(i.e. the bicharacteristic of $\sigma = \sqrt{\mu}$); the geodesic, minus, perhaps, its endpoints, must lie in $S^*(X \setminus Y)$; t (the length) is the natural parameter along the geodesic.

By a <u>geodesic billiard</u> we mean a curve lying in S^*X consisting of segments of geodesics; the endpoint of the precedent segment and the starting point of the consequent must belong to $S^*X|_Y$ and be equivalent (i.e. identified); t (the length) is the natural parameter along geodesic billiards.

It is easy to show that there exists a set $\Sigma \subset S^*X$, of first Baire category and measure zero, such that through each point of $S^*X \setminus \Sigma$ a geodesic billiard of infinite length in both directions can be passed in such a way that each of its intervals of fi-

---

*) Here $O(k^0) \overset{def}{=} O(\ln k)$

nite length contains a finite number of segments, and all the geodesics included in it are transversal to the boundary. Thus, on $S^*X\setminus\Sigma$ a continuous measure - preserving flow $\Phi(t)$ is given. *)

We call a point $\rho\in S^*X\setminus\Sigma$ periodic if there exists a $T\neq 0$ such that $\Phi(T)\rho=\rho$; T is the period of point ρ.

THEOREM 0.5. Let $\Pi\Pi^{\pm}$ be not a singular Green operator and condition (H.2) be satisfied. Assume that the set of periodic points has measure zero. Then the following asymptotics holds:

$$(0.5)\quad N_H^{\pm}(k)=\varkappa_0^{\pm}k^d+(\varkappa_1^{\pm}+\varkappa_1^{\prime\prime\pm})k^{d-1}+o(k^{d-1})\qquad k\to\infty\,,$$

where $\varkappa_0^{\pm}$, $\varkappa_1^{\pm}$ are given by (0.4), (0.6), $\varkappa_1^{\prime\prime\pm}$ depends only on $a|_Y$, b, $\Pi\Pi^{\pm}|_Y$, $(\Pi\Pi^{\pm})_0'$.

0.6. As we mentioned above, the asymptotics of $e_H^{\pm}(x,x,k)$ near boundary has a more complicated character. Let us identify some neighbourhood of Y with $[0,\delta)\times Y$; then point x will be identified with (x_1,x'), where x_1 is such as above, $x'\in Y$. If condition (H.2) is satisfied then X is a Riemannian manifold and this identification is canonical, $x_1=\mathrm{dist}(x,Y)$.

THEOREM 0.6. Let $\Pi\Pi^{\pm}$ be not a singular Green operator and condition (H.2) be satisfied. Then in neighbourhood of Y the following asymptotics holds:

$$(0.8)\quad e_H^{\pm}(x,x,k)=(\alpha_0^{\pm}(x)+Q_0^{\pm}(x',x_1k))k^d+O(k^{d-1})$$

as $k\to\infty$ uniformly with respect to x , where $\alpha_0^{\pm}(x)$ is given by (0.3),

$$Q_0^{\pm}\in C^{\infty}(Y\times\bar{\mathbb{R}}^+),$$

$$(0.9)\quad D_{x'}^{\alpha'}D_s^{i}\,Q_0^{\pm}(x',s)=O(s^{\frac{d+1}{2}})$$

as $s\to\infty$ uniformly with respect to $x'\in Y$ for every α', i.

THEOREM 0.7. Let $\Pi\Pi^{\pm}$ be not a singular Green operator and the following conditions be satisfied:

(H.3) For every point $\rho^*\in T^*Y\setminus 0$ we can find $\mu\in\mathbb{R}$ such that at all points of $j^{-1}\rho^*\cap\{g_{\pm}(\rho,1)=0\}$ the following inequalities hold:

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*) The simple proof of these statements is published in [21] ; it coincides with the unpublished proof by the author [56].

$$(0.10)\qquad \theta_{\pm}^{-1}(\rho)\left(\frac{\partial}{\partial\tau}\right)^{i}\left(\frac{\partial}{\partial\tau}+(\xi_1-\mu)\frac{\partial}{\partial\xi_1}\right)^{l-i} g_{\pm}(\rho,\tau)\Big|_{\tau=1}>0$$

$$\forall i=0,\dots,l-1\;;\; l=l_{\pm}(\rho)\;;$$

(H.4) $\xi_1=\lambda_i(x,\xi')$ - all real roots of the $g_{\pm}(x,\xi',\xi_1,1)$ - have constant multiplicities in the neighbourhood of $N^*Y=\{x_1=\xi'=0\}$ and

$$(0.11)\qquad \frac{\partial\lambda_i}{\partial\xi_k}\Big|_{x_1=\xi'=0}\qquad \text{do not depend on } i\,,$$

$$(0.12)\qquad \operatorname{sign}\lambda_i\left(\frac{\partial^2\lambda_i}{\partial\xi_k\,\partial\xi_l}\Big|_{x_1=\xi'=0}\right)_{k,l=2,\dots,d}$$

are negative definite $(d-1)\times(d-1)$-matrices.

Then for $d\geqslant 3$ the asymptotics (0.9) holds; for $d=2$ the following asymptotics holds:

$$(0.8)\qquad e^{\pm}(x,x,k)=(\alpha_0^{\pm}(x)+Q_0^{\pm}(x',x_1k))k^2+O(k^{3/2})$$

as $k\to\infty$

uniformly with respect to x_+ where $Q_0^{\pm}$ satisfies (0.9).

REMARK. In theorems 0.6, 0.7 $Q_0^{\pm}(x',\cdot)$ depends only on $a(0,x',\cdot)$, $b(x',\cdot)$, $\Pi\Pi^{\pm}(0,x',\cdot)$, $(\Pi\Pi^{\pm})_0'(x',\cdot)$. Moreover

$$\varkappa_1''^{\pm}=\int_Y\int_0^{\infty}\operatorname{tr}Q^{\pm}(x',s)\,ds\,dx'$$

where $dx'=\frac{dx}{dx_1}\Big|_Y$ is a C^{∞}-density on Y.

COROLLARY. If the conditions of the theorem 0.6 hold or if the conditions of the theorem 0.7 hold and $d\geqslant 3$ then the asymptotics (0.2) holds outside the boundary layer $x_1\leqslant k^{-(d-1)/(d+1)}$.

THEOREM 0.8. Let $\Pi\Pi^{\pm}$ be a singular Green operator but not a smoothing operator. Then the following asymptotics holds:

$$(0.13)\qquad e_H^{\pm}(x,x,k)=k^d Q_0^{\pm}(x',x_1k)+O(\min(k^{d-1},x^{-d+1}))$$

as $k\to\infty$

uniformly with respect to x

where $Q_0^{\pm}\in C^{\infty}(Y\times\overline{\mathbb{R}}^+)$,

$$(0.14)\qquad \mathcal{D}_{x'}^{\alpha'}\mathcal{D}_s^{i}Q_0^{\pm}(x',s)=O(s^{-d-i})$$

as $s\to\infty$ uniformly with respect to $x'\in Y$ for every α', i;

$Q_0^{\pm}(x',\cdot)$ depends only on $a(0,x',\cdot)$, $b(x',\cdot)$, $(\Pi\Pi^{\pm})_0(x',\cdot)$. Moreover,

$$x_1'^{\pm}=\iint\limits_{Y\,0}^{\infty} \operatorname{tr} Q_0^{\pm}(x',s)\,ds\,dx' .$$

It is interesting that on Y $e^{\pm}(x,x,k)$ has degree-like asymptotics again.

THEOREM 0.9. Let $\Pi\Pi^{\pm}$ be not a smoothing operator and $B=i\tau\mathcal{D}_1+$ $+B_1\tau$. Then the following asymptotics holds:

$$e_H^{\pm}(x,x,k)=\varkappa_0'^{\pm}(x)k^d+O(k^{d-1}) \tag{0.15}$$

as $k\to\infty$ uniformly with respect to $x\in Y$ where $\varkappa_0'^{\pm}(x)$ depends only on $a(x,\cdot)$, $b(x,\cdot)$, $\Pi\Pi^{\pm}(x,\cdot)$, $(\Pi\Pi^{\pm})_0'(x,\cdot)$.

REMARK. In reality theorems 0.6-0.9 have the local character and therefore if we wish to derive the asymptotics of $e^{\pm}(x,x,k)$ on some subset $\Omega\subset X$ then we need the fulfillment of conditions of these theorems only on $\Omega\cap Y$.

0.7. As an illustration we shall consider the Laplace - Beltrami operator (or its certain generalization) on the Riemannian manifold with the boundary.

Thus, let X be a compact Riemannian manifold with the boundary Y, dx and dx' be natural measures on X and Y respectively, E a Hermitian fibering over X. Let $\mathcal{A}:C^{\infty}(X,E)\to C^{\infty}(X,E)$ be a formally selfadjoint differential operator with the principal part $-\Delta Id$,*) where Δ is the Laplace - Beltrami on X, $Id:E\to E$ is the identical mapping. We denote by A_D, A_N, A_G an operator A_B if the boundary operator is $B=\tau$, $B=\tau\frac{\partial}{\partial n}+i\varphi\tau$, $B=\tau\frac{\partial}{\partial n}+(i\varphi+B_1)\tau$ respectively where n is the interior unit normal to Y, φ is a smooth matrix such that A_N is the selfadjoint operator, $B_1:C^{\infty}(Y,E)\longrightarrow C^{\infty}(Y,E)$ is a first-order classical pseudo-differential operator on Y with the principal symbol β. Then A_G is selfadjoint if and only if B_1 is symmetric. Let $\beta_K (K=1,\dots,D)$ be the eigenvalues of β Then Šapiro-Lopatinskii condition means precisely that

$$\beta_K\neq 1 \quad \text{on} \quad S^*Y \quad \forall K=1,\dots,D. \tag{0.16}$$

When (0.16) holds then A_G is semibounded from below if and only if

$$\beta_K<1 \quad \text{on} \quad S^*Y \quad \forall K=1,\dots,D. \tag{0.17}$$

Let $\Pi=I$. One can concretize the statements of the theorems 0.3, 0.5, 0.6, 0.9:

THEOREM 0.10 (i). The following asymptotics hold as $k\to\infty$;:

*) Certainly, $-\Delta Id$ is not an invariant object ixcept in the case of $E=X\times\mathbb{C}^D$.

$$N^+(k) = \text{æ}_0^+ k^d + O(k^{d-1}),$$

$$e^+(x,x,k) = (\alpha_0^+ + Q_0^+(x', z x_1 k))k^d + O(k^{d-1}),$$

where $\text{æ}_0^+ = (2\pi)^{-d}\omega_d D\, vol\, X$, $\alpha_0^+ = (2\pi)^{-d}\omega_d Id$, ω_ℓ is the volume of the unit ball in $\mathbb{R}^\ell$,

$$Q_0^+(x', s) = (2\pi)^{-d+1}\int_{S^{d-2}} \mathcal{K}_d(\beta, s)\, d\theta,$$

$d\theta$ is the Lebesque measure on

$$S^{d-2} \ni \theta, \quad \beta = \beta(x', \theta),$$

$$\mathcal{K}_d(\beta, s) =$$

$$= -\frac{1}{2\pi}\int_\gamma (-i\omega + \beta\sqrt{1-\omega^2})^{-1}(i\omega + \beta\sqrt{1-\omega^2})(1-\omega^2)^{\frac{d-3}{2}}(-i\omega)^{-d}\left(\frac{\partial}{\partial s}\right)^{d-1}\frac{1}{s}(e^{-is\omega}-1)\,d\omega$$

for $A = A_G$,

$$\mathcal{K}_d(s) = \pm\frac{1}{2\pi}\int_\gamma (1-\omega^2)^{\frac{d-3}{2}}(-i\omega)^{-d}\left(\frac{\partial}{\partial s}\right)^{d-1}\frac{1}{s}(e^{-is\omega}-1)\,d\omega$$

for $A = A_N$ and $A = A_D$; here and below, the signs (+) and (-) correspond to $A = A_N$ and $A = A_D$ respectively, $Re(1-\omega^2)^{1/2} > 0$, γ is a contour with the starting point -1 and the endpoint +1 lying in the lower complex half-plane $Re\,\omega < 0$ below all points $\omega_k = -i\beta_k/\sqrt{1-\beta_k^2}$ with $0 \leqslant \beta_k < 1$.

(ii) Assume that the set of points periodic with respect to geodesic flow with reflection at the boundary has measure zero. Then the following asymptotics holds: $N^+(k) = \text{æ}_0^+ k^d + \text{æ}_1^+ k^{d-1} + o(k^{d-1})$ as $k \to \infty$, where

$$\text{æ}_1^+ = -\frac{(2\pi)^{-d}}{2i(d-1)}\int_{S^*Y}\int_\gamma tr(-i\omega+\beta\sqrt{1-\omega^2})^{-1}(i\omega+\beta\sqrt{1-\omega^2})(1-\omega^2)^{\frac{d-3}{2}}\omega^{-1}\,d\omega\, d\theta\, dy$$

for $A = A_G$, $\text{æ}_1^\pm = \pm\frac{1}{4}(2\pi)^{-d+1}\omega_{d-1} vol\, Y$ for $A = A_N$ and $A = A_D$.

COROBLAIRES AND REMARKS.

(i) As before

$$\mathcal{K}_d(\beta, s) = O(s^{-\frac{d+1}{2}}) \quad \text{as} \quad s \to +\infty.$$

(ii) The following asymptotics holds as $s \to +\infty$:

$$\mathcal{K}_d(\beta, s) = \frac{1}{4\pi}\Gamma\left(\frac{d-1}{2}\right) 2^{\frac{d+1}{2}} \cos\left(s - \frac{\pi}{4}(d-3)\right) + O\left(s^{-\frac{d+2}{2}}\right)$$

for $A = A_G$,

$$\mathcal{K}_d(s) = \pm\frac{1}{4\pi}\Gamma\left(\frac{d-1}{2}\right) 2^{\frac{d+1}{2}} \cos\left(s - \frac{\pi}{4}(d-3)\right) + O\left(s^{-\frac{d+3}{2}}\right)$$

for $A = A_N$ and $A = A_D$; therefore on the greater part of boundary layer we can simplify the expression for Q_0^+.

(iii) If d is odd then for $A = A_N$ and $A = A_D$

$$Q_0^+(s) = \pm 2(2\pi)^{-\frac{d+1}{2}}\left(-\frac{1}{s}\frac{\partial}{\partial s}\right)^{\frac{d-1}{2}}\frac{\sin s}{s}.$$

(iv). If d is even then for $A = A_N$ and $A = A_D$

$$Q_0^+(s) = \pm 2(2\pi)^{-\frac{d}{2}-1}\left(-\frac{1}{s}\frac{\partial}{\partial s}\right)^{\frac{d-1}{2}}\int_{-1}^{1}\left(\frac{\sin ws}{ws} + \frac{\cos ws - 1}{w^2 s^2}\right)\left(1-w^2\right)^{-1/2} dw.$$

(v) The asymptotics (0.15) holds with

$$\alpha_0'^{+} = (2\pi)^{-d}\left\{\omega_d - \frac{1}{d}\int_{S^{d-2}}\int_{\gamma}\left(-iw+\beta\sqrt{1-w^2}\right)^{-1}\left(iw+\beta\sqrt{1-w^2}\right)\left(1-w^2\right)^{\frac{d-3}{2}} dw\, d\theta\right\};$$

in particular, $\alpha_0'^{+} = 2(2\pi)^{-d}\omega_d \operatorname{Id}$ for $A = A_N$.

(vi) Degree-like and boundary layer type terms in asymptotics for $e^+(x,x,k)$ generate the first and the second terms in the asymptotics for $N^+(k)$ (after integrating with respect to dx), but the asymptotics for $N^+(k)$ with the second term does not follow from the asymptotics for $e^+(x,x,k)$.

(vii) If β is odd with respect to θ , then

$$\varkappa_1^+ = -\frac{1}{4}(2\pi)^{-d+1}\omega_{d-1}\operatorname{vol} Y + \sum_{j:\,|\beta_j|<1}\frac{(2\pi)^{-d+1}}{2(d-1)}\int_{S^*Y}\left(1-\beta_j^2\right)^{-\frac{d+1}{2}} d\theta\, dx'.$$

One can concretize the statements of the theorems 0.4, 0.8 and 0.9 again.

THEOREM 0.11. The following asymptotics hold as $k \to \infty$:

$$N^-(k) = \varkappa_1^- k^{d-1} + O(k^{d-2}), \qquad ^{*)}$$

*) Here $O(k^0) = O(1)$.