Theory of Vibration with Applications

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Preface

The subject of vibrations has a unique fascination. It is a logical subject explainable by basic principles of mechanics. Unlike some subjects, its mathematical concepts are all associated with physical phenomena which can be experienced and measured. It is a satisfying subject to teach and to share with students. From the first elementary text, *Mechanical Vibrations*, published in 1948, the author has attempted to improve its presentation in keeping with technological advances and experience gained by teaching and practice. In this respect, many teachers and students have contributed with suggestions and interactions over the years.

This new text, which has been almost entirely rewritten, is again a desire on the author's part towards clearer presentation with modern techniques which have become commonplace. In the first five chapters, which deal with single degree of freedom systems and with two degrees of freedom systems, the simplicity of the previous text has been adhered to and, hopefully, improved upon. Since the digital computer is now a commonly available facility, its use in the vibration field is encouraged by some simple examples. In spite of the versatility of the digital computer, the analog computer is still a useful tool, and, in many cases, its use is fully justified. The first five chapters, which keep two degrees of freedom systems on a simple and physical basis, form a background for the understanding of the basic subject of vibrations, which can be covered in a quarter or a semester in a first course on vibration.

In Chapter 6 the concepts of the two degrees of freedom systems are generalized to those of multidegree of freedom systems. The emphasis of this chapter is theory, and, with the aid of matrix algebra, the extension to multidegree of freedom systems can be presented elegantly. All of the basis for coordinate decoupling becomes clear with matrices. Some uncommon ideas of normal modes in forced vibration and the method of state space used commonly in control theory are introduced.

There are many analytical approaches to vibration analysis of complex structures of many degrees of freedom. Chapter 7 presents some of the more useful procedures, and, although most multidegree of freedom systems are today solved on the digital computer, one still needs to know how to formulate such problems for efficient computation and to know some of the approximations which can be made to check the calculations. All of the problems here can be programmed for the computer, but the theory behind the computations must be understood. A digital computation of a Holzer-type problem is illustrated.

Chapter 8 deals with continuous sytems, or those problems associated with partial differential equations. A finite difference approach to beam problems offers an opportunity to solve such problems on the digital computer.

Lagrange's equations, covered in Chapter 9, strengthen again the understanding of dynamical systems presented earlier and broaden one's view for other extensions. For example, the important concepts of the mode summation procedure is a natural consequence of the Lagrangian generalized coordinates. The meaning of constraint equations as physical boundary conditions for modal synthesis is again logically understood through Lagrange's theory.

Chapter 10 treats dynamical systems excited by random forces or displacements. Such problems must be examined from a statistical point of view and, in many cases, the probability density of the random excitation is normally distributed. The point of view taken here is that, given a random record, an autocorrelation can be easily determined from which the spectral density and mean square response can be calculated. The digital computer is again essential for the numerical work.

In Chapter 11 the treatment of nonlinear systems is introduced with emphasis on the phase plane method. When the nonlinearities are small, the methods of perturbation or iteration offer an analytical approach. Results of machine computations for a nonlinear system illustrate what can be done.

Chapters 6 through 11 represent subject matter appropriate for a second course in vibration, which may be covered at the graduate level.

WILLIAM T THOMSON

Contents

1

Oscillatory Motion

1.1 INTRODUCTION

The study of vibration is concerned with the oscillatory motions of bodies and the forces associated with them. All bodies possessing mass and elasticity are capable of vibration. Thus most engineering machines and structures experience vibration to some degree, and their design generally requires consideration of their oscillatory behavior.

Oscillatory systems can be broadly characterized as *linear* or *nonlinear*. For linear systems the principle of superposition holds, and the mathematical techniques available for their treatment are well-developed. In contrast, techniques for the analysis of nonlinear systems are less well known, and difficult to apply. However, some knowledge of nonlinear systems is desirable, since all systems tend to become nonlinear with increasing amplitude of oscillation.

There are two general classes of vibrations—free and forced. *Free vibration* takes place when a system oscillates under the action of forces inherent in the system itself, and when external impressed forces are absent. The system under free vibration will vibrate at one or more of its *natural frequencies*, which are properties of the dynamical system established by its mass and stiffness distribution.

Vibration that takes place under the excitation of external forces is called *forced vibration*. When the excitation is oscillatory, the system is forced to vibrate at the excitation frequency. If the frequency of excitation coincides with one of the natural frequencies of the system, a condition of *resonance* is encountered, and dangerously large oscillations may result. The failure of major structures, such as bridges, buildings, or airplane wings, is an awesome possibility under resonance. Thus, the calculation of the natural frequencies is of major importance in the study of vibrations.

Vibrating systems are all subject to *damping* to some degree because energy is dissipated by friction and other resistances. If the damping is small, it has very little influence on the natural frequencies of the system, and hence the calculations for the natural frequencies are generally made on the basis of no damping. On the other hand, damping is of great importance in limiting the amplitude of oscillation at resonance.

The number of independent coordinates required to describe the motion of a system is called the *degrees of freedom* of the system. Thus a free particle undergoing general motion in space will have three degrees of freedom, while a rigid body will have six degrees of freedom, i.e., three components of position and three angles defining its orientation. Furthermore, a continuous elastic body will require an infinite number of coordinates (three for each point on the body) to describe its motion; hence its degrees of freedom must be infinite. However, in many cases, parts of such bodies may be assumed to be rigid, and the system may be considered to be dynamically equivalent to one having finite degrees of freedom. In fact, a surprisingly large number of vibration problems can be treated with sufficient accuracy by reducing the system to one having a single degree of freedom.

1.2 HARMONIC MOTION

Oscillatory motion may repeat itself regularly, as in the balance wheel of a watch, or display considerable irregularity, as in earthquakes. When the motion is repeated in equal intervals of time τ, it is called *periodic motion*. The repetition time τ is called the *period* of the oscillation, and its reciprocal, $f = 1/\tau$, is called the *frequency*. If the motion is designated by the time function $x(t)$, then any periodic motion must satisfy the relationship $x(t) = x(t + \tau)$.

Irregular motions, which appear to possess no definite period, can be considered to be the sum of a very large number of regular motions of different frequencies. Properties of such motion can be described statistically; the discussion of these properties will be deferred to a later section.

The simplest form of periodic motion is *harmonic motion*. It can be demonstrated by a mass suspended from a light spring, as shown in Fig. 1.2-1. If the mass is displaced from its rest position and released, it will oscillate up and down. By placing a light source on the oscillating mass, its motion can be recorded on a light-sensitive film strip which is made to move past it at constant speed.

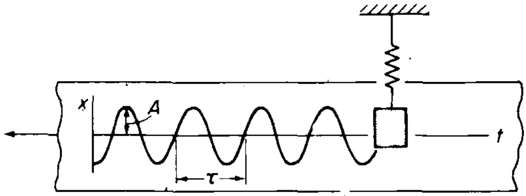

Figure 1.2-1. Recording of harmonic motion.

The motion recorded on the film strip can be expressed by the equation

$$x = A \sin 2\pi \frac{t}{\tau} \tag{1.2-1}$$

where A is the amplitude of oscillation, measured from the equilibrium position of the mass, and τ is the period. The motion is repeated when $t = \tau$.

Harmonic motion is often represented as the projection on a straight line of a point that is moving on a circle at constant speed, as shown in Fig. 1.2-2. With the angular speed of the line op designated by ω, the displacement x can be written as

$$x = A \sin \omega t \tag{1.2-2}$$

The quantity ω is generally measured in radians per second, and is referred

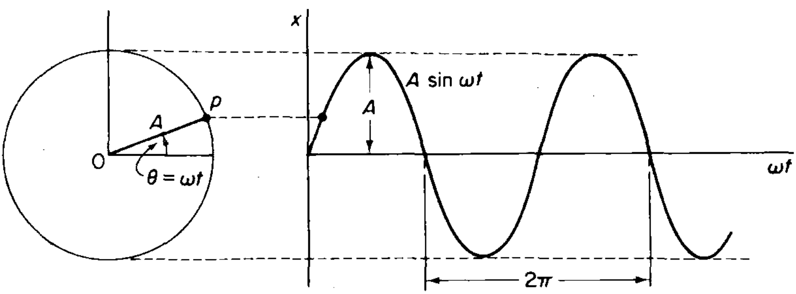

Figure 1.2-2. Harmonic motion as projection of a point moving on a circle.

to as the *circular frequency*. Since the motion repeats itself in 2π radians, we have the relationship

$$\omega = \frac{2\pi}{\tau} = 2\pi f \qquad (1.2\text{-}3)$$

where τ and f are the period and frequency of the harmonic motion, usually measured in seconds and cycles per second respectively.

For the motion of a point around a circle, it is convenient to use an imaginary axis i and let the radius of the circle be represented by a complex quantity z called a *phasor*.

The phasor z is expressed by the equation

$$z = Ae^{i\theta} = A\cos\theta + iA\sin\theta \qquad (1.2\text{-}4)$$

which define the real and imaginary components. With $\theta = \omega t$, the components vary sinusoidally with time

$$\text{Re}\, z = A\cos\omega t$$
$$\text{Im}\, z = A\sin\omega t$$

It is often necessary to consider two harmonic motions of the same frequency but differing in phase by ϕ. The two motions may be expressed by the phasors

$$z_1 = A_1 e^{i\omega t}$$
$$z_2 = A_2 e^{i(\omega t + \phi)}$$

where A_1 and A_2 are real numbers. The second phasor can be further rewritten as

$$z_2 = A_2 e^{i\phi} e^{i\omega t} = \bar{A}_2 e^{i\omega t} \qquad (1.2\text{-}5)$$

where $\bar{A}_2$ is now a complex number. This form is often useful in problems involving harmonic motion.

The addition, multiplication, or raising to powers of phasors follow simple rules which are given in Appendix A. With harmonic motion expressed as phasors, their manipulations are easily carried out.

The velocity and acceleration of harmonic motion can be simply determined by differentiation of Eq. (1.2-2). Using the dot notation for the derivative, we obtain

$$\dot{x} = \omega A\cos\omega t = \omega A\sin\left(\omega t + \frac{\pi}{2}\right) \qquad (1.2\text{-}6)$$

$$\ddot{x} = -\omega^2 A\sin\omega t = \omega^2 A\sin(\omega t + \pi) \qquad (1.2\text{-}7)$$

Thus the velocity and acceleration are also harmonic with the same frequency of oscillation, but lead the displacement by $\pi/2$ and π radians respectively, as shown in Fig. 1.2-3.

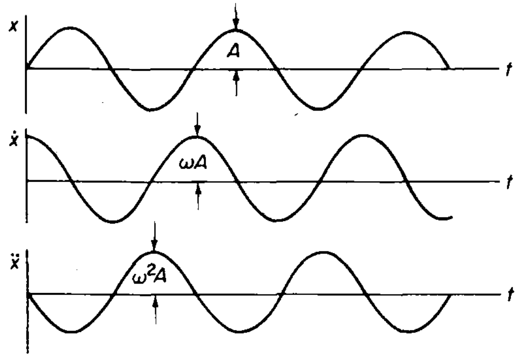

Figure 1.2-3. In harmonic motion, the velocity and acceleration lead the displacement by $\pi/2$ and π.

Examination of Eqs. (1.2-2) and (1.2-7) reveals that

$$\ddot{x} = -\omega^2 x \tag{1.2-8}$$

so that in harmonic motion the acceleration is proportional to the displacement and is directed towards the origin. Since Newton's second law of motion states that the acceleration is proportional to the force, harmonic motion can be expected for systems with linear springs with force varying as kx.

1.3 HARMONIC ANALYSIS

It is quite common for vibrations of several different frequencies to exist simultaneously. For example, the vibration of a violin string is composed of the fundamental frequency f and all its harmonics $2f$, $3f$, etc. Another example is the free vibration of a multidegree-of-freedom system, to which the vibrations at each natural frequency contribute. Such vibrations result in a complex waveform which is repeated periodically as shown in Fig. 1.3-1.

The French mathematician J. Fourier (1768–1830) showed that any periodic motion can be represented by a series of sines and cosines which are harmonically related. If $x(t)$ is a periodic function of the period τ, it is represented by the Fourier series

$$\begin{aligned} x(t) = \frac{a_0}{2} &+ a_1 \cos \omega_1 t + a_2 \cos 2\omega_1 t + \cdots \\ &+ b_1 \sin \omega_1 t + b_2 \sin 2\omega_1 t + \cdots \end{aligned} \tag{1.3-1}$$

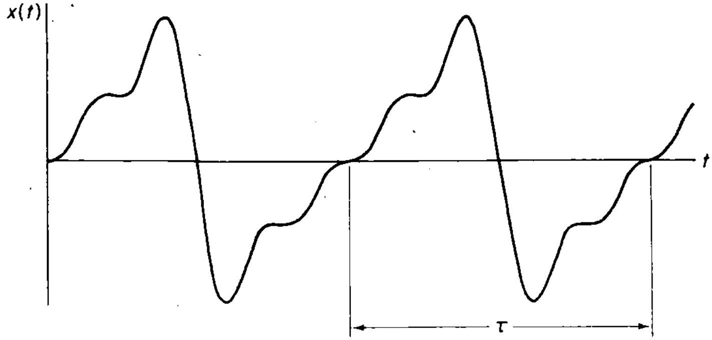

Figure 1.3-1. Complex periodic motion of period τ.

where $\omega_1 = 2\pi/\tau$ is the fundamental frequency. To determine the coefficients a_n and b_n, we multiply both sides of Eq. (1.3-1) by $\cos n\omega_1 t$ or $\sin n\omega_1 t$, and integrate each term over the period τ. Recognizing the following relations,

$$\int_{-\tau/2}^{\tau/2} \cos n\omega_1 t \cos m\omega_1 t \, dt = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{\pi}{\omega_1} & \text{if } m = n \end{cases}$$

$$\int_{-\tau/2}^{\tau/2} \sin n\omega_1 t \sin m\omega_1 t \, dt = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{\pi}{\omega_1} & \text{if } m = n \end{cases}$$

$$\int_{-\tau/2}^{\tau/2} \cos n\omega_1 t \sin m\omega_1 t \, dt = \begin{cases} 0 & \text{if } m \neq n \\ 0 & \text{if } m = n \end{cases}$$

all terms except one on the right side of the equation will be zero, and we obtain the results

$$a_n = \frac{\omega_1}{\pi} \int_{-\tau/2}^{\tau/2} x(t) \cos n\omega_1 t \, dt \qquad (1.3\text{-}2)$$

$$b_n = \frac{\omega_1}{\pi} \int_{-\tau/2}^{\tau/2} x(t) \sin n\omega_1 t \, dt \qquad (1.3\text{-}3)$$

Returning to Eq. (1.3-1) and examining the two terms at one of the frequencies, $n\omega_1$, their sum can be written as

$$\begin{aligned} & a_n \cos n\omega_1 t + b_n \sin n\omega_1 t \\ &= \sqrt{a_n^2 + b_n^2}\left\{\frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos n\omega_1 t + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin n\omega_1 t\right\} \\ &= c_n \cos(n\omega_1 t - \phi_n) \end{aligned}$$

where

$$c_n = \sqrt{a_n^2 + b_n^2} \tag{1.3-4}$$

and

$$\tan \phi = \frac{b_n}{a_n} \tag{1.3-5}$$

Thus c_n and ϕ_n (or a_n and b_n) completely define the harmonic contribution of the periodic wave.

When c_n and ϕ_n are plotted against the frequency $n\omega_1$ for all n, the result is a series of discrete lines at ω_1, $2\omega_1$, $3\omega_1$, etc., as shown in Fig. 1.3-2. Such plots are called the *Fourier spectrum* of the waveform.

With the aid of the digital computer, harmonic analysis today is efficiently carried out in minimum time. A new computer algorithm introduced recently, known as the Fast Fourier Transform,* further reduces the computational time.

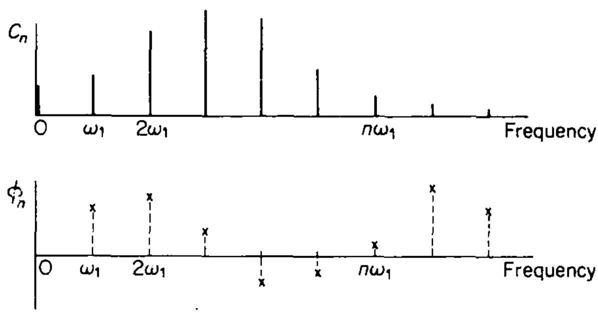

Figure 1.3-2. Fourier spectrum for a periodic time function.

1.4 TRANSIENT TIME FUNCTION

A function that exists only for a limited time and is zero at all other times is called a *transient time function.* Such functions are not periodic. Figure 1.4-1 shows a typical pressure variation of a sonic boom that is a transient time function. The force of impact during a collision of two bodies is another example.

*J. W. Cooley and J. W. Tukey, "An Algorithm for the Machine Calculation of Complex Fourier Series," *Mathematics of Computation* 19; 90 (April 1965), pp. 297–301.

See also—"Special Issue on Fast Fourier Transform," *IEEE Trans. on Audio & Electroacoustics*, Vol. AU-15, No. 2 (1967).

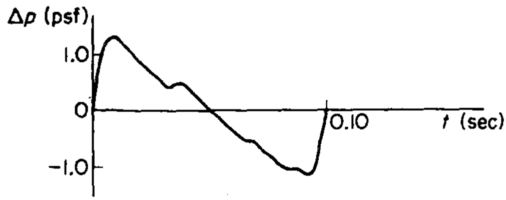

Figure 1.4-1. The sonic boom (N-Wave) is a transient time function.

The response of a mechanical system to an impulse or shock is generally referred to as a *transient response*. Due to the presence of damping, such vibrations will die down after the excitation is over.

Since transient waves are not periodic, the method of Fourier series is *not* applicable. However, nonperiodic functions can be analyzed for their frequency content by the method of Fourier Transforms (see Chapter 10). In contrast to the discrete frequency spectrum of the periodic function, the frequency spectrum of a transient time function is continuous.

1.5 RANDOM TIME FUNCTION

The types of functions we have considered up to now can be classified as deterministic, i.e., mathematical expressions can be written which will determine their instantaneous values at any time t. There are, however, a number of physical phenomena that result in nondeterministic data where future instantaneous values cannot be predicted in a deterministic sense. As examples, we can mention the output of a noise generator, the heights of waves in a choppy sea, ground motion during an earthquake, and pressure gusts encountered by an airplane in flight. These phenomena all have one thing in common: the unpredictability of their instantaneous value at any future time. Nondeterministic data of this type are referred to as *random time functions.*

A sample of a typical random time function is shown in Fig. 1.5-1. In spite of the irregular character of the function, certain averaging proce-

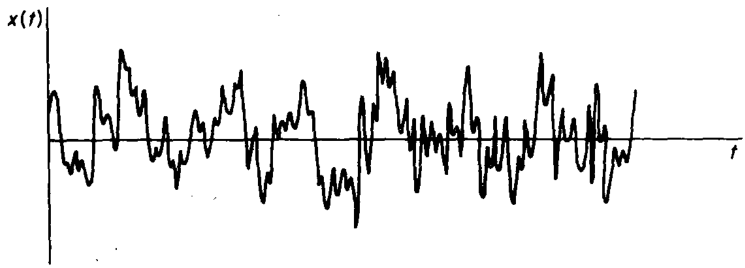

Figure 1.5-1. A record of random time function.