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S. Busenberg B. Forte H.K. Kuiken

Mathematical Modelling of Industrial Processes

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PREFACE

The role that university mathematicians can play in solving problems from real life has been the subject of discussions, round tables and symposia which are becoming more and more frequent. Certainly this matter does not belong specifically to our days: there are so many instances of illustrious mathematicians of the past who have used or even created mathematical tools for investigating problems of engineering, physics, chemistry, biology etc. The reason for such growing interest lies on the one hand in the increasing complexity of technological problems which require more mathematics, and on the other hand in the fact that university mathematicians are gradually rediscovering applied sciences as an inexhaustible source of appealing and challenging mathematical problems.

This set of CIME Courses demonstrated effectively such traits of so-called Industrial Mathematics. The lecturers are well-known mathematicians with a very large experience in the application of mathematics to problems submitted by industrial companies.

They described a variety of problems arising in different fields. All of them being brilliant teachers, their lectures were highly stimulating and very appropriate to illustrate one fundamental point: Industrial Mathematics IS FIRST OF ALL MATHEMATICS, not just modelling (although the role of mathematicians is often crucial in this stage), not just computing (usually the final stage of such kind of research), despite the fact that often a theorem has to be transformed into innovative software in order to become the terminal product "sold" to the company.

We are proud to have organised this CIME Session and we thank the Director and the staff of CIME for their support, and TECNOPOLIS for having offered the lecture room as well as financial and logistic help.

We are extremely grateful to the lecturers for their efforts in selecting the most appropriate material and for drawing such a clear picture of what Industrial Mathematics is today.

Vincenzo Capasso　　　　Antonio Fasano

TABLE OF CONTENTS

Mathematical Modelling of Industrial Processes

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Contents

Industrial Mathematics

What is industrial mathematics?

I am not sure who was the first to come up with the term industrial mathematics. The first time I heard it was during a visit to Bell Labs in Murray Hill in the summer of 1974. People like Henry Pollak of Bell Labs or Murray Klamkin of the Ford Motor Company used it freely in various publications. It is possible that the term was already used by Fry who headed the Math group of Bell Labs before WW II. Of course, the 'I' in the acronym SIAM stands for industry. Yet, most members of that society are applied mathematicians in the sense I shall define later on, rather than industrial mathematicians. When I subsequently mentioned the term in Europe, people looked at me in disbelief and dismissed it as an objectionable americanism. However, as happens over and over again with neologisms that seem to meet an existing need, the term was eventually widely accepted, also in Europe. Many universities have since created chairs of industrial mathematics. Societies such as ECMI (European Consortium for Mathematics in Industry) have been founded which profess their devotion to the application of mathematics in industry.

Hearing the expression 'industrial mathematics' for the first time, one can justifiably ask oneself whether this means anything different from the mathematics people use in non-industrial circles. In general, one should say, mathematics, whether this be topology or numerical analysis, does not depend upon the place where it is practised. The response of the industrial mathematician will be that the term does not specify a novel kind of mathematics, but rather emphasizes the fact that the application of mathematics in non-mathematical disciplines is the central theme. First, there is a problem arising outside mathematics and then whatever mathematics is needed is brought to bear upon it. Here mathematics is not a goal in itself, but rather a means to get answers in a non-mathematical world. Industrial labs such as Bell Labs in America or Philips Research Labs in Europe and, of course, there are many many more, are devoted to fundamental and basic research in many non-mathematical areas. Nevertheless, this makes them into veritable feeding grounds for the application of all kinds of mathematics.

Of course, the term 'applied mathematics' has been known to us for a much longer time and there seems to be good reason to question the necessity of introducing yet another expression for an activity which seems no different from ordinary applied mathematics. I feel that an explanation can be given when we consider the historical development of mathematics. Mathematics came into being because man felt the need to better govern the world around him and to improve his ability to predict phenomena that determined his life. Geometry was created in ancient Egypt for people to be able to cope with the problem of finding again each plot of land after the annual flooding of the Nile. The ancients of Mesopotamia were masters at devising tables with which they could predict conjunctions of the planets which, so they believed, determined the fate of nations. We shall not elaborate on this issue here. It should be clear, however, that these early, anonymous mathematicians were also physicists and engineers *avant la lettre*.

It would seem that this situation persisted until quite late in the nineteenth century. Great mathematicians such as Newton or Euler relied heavily on what they saw in the world around them for their mathematical inspirations. Indeed, it is very difficult to distinguish between Newton the Physicist and Newton the Mathematician. Mathematicians regard him as one of their leaders, because he is the (co-)inventor of differential calculus. Physicists call him their greatest on account of his three famous laws of mechanics. His *Principia* is good reading for physicist and mathematician alike. Only late in the nineteenth century do we see the emergence of a breed of investigators that we now call pure mathematicians. These people seem to be interested solely in the axiomatic worlds they themselves created. For their inspiration they do not rely on the world in which they live, but only on the world in which they think. This development ultimately leads to the likes of G.H. Hardy, who complimented himself on never having achieved anything of practical value, and Paul Halmos who tried to convince an audience that any mathematics that can be applied is dirty or ugly by definition. Quite amusingly, the fact of the matter is that most, and perhaps all, mathematics such people create is or will be used eventually by others to elucidate real-world problems. Sometimes this happens so fast that they live to see it.

This development of mathematics from applied to pure, which meant a turning away from the real world, an introspective move so to speak, did not mean that the application of mathematics got into low gear. On the contrary, many fields of science which had always had a verbal character to them, meaning that their truths were presented in plain language, were rapidly mathematized. However, people who called themselves mathematicians seemed to lose touch with this development. This game is now played by the physicists, the chemists, the engineers themselves. In the meantime, the mathematicians, at least the purer ones among them, are dreaming away in a world of their own making.

The central tenet of pure mathematics is formal proof. The sentiment about formal proof has become so strong among pure mathematicians that many of them will not accept as true any mathematics which has not stood this test. The difficulty is that most of the problems which arise outside mathematics are so complicated that it is impossible, at least for the time being, to give formal proof. As an example we could mention the mathematics that is used to describe flows around aeroplanes. Long before a real aeroplane is actually flown, its mathematical model has already been tested to the full. It is unthinkable that any modern aeroplane could be developed without the aid of extensive mathematical modelling. Yet, the correctness of these models has not been fully established. Even so, mathematicians, both pure and applied, use these machines for their long-distance travels. Only a handful of them will question the unproven safety these vehicles afford.

Who then are the inheritors in our day and age of the long tradition that began with our anonymous friends of the Nile and the Euphrates, all the way to Newton, the Bernoulli's, Euler, etc.? Indeed, there are such contemporaries, but if there had been a Nobel prize for Mathematics, which there is not, thanks to the strained relations between Nobel and Mittag-Leffler, then these people would not be among the laureates, for the simple reason that they would not be recognized as mathematicians. I mean people such as Theodore von Kármán and Geoffrey Taylor. The first of these applied his mathematical genious to almost every branch of engineering science. The same is true for Taylor who also had a knack for doing beautiful experiments and making simple but very elucidating mathematical models for them. We could also think of someone like John von Neumann. Pure mathematicians also regard him as one of their heros. Contrary to Hardy or Halmos, he also used his enormous talent to clarify problems emanating from other disciplines.

Again, I pose the question: is industrial mathematics any different from applied mathematics? No, not if we mean applied mathematics in the classical sense, linked with names such as Newton or Taylor. Of course, in the old days there were no industries to speak of, at least not ones that needed mathematics on a large scale. Modern industries are simply concentrated worlds within our world, where all kinds of problems arise that mathematics may be successfully applied to. Archimedes, Huygens, Newton would probably have taken a great interest in them. In principle, the manner in which mathematics is applied in these industries is not different from the way the classics operated. The difficulty with the term applied mathematics is that it is used nowadays to signify a different activity. Someone who studies properties of differential equations, although he/she may never solve a real-world problem, is called an applied mathematician, for the simple reason that differential equations are used to tackle problems arising outside mathematics. The same is true for someone interested in operator theory, Lie groups or numerical analysis. Clearly, in this sense applied mathematics is concerned with problems that arise within the mathematical world itself. It goes without saying that the fruits of this type of research are very useful for those mathematicians who tackle real-world problems.

Henry Pollak of Bell Labs distinguishes five stages in any separate activity of an industrial mathematical nature, which I shall repeat here in my own phraseology.

- Stage 1, *a problem arises outside mathematics.*
  This is the interface of mathematics and the real world. It often happens that the mathematician working in industry is approached by one of his/her non-mathematical colleagues who puts a technical problem before him/her. By problem I do not mean that the person in question wants to solve a differential equation or evaluate an integral. Of course, occasionally the industrial mathematician may help out with problems such as these, but this is not what he is in industry for. No, by technical problem I mean that the colleague describes a phenomenon or an experiment, and that he/she feels that the situation may be clarified by bringing mathematics to bear upon it. During this first encounter the discussion will be conducted mostly in ordinary language, although it may be interspersed with the jargon of the field the technical problem arose from. Depending upon the mathematical skills of the one who brought the problem, the discussion may take a mathematical twist. It should be clear that one of the prerequisites for this encounter to lead to any success or progress is that the industrial mathematician must be able to understand the language of the person who consults him/her. If his/her head is always high in the mathematical clouds, he/she will probably fail as an industrial mathematician. The successful industrial mathematician must therefore be interested in at least a few disciplines other than mathematics itself.

- Stage 2, *mathematical modelling.*
  This is where the original technical problem is cast into mathematical terms. I shall devote a separate section to describing what I think a mathematical model is all about. Let me state here that there is not one single mathematical model for each particular technical problem, but that we have a whole hierarchy of them. Aris describes this at length.

- Stage 3, *the analysis of the problem.*
  The purpose of the model is to obtain understanding. This understanding is obtained by analysis. These days, with all the computing power available, many industrial mathematicians are tempted to take the model to the computer as fast as they can, churning out numbers in endless series. Because these computers are so powerful, there is a tendency to devise models of ever-increasing complexity. The world models developed by economists are good examples of where this trend may lead us. Many hundreds or even thousands of parameters can be varied simultaneously, yielding a multitude of different answers. Although it is possible that this

is sometimes the only realistic way to proceed, I should like to state here that a professional industrial mathematician ought to postpone the use of the computer as long as he/she can. First, he/she should apply all the mathematical craft he/she can muster to work with the model, to modify it, to reduce it, to simplify it. If the primary model is dimensional, render it dimensionless. Then look at the dimensionless parameters. Are they small or large? Can the model be reduced by means of asymptotic methods? If the model is too complicated, can a simpler model give some clues? And so on and so forth.

- Stage 4, *numerical evaluation of results.*
  Except in the simplest of cases most models will eventually require the use of computers. This can be a simple matter of evaluating the value of an integral or an infinite series, or it can be a very complicated exercise in finite elements. In any case, the problem that is taken to the computer should be well balanced and computing time should be used economically. For instance, if someone has to calculate the value of a slowly-converging series which needs the evaluation of, say, one million terms to reach four-digit accuracy, then he/she has a very lazy mind if he/she writes a direct algorithm. First, one has to investigate whether convergence can be accelerated.

- Stage 5, *communication.*
  Since the problem came from outside, the results have to be returned to the outside world in a way that can be understood by outsiders. This aspect is neglected in many works which pretend to be applied or even industrial mathematics. Most mathematicians are happy and content when they have solved a problem their way, meaning that they have finally produced an intricate formula, a nice algorithm or splendid proof. However, this language is rarely understood by non-mathematicians. The successful industrial mathematician, on the contrary, will spend a great deal of his/her energy on making clear graphs, fine tables and a lucid verbal description of the results obtained. An accomplished industrial-mathematics report will contain a clear description put in ordinary language of the problem to be tackled. Then follows a description of the model and a sketch of the methods that were used to obtain the solution. Extensive calculations which would not interest anyone but the real experts are best reported in appendices. The report ends with one or two sections on results and a discussion of the results. Again the emphasis should be on the use of ordinary plain language.

What is a mathematical model?

A mathematical model is an imperfect image of a part of the world around us, in which use is made of mathematical symbolism. The model employs mathematical representations of the basic laws of nature, for instance the conservation laws. In this respect it may be useful to remark that the word 'nature' is to be interpreted in the widest sense possible. It is not restricted to what physicists understand by nature, but it includes also things abstract; to put it shortly, everything our brain can grasp. As a special feature, a mathematical model, despite its imperfections, provides us with an insight into those parts of the 'world' which are inaccessible to us, either for the time being or permanently. Mathematical models have been made to describe conditions in the core of the sun or the earth. Clearly, we shall probably never have bodily access to those parts of the world, and neither will it be possible, at least for the time being, to carry out experiments there. Nevertheless, we feel that these models enable us to predict what happens in those remote places. Mathematical models are also helpful in predicting things which are still to happen, to predict future occurrences, so to speak. According to a set of assumptions, different scenarios are presented to politicians, managers, the military, whose decision on what course of action is to be taken depends upon what the models foretell.

We have already pointed out the imperfection of mathematical models. This has to do with the fact that in building such a model we can represent only a very limited number of aspects of the part of the world we wish to know more about. Depending upon whether our approach is cruder or less crude, we may refer to our models as very imperfect or less imperfect. The euphemism 'refined' is sometimes used for models which, in reality, are only less imperfect. However this may be, it should be clear that each part of our world can be modelled in a great many different ways. In each particular instance we can distinguish a complete hierarchy of mathematical models from crude to refined. The level of refinement we can reach may depend upon the time we can spare, upon our financial means and, last but not least, upon the limitations of our brain power.

Apart from mathematical models we also know physical models. Whereas with a mathematical model we are concerned with an abstract representation of some part of our world, be it abstract or tangible, a physical model is a tangible representation of a tangible part of the world. That particular tangible part of the world is copied on a reduced and simplified scale. Architects use such models. Civil engineers have a great tradition in this particular area. They have earned themselves fame with their physical models of tidal systems for low-lying coastal areas. Sometimes the sizes of these models are gigantic, occupying many thousands of square metres. Nevertheless, despite the great sums that are spent on these physical models, they do have very definite drawbacks. The most important of these is that it is never possible to scale down all variables of the modelled system in the same fashion. For instance, for practical reasons the rivers in the aforesaid tidal models are always much deeper than they ought to be. If these depths were modelled to scale, then the surface tension of the water would affect the working of the model in an intolerable fashion. Scaling up and scaling down are very tricky operations. It is precisely through the study of mathematical modelling that we know that a scaled-down version of a physical reality can be in a theoretical regime which is completely different from that of the physical reality itself. Of course, nature itself has known this all along: There is no such thing as the one-millimetre elephant, nor is there, contrary to what some science-fiction movies would have us believe, room in our world for the one-metre ant. Be this as it may, it would seem that physical models are things of the past. Yet, with man's disposition being such that, on the whole, he is inclined to favour things tangible, these physical models are likely to remain with us for a long time.

We remarked earlier that in making a mathematical model we adopt a selection process in which only certain attributes of the world around us are seen fit to be represented in the model, whereas others are left out of consideration. For the latter we use the word 'neglected'. It often happens that after the model has been set up, the process of neglecting certain effects is carried still further on the basis of a process of mathematical reasoning conducted within the model. Although after the making of the model, physical reality is represented by a purely mathematical object which, strictly speaking, we can talk about in mathematical terms only, it can sometimes be helpful to refer back to the physical reality the model is thought to be an image of. We have to be careful, however, not to refer to physical entities that are not represented in the model. This would seem to be a matter of course. However, the following anecdote serves to show that confusion can arise if one is not careful in one's reasoning:

A mathematician working in industry had made a model for the diffusion of electrons in a layer of a few thousand ångströms. In his model he made use of the well-known diffusion equation for continuous media. This would seem a reasonable thing to do for a layer that was many thousands of atoms thick. He presented the fruits of his research to an audience that was composed mainly of physicists. He told them that he had used finite differences to solve the problem and that he had found it to be necessary to divide the layer into many thousands of subintervals in order to attain the required accuracy. Someone in the audience questioned the validity of the model, since each subinterval was of subatomic length. Continuous models are not valid then. This remark baffled the speaker, who could not come up with a convincing answer. The audience left the lecture room, being more convinced than ever that these mathematicians had nothing useful to offer. Of course, after a moment's thought, the correct answer is easily found. The subdivision of the interval occurs *within* the mathematical model. Although the subintervals are deceptively like atoms, they have nothing to do with them. If the physicist agrees that a continuous model is valid when the modelled physical entity contains many thousands of atoms, then a mathematician can prove to him that the solution to that valid model will be approximated more and more accurately when he applies finer and finer meshes.

Finally we should be careful not to overestimate the power of a model and not to be too absolute about the results derived from it. The model is as good or as bad as the assumptions on which it is founded. If it is at all possible, one should test its validity by means of experiments. In the event that the quality of the results is inadequate, one might consider refining the model. Of course, all depends upon one's goals. If one's goal was the procurement of qualitative insight, for instance about the direction in which a process will run or about the order of magnitude of certain effects, then a crude model may do. If one needs numerical answers that have a certain measure of accuracy, then one may opt for mathematical models that are of the less imperfect kind, refined if you like. A report which appeared in the Dutch newspaper NRC Handelsblad on October 29, 1986, is quite illustrative in this respect. The newspaper article dealt with some unexpected flow levels in a newly dug canal in the south-west of the Netherlands: I shall present an English translation of the Dutch original:

Shipping in the Rhine-Scheldt canal is severely hampered by unexpectedly high flow velocities in this canal. These result from the closing last week of the Oesterdam (Oyster dam) between Zuid-Beveland and Tholen ...
... Mr. Hamer (M. Eng.) of the Ministry of Public Works said this morning that computer models have led him to expect that the flow velocities would have increased from 0.5

metres per second to 1.5 metres per second. Now this has turned out to be 2 metres per second. This can cause problems, since the transport of soil from the canal bed increases disproportionately. In the calculations use was made of both a physical model and a one-dimensional mathematical model. Hamer felt that it would be unjustified to state that the models were flawed or that nature had played yet another trick on technology. "We shall proceed by organizing all our data and making the right comparisons. Only then shall we be able to draw the right conclusions", he said.

His remarks about the surprises nature has in store for us show that Mr. Hamer seriously overestimated the power of his mathematical models. Of course, it is quite inconceivable that one should be able to simulate accurately a flow system as complicated as the Rhine-Scheldt delta by means of a one-dimensional model. Qualitative answers are the best one can expect from such a simple model. On the contrary, it is surprising that the Public-Works people, using their simple model, should have been able not only to predict the direction in which the flow field would change, but also two-thirds of its magnitude. This is ample evidence of the expertise these people have acquired over the years.

What about a lecture series on industrial mathematics?

I have been asked to say a few things here in Bari about the application of mathematics in industry, so as to give the audience an idea of what industrial mathematics is all about. From what I have said up to now it should have become clear that this is an impossible task. In principle, the subject encompasses almost everything the human mind can think of. It is not for one person to present an overview of such a wide field, and certainly not if he is allowed only one week to do it in. All I can do is talk about a few things that have come my way. Next year when you invite someone else, you will hear a completely different story. But then, looking at one painting will not make one an expert on painting, nor will reading one poem transform the reader into a master of poetry. Of course, if you never look at a painting or read a poem, ...

Suggestions for further reading

- R. Aris, *Mathematical modelling techniques*, Pitman, 1978.
- W.E. Boyce (Editor), *Case studies in mathematical modeling*, Pitman, 1981.
- M.S. Klamkin, On the ideal role of an industrial mathematician and its educational implications. Educ. Stud. in Math. **3** 244-269, 1971.
- C.C. Lin and L.A. Segel, *Mathematics applied to deterministic problems in the natural sciences*, Macmillan, 1974.
- H.O. Pollak, How can we teach applications of mathematics? Educ. Stud. in Math. **2** 393-404, 1969.
- A.B. Tayler, *Mathematical models in applied mechanics*, Clarendon, 1986.

Some relevant journals

- Journal of Engineering Mathematics (quarterly)
- Mathematical and Computer Modelling (monthly)
- Mathematical Engineering in Industry (quarterly)
- SIAM Journal of Applied Mathematics (bi-monthly)

Temperature distribution within a crystal-growing furnace

Motivation

The purpose of this chapter is to illustrate the process of reduction, which is an important element in the development of a mathematical model. The mathematical model is an image of a part of the world around us. If this part is in itself rather complicated, then a truthful representation of it in mathematical terms means that we must consider an equally complicated mathematical model. This is not always what we want. Therefore, as occurs frequently when making a mathematical model, we follow a process of reduction. The final model is always a compromise between what we would like to achieve and what we can actually realize.

Discussion of the technical background

Crystals are very important semi-manufactured products in the electronics industry. For instance, silicon crystals are used as a basic material for the manufacture of chips. Materials such as gallium arsenide (GaAs) and cadmium telluride (CdTe) are applied in lasers. The basic property of crystals which makes them so desirable is that their atoms are ordered. The way the atoms are ordered depends upon the material in question. We distinguish cubic, hexagonal, rhombic, etc., crystallographic structures. Quite frequently, industry produces large so-called single crystals, the sizes of which can range from a few to many tens of centimetres. In such a single crystal the basic structure and orientation are the same throughout the crystal. The aforementioned products are made by cutting a large single crystal into smaller parts (chips). A basic problem is that these large single crystals are hardly ever perfect. The ideal crystallographic structure is interrupted by faults which are called defects. These defects are created during the crystal-growing process, owing to induced stress levels. Mostly, thermal stresses are to be blamed. These defects often show a tendency to propagate themselves by diffusion throughout the crystal, thereby completely spoiling it. It would seem, therefore, that temperature control is of the utmost importance in the aforesaid production processes.

A few methods are available for the production of these large, mostly cylindrical, single crystals. One of these is the so-called Bridgman-Stockbarger technique. In this process a powdery mixture of the materials the crystal is to be composed of is put into a cylindrical shell or container, which is then closed. This container or crucible is then slowly moved downwards through a vertically positioned furnace. Within this furnace there is a special hot zone in which the temperature is high enough to melt the powder. On leaving the hot zone, the melt recrystallizes, hopefully as the desired perfect single crystal. Some crystal growers favour a narrow but intense hot zone, others an elongated mild one. The present study arose from precisely this conflict.

One of the problems facing the crystal grower is that it is very difficult or even impossible, because of the extremely high temperatures, to obtain direct experimental insight

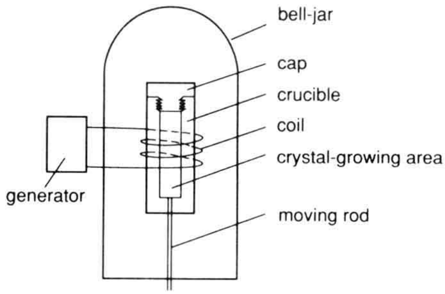

Figure 1: Sketch of crystal-growth system.

into what happens in the hot zone. Moreover, the processes of melting and recrystallization occur within a closed container. Nevertheless, it is still very important for the crystal grower to know the isotherm structure around the solid-liquid interface. A highly curved s-l interface may give rise to induced thermal stresses and hence to the creation of defects. The ideal situation is one with a flat interface. The question is how to bring this about. Often different ways of heating will be necessary to achieve the ideal s-l interface shape during the various stages of the crystal-growing process.

It should be clear that a mathematical model may offer a way out where experimental methods fail. Our purpose will be to write down a set of equations which govern heat transfer around and inside the crucible. Moreover, we shall have to describe boundary conditions which determine the exchange of heat with the surrounding world. It will not come as a surprise that, because of both the complicated geometry and a multitude of phenomena that are physically relevant, this may give rise to models that are extremely complicated. We must therefore ask ourselves right at the beginning of our enterprise whether a broad approach is really what we want, assuming of course that we shall be able to formulate and then solve an all-encompassing model. The average crystal grower is often simply groping about in the dark concerning even the most basic aspects of his process. Therefore, models that provide insight will be most welcome. The maker of a suitable mathematical model therefore faces the task of reducing the problem definition to such an extent that the ensuing model will be manageable, without trivializing the subject matter in an intolerable fashion.

Let us now put these principles to the test, using a Bridgman-Stockbarger configuration with a narrow heating zone. Fig. 1 gives a rough sketch of the furnace system. The crucible is in the centre of the furnace. During the crystal-growing process it moves in a vertical direction, mostly downwards. Sizes of crucibles are from ten to a few tens of centimetres. Widths may range from two to five centimetres. Ideally a crucible is quite a bit longer than it is wide. Surrounding the crucible is a co-called r-f coil. This is a hollow curled-up tube, mostly made of copper, with a diameter of about half a centimetre. Water flows through the tube for cooling purposes. The coil is connected to an electric element which produces a high-frequency current within it. This alternating current in turn pro-

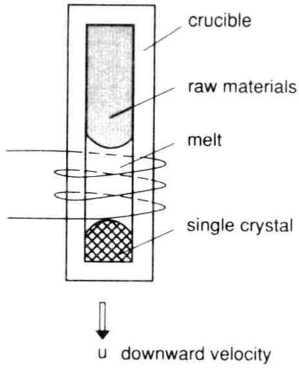

Figure 2: Simplified model of crucible.

duces a strong electromagnetic field in the immediate vicinity of the coil. If the crucible is composed of a suitable material such as graphite, the field will produce intense eddy currents within it. In the majority of cases the penetration depth of the field is limited, so that the currents are strong only in the part of the crucible that is close to the coil. The eddy currents cause dissipation (Joule heating), thereby heating up the crucible and melting the powder inside. When the crucible is moved downwards through the coil, we first see a melting of the powder just above or in the upper reaches of the coil, followed by a recrystallization, hopefully as a perfect single crystal, below the coil (Fig. 2).

The crucible and the coil are completely surrounded by a perfectly sealed bell-shaped structure. It is often necessary to prevent certain gases, e.g. oxygen, from coming into contact with the crucible. Not infrequently the pressure within the bell-jar may exceed the atmospheric pressure many times. The reason is that the vapour pressure of the molten crystal material increases rapidly with temperature. When it is perfectly sealed, this may give rise to high pressures within the crucible. In the case of an imperfect seal, volatilization may occur. A high pressure within the bell-jar may compensate for these detrimental effects.

It would seem that in making a mathematical model we should at least consider the following effects and phenomena:

1. Generation of the electromagnetic field and the manner in which this field gives rise to eddy currents within the graphite the crucible is made of;

2. The production of Joule heat by these eddy currents;

3. The conduction of heat through the crucible to the charge (powder + melt + crystal), then through the charge, and finally towards the colder parts of the crucible away from the coil.

4. Heat transfer from the outer surface of the crucible to the surroundings. This occurs through radiation and convection.