# Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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# I. INTRODUCTION

Take an initially straight rod of circular cross section which is composed of an isotropic material. Apply an axially symmetric compressive load to it. In general, the rod will assume an equilibrating configuration. However, this equilibrium is not isolated. Because of the symmetry of the rod and the load, the configuration will determine an entire family or "orbit" of other equilibrating configurations which is gained by rotating the image of the original configuration about the axis of symmetry through any angle.

Now perturb the compressive load by additional loads which break the axial symmetry. How does this perturbation alter the orbit of equilibrating configurations for the unperturbed problem? Will the perturbed problem also have an orbit of equilibrating configurations which is in bijective correspondence with the original orbit, or will the original orbit break or disappear with the perturbation? To what extent does the alteration of the original orbit depend upon the material comprising the rod? To what extent does it depend upon the manner in which the perturbing load breaks the axial symmetry of the original load?

This work indicates how we can address these questions using the methods of modern analysis and the theory of singularities or bifurcations in the presence of symmetry. We direct the work towards two groups of researchers: mathematicians and mechanicians. For the mathematician it illustrates how these tools can contribute greatly towards resolving problems of current interest in mechanics. Conversely, the rod problem gives the mechanician a concrete context in which to learn how to apply these nonlinear mathematical tools.

Generally speaking, we can isolate three aspects of the rod problem we've proposed. First, there is a "symmetric buckling" problem. It manifests itself in the buckling of the rod under axially-symmetric compression. Second, there is a "pure orbit-breaking" problem. It occurs prior to the buckling of the rod, when we fracture the orbit of equilibrating configurations to the symmetric problem by applying an asymmetric perturbing load. Finally, there is the "full" or "coupled" problem. We compress the rod sufficiently to buckle it, while simultaneously exerting perturbing loads which break the axial symmetry.

The analysis of the symmetric buckling problem has its origins in Euler's study of the elastica, with the establishment of the existence of a positive compression $p_0$ at which buckling of a centerline of a rod first occurs. By 1976 S. Antman and his associates had begun the analysis of the buckling of a nonlinear rod with circular cross section under axially-symmetric compression (see [5], [9]). They used the bifurcation theory of Crandall and Rabinowitz. In 1985, E. Buzano, G. Geymonat, and T. Poston applied to S. Antman's model the symmetric bifurcation theory developed by M. Golubitsky and D. Schaeffer [23] to study the buckling of a prismatic rod subject to axially symmetric compression [4]. In each case there is a distinguishing mathematical feature of the development. While the equations governing the behavior of the rod maintain their symmetry as the load increases, equilibrating configurations arise which exhibit less symmetry than the equations.

The "pure orbit-breaking" problem differs from the first problem in that the perturbing load actually breaks the symmetry of the equations governing the behavior of the rod. The perspective which we wish to use to analyze it was developed in 1983 by D. Chillingworth, J. Marsden, and Y. Wan in [1, 2, 3] to study Stoppelli's Problem, which is a problem in the three-dimensional theory of elasticity where analogous questions arise. Their contribution was to reformulate the symmetry-breaking problem as a problem of bifurcation defined on a manifold which was a group. The formulation then allows us to use the theory of singularities and bifurcation to carry out the analysis.

The results for these two aspects suggest that we may analyze the full problem by formulating it as a bifurcation problem on a space which is a semidirect product of a group and a vector space on which the group acts by means of a representation.

In this work we take the first step towards analyzing the full problem. First, we formulate as bifurcation problems all three aspects of the rod problem, under the assumptions of the Kirchhoff rod theory. We present the model for the rod in the Kirchhoff theory, formulate the equilibrium problem in a variational setting, and extract from the equilibrium problem the pure orbit-breaking problem, the pure symmetric buckling problem, and the full problem as bifurcation problems.

We then adapt the approach of [1, 2, 3] to analyze the pure orbit-breaking problem for the rod. The analysis predicts how an orbit of equilibrating configurations generated by the straight configuration for a rod in the Kirchhoff theory subject to an axially symmetric compressive load alters when we perturb the given load by dead loads which break the axial symmetry. The methods lead to a classification of the perturbing loads. For each type of perturbing load in the classification we determine whether or not the orbit of equilibrating configurations for the unperturbed problem breaks. If it does, we determine qualitatively how the orbit alters. We also determine whether or not the alteration depends upon the material comprising the rod. If it does, we examine whether the alteration is determined by the first-order (linear) approximation to the response of the material, or by the higher-order (nonlinear) approximations to the response. We illustrate these conclusions using specific perturbing loads.

We then comment on how we may use the notion of "unfoldings" in the theory of bifurcation to begin to analyze the full problem. The comments indicate what obstructions we encounter when we try to carry out the analysis. We close by summarizing the open questions, and by indicating some avenues for further investigation.

To apply the methods of the singularity theory we must formulate the rod problem in the language of modern analysis.

Consequently, in Sections 2 through 4 we develop in detail the equilibrium problem for a rod in the special Cosserat and the Kirchhoff theories as a problem involving a mapping between manifolds of functions. The formulation extends to a global setting the analytical formulation of the equilibrium problem for the two theories given by S. Antman in [9]. The models are examples of the general Hamiltonian structure for a rod model in the convective representation of [33] §6.

In Section 2 we present the geometry of the manifolds of configurations for a rod in the two theories. In Section 3 we specify the kind of perturbing loads we will examine (Assumption 3.2), and present the geometry of the spaces of admissible loads for each of the two theories. In Section 4 for each theory we show how the system of ordinary differential equations specifying an equilibrium configuration for the rod determines a mapping from the manifold of configurations into the space of loads (Theorem 4.20). Restricting attention to the Kirchhoff theory and assuming the material comprising the rod is hyperelastic, we extract from the equilibrium problem the general symmetry-breaking problem of interest as a problem of finding the singularities of a function (Problem 4.25), which is a particular type of bifurcation problem.

The development in Sections 2 and 3 is directed prinicipally towards mechanicians. Readers interested principally in the analysis of the mathematical model, and not in its development may pass directly to Section 4 with only a modicum of discomfort.

In Sections 5 through 7 we analyze Problem 4.25. In Section 5 we reduce the problem from one defined on a function space to one which is specified on a finite dimensional space. How the problem reduces depends upon the pressure of the compressive load. We examine the reduction for two cases: when the pressure approaches the value at which the rod first begins to buckle (Theorem 5.18), and when the pressure is at the first buckling value (Theorem 5.24). From the two reductions we formulate three finite-dimensional bifurcation problems (Problems 5.20, 5.26, and 5.27). They constitute the three aspects of the rod problem we presented at the beginning of the introduction.

We then restrict attention to the analysis of the pure orbit-breaking problem (Problem 5.20). We use the orbit generated by the straight configuration of the rod as the trivial orbit of equilibrating configurations for the bifurcation problem. How we proceed to resolve the reduced problem depends upon the nature of the perturbing load. In Section 6 we produce a classification for the perturbing loads (Theorem 6.20). In Section 7 we determine how the trivial orbit of equilibrating configurations for the symmetrically loaded problem alters for each type of perturbing load arising from the classification theorem (Theorems 7.6, 7.10, and 7.11). Some types of perturbing loads alter the orbit in a way which is independent of the material comprising the rod. Other types alter the orbit in ways which are determined by the first-order (linear) approximation of the response of the material comprising the rod, where the approximation is taken relative to the trivial configuration. Still other types of perturbing loads alter the orbit in ways which are determined by the higher-order (nonlinear) approximations to the response of the material comprising the rod taken relative to the trivial configuration.

To aid the comprehension we illustrate the principal results of Sections 3, 6 and 7 using specific perturbing loads. In Section 3 we present a variety of three-dimensional force distributions which produce the kind of rod loads we are admitting as perturbations (Examples 3.18-3.21). In Section 6 we illustrate the classification theorem by classifying the rod loads that were presented in Section 3 (Examples 6.21-6.24). In Section 7 we illustrate the various conclusions about how the orbit alters and what factors influence the alteration using the specific rod loads which were classified in Section 6 (Examples 7.14-7.17). Example 7.17 and the subsequent remarks are particularly worthy of note.

In Section 8 we comment on how we may begin to analyze the full problem (Problem 5.26) using the notion of unfoldings in the bifurcation theory. The comments indicate what obstructions we encounter using this perspective, and ways by which we may overcome them. We close by some other problems which may be investigated using the singularity theory.

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## II. THE SPACES OF CONFIGURATIONS

In this section we specify the spaces of configurations and strains for the rod in the special Cosserat and the Kirchhoff theories. We obtain kinematic models for the two rod theories in the spatial and convective representations of [33]. We relate their descriptions to the more classical descriptions in terms of director vectors. We examine some geometric features of the spaces which will be of importance in the latter sections. Finally, we generalize the differentiability class of the configurations.

### II.1. *The Spaces of Classical Configurations for a Rod*

Fix an origin and a triad $\{ \underline{e}_j \mid j = 1, 2, 3 \}$ of orthonormal vectors in the physical space $E^3$. View a rod as a slender three-dimensional body $\mathcal{B}$ whose image in a reference configuration is a circularly cylindrical solid of length 1 and radius R, and whose line of centroids lies along the $\underline{e}_3$ axis with its left end at the origin. Identify a material point p in $\mathcal{B}$ with its coordinates $X = (X^1,X^2,S)$ in the reference configuration. Identify the parameter S, $0 \le S \le 1$, with the corresponding point (0,0,S) on the line of centroids, or centerline for the rod. For S fixed, call the planar surface

$$\mathcal{B}(S) = \{ X = (X^1,X^2,S) \mid (X^1)^2 + (X^2)^2 \le R^2 \}$$

the *material cross section* for the rod at the point S on the centerline.

As in [6] view a rod theory as describing the behavior of a constrained three-dimensional material body. Assume that a configuration such a constrained body can attain is determined by specifying a position vector function x(S) and a pair of orthonormal vector functions $\underline{d}_\alpha(S)$, $\alpha = 1, 2$. Interpret x(S) as specifying the position of the centerline points in the new configuration, and interpret the $\underline{d}_\alpha(S)$ as determining the orientation of the plane of the section $\mathcal{B}(S)$ in the new configuration and a line in the plane.

The relation expressing how x and the $\underline{d}_\alpha$ specify a configuration $\mathfrak{X}$ for the three-dimensional body can be quite general (see [6], p. 323). For convenience, assume a particularly simple, but acceptable relation:

$$\mathfrak{X}(X^1,X^2,S) = x(S) + \phi^\alpha(X^1,X^2,S)\underline{d}_\alpha(S), \tag{2.1}$$

where the summation is implied. The development we present remains valid for more general expressions.

As (2.1) indicates, the vector functions characterize those three-dimensional configurations for the rod for which the centerline may flex, twist, and elongate, and the cross sections for the rod may rotate and shear relative to the centerline. However, the cross sections remain planar and undistorted in shape.

From the three-dimensional invertibility condition ([6], p. 312) require

$$x'(S)\cdot\underline{d}_1(S) \times \underline{d}_2(S) \neq 0, \tag{2.2}$$

or that the tangent to the centerline not lie in the plane of the cross section in any configuration.

For x, $\underline{d}_\alpha$ satisfying (2.2), define

$$\underline{d}_3(S) = \operatorname{sgn}(x'(S)\cdot\underline{d}_1(S) \times \underline{d}_2(S))\underline{d}_1(S) \times \underline{d}_2(S). \tag{2.3}$$

Then $\{\underline{d}_j \mid j = 1, 2, 3\}$ form an orthonormal triad of vector functions, and

$$x'(S)\cdot\underline{d}_3(S) > 0. \tag{2.4}$$

Call a collection $\{ x, \underline{d}_j, j = 1, 2, 3 \}$ of $C^k$ vector functions satisfying (2.2) through (2.4) a $C^k$ *configuration* for the rod in the *special Cosserat theory.*

A particular case of (2.2) is the requirement

$$| x'(S) \cdot \underline{d}_1(S) \times \underline{d}_2(S) | \equiv 1. \tag{2.5}$$

As (2.1) indicates, vector functions satisfying (2.5) describe configurations for the rod in which the centerline is inextensible, and cross sections do not shear relative to the centerline. Call (2.5) the *Kirchhoff hypothesis,* and call a collection $\{ x, \underline{d}_j, j = 1, 2, 3 \}$ of $C^k$ vector functions satisfying (2.3) through (2.5) a $C^k$ *configuration* for the rod in the *Kirchhoff theory.* Notice that $x'(S) = \underline{d}_3(S)$ for such a configuration.

We now represent the space of all $C^k$ configurations for the rod in either theory in a manner which will allow us to take advantage of the elements of modern analysis. First, we simplify the description of a rod configuration by introducing orthogonal transformation-valued functions.

**2.1 Lemma.** *Let $k$ be an integer, $k \geq 1$.*

*a) Let $O(E^3)$ be the space of orthogonal linear transformations of $E^3$ for the given origin. Let $\gamma(S) \in O(E^3)$ be a $C^k$ function on $I = [0,1]$. Then $\gamma$ determines a unique $C^k$ configuration in the Kirchhoff theory for which the left end of the centerline for the rod is fixed at the origin, and conversely.*

*b) Let $\chi = (x,\gamma)$, where $x(S) \in E^3$ and $\gamma(S) \in O(E^3)$ are $C^k$ functions on $I$ satisfying*

$$\gamma(S)\underline{e}_3 \cdot x'(S) > 0. \tag{2.6}$$

*Then $\chi$ determines a unique $C^k$ configuration in the special Cosserat theory, and conversely.*

**Proof.**

a) Given $\gamma$, define $x(S)$ and $\underline{d}_j(S)$ by requiring

$$\underline{d}_j(S) = \gamma(S)\underline{e}_j, \quad j = 1, 2, 3, \tag{2.7}$$

$$x(S) = \int_0^S \underline{d}_3(t)dt = \int_0^S \gamma(t)\underline{e}_3 dt. \tag{2.8}$$

Then $x'(S) = \underline{d}_3(S)$, and conditions (2.3) through (2.5) follow. Also, $x(0) = 0 \in E^3$. Conversely, given (2.3) through (2.5) and $x(0) = 0$, we can solve (2.7) uniquely for $\gamma(S)$. Since $x'(S) = \underline{d}_3(S)$, (2.8) follows. If the vector functions are $C^k$, then $\gamma$ is also.

b) Given $\chi = (x,\gamma)$, define the vector functions by (2.7). By (2.6), the vector functions will satisfy (2.2) through (2.4). Conversely, given the vector functions, (2.7) specifies $\chi$. If the vector functions are $C^k$, then $\chi$ is also. ∎

**2.2 Definition.** *Let $k \geq 1$, and let $I = [0,1]$.*

*a) Take the space of $C^k$ configurations in the Kirchhoff theory for a rod with its left end fixed at $0 \in E^3$ to be*

$$M = C^k(I, O(E^3)).$$

*b) Take the space of $C^k$ configurations for a rod in the special Cosserat theory to be*

$$N = \{\ \chi = (x,\gamma) \in C^k(I, E^3 \times O(E^3))\ |\ (2.6) \text{ holds } \}.$$ ∎

**2.3 Lemma.** *For $k \geq 1$, $M$ and $N$ are differentiable manifolds.*

**Proof.**

a) By [7] § 4.4, $O(E^3)$ is a closed submanifold of the Banach space $L(E^3)$. By [8] § 13, it then follows that M is a closed submanifold of the Banach space $C^k(I,L(E^3))$.

b) From part a), $C^k(I, E^3 \times O(E^3)) = C^k(I,E^3) \times C^k(I,O(E^3))$ is a product manifold. Condition (2.6) characterizes N as an open set in this manifold; hence, it is a differentiable manifold. ∎

*Remark.* The Euler angles are a coordinate system on a portion of $O(E^3)$. By [8], p. 50, $C^k$ curves of Euler angles can be used to construct a coordinate system over a portion of $C^k(I,O(E^3))$. It is this coordinate representation on M which is used in [4], [9], and [10] in their studies of rod problems.

We close the subsection by identifying groups of transformations which act effectively on M and N.

**2.4 Definition.** *Let*

$$SO(3) = \{ Q \in O(E^3) \mid \det Q = +1 \},$$
$$O(2) = \{ Q \in O(E^3) \mid Q\underline{e}_3 = \underline{e}_3 \},$$
$$SO(2) = O(2) \cap SO(3).$$

*Let* $R_\pi$, $J \in O(2)$, $\Sigma_3 \in O(E^3)$ *be defined by*

$$R_\pi \underline{e}_\alpha = -\underline{e}_\alpha,$$
$$J\underline{e}_1 = -\underline{e}_1, \quad J\underline{e}_2 = \underline{e}_2,$$
$$\Sigma_3 \underline{e}_\alpha = \underline{e}_\alpha, \quad \Sigma_3 \underline{e}_3 = -\underline{e}_3,$$

*for* $\alpha = 1, 2$. *Denote by* $\langle\Sigma_3\rangle$ *and* $\langle J\Sigma_3\rangle$ *the subgroups generated by the elements enclosed in the brackets (or equivalently, the smallest subgroups of* $O(E^3)$ *containing the elements enclosed in the brackets. Set*

$$G_s = \langle O(2), \Sigma_3\rangle,$$
$$\Gamma = G_s \cap SO(3),$$
$$\Pi_1 = O(E^3) \times G_s,$$
$$\Pi = O(E^3) \times \Gamma,$$
$$\Pi_s = SO(3) \times G_s,$$

*where the latter three groups are external direct product groups, and the latter two groups are subgroups of* $\Pi_1$.

*a) For* $g = (Q_1, Q_2) \in \Pi_1$, *for* $T \in L(E^3)$, *define* $g\cdot T \in L(E^3)$ *by*

$$g\cdot T = Q_1 T Q_2^T.$$

*b) For $g \in \Pi_1$, $\gamma \in M$ and $\chi = (x,\gamma) \in N$, define $g\gamma \equiv \mathcal{T}_g\gamma \in M$ and $g\chi \equiv \mathcal{T}_g\chi \in N$ by*

$$(\mathcal{T}_g\gamma)(S) = g\cdot\gamma(S) \tag{2.9}$$

$$(\mathcal{T}_g\chi)(S) = (Q_1x(S), g\gamma(S)). \blacksquare \tag{2.10}$$

**2.5 Lemma.** *$\Pi_1$ acts on $M$ and $N$ as a group of transformations.*

*a) $\Pi_1$ acts effectively on $N$, in that $\mathcal{T}_g\chi = \chi$, $\forall\chi \in N$ iff $g = (1,1) \in \Pi_1$.*

*b) Let*

$$H = \{\ g \in \Pi_1 \mid \mathcal{T}_g\gamma = \gamma,\ \forall\gamma \in M\ \}.$$

*Then $H = \langle(-1,-1)\rangle$.*

*c) $\Pi$ and $\Pi_s$ each act effectively on $M$.*

**Proof.** Equations (2.9) and (2.10) imply that $\Pi_1$ acts as a group of transformations on M and N. Since $O(E^3)$ acts effectively on $\mathbb{R}^3$, the equations also imply that $\Pi_1$ acts effectively on N.

b) Let $g = (Q_1,Q_2)$. By choosing in turn $\gamma(S) \equiv Q_1^T$ and $\gamma(S) \equiv Q$ fixed, but arbitrary, (2.9) implies $g \in H$ if and only if $Q_1 = Q_2$ and $Q_1$ commutes with all elements of $O(E^3)$, or equivalently, belongs to the center of $O(E^3)$. As $\langle -1\rangle$ is the center of $O(E^3)$ (see [7], chapter 4), part b) follows.

c) As H is a normal subgroup of $\Pi_1$, the factor group $\Pi_1/H$ acts effectively on M under the induced action. Hence, it suffices to show that $\Pi_1$ is isomorphic to the direct sums

$$\Pi_1 \approx \Pi \oplus H \approx \Pi_s \oplus H.$$

We establish the first direct sum. The second follows by an analogous argument. As H constitutes the center of $\Pi_1$, the subgroups $\Pi$ and H commute. Since $\Gamma$ is contained in SO(3), the intersection of $\Pi$ and H is trivial. So $\Pi \oplus H$ is a subgroup of $\Pi_1$. Finally, for $g = (Q_1,Q_2) \in \Pi_1$, $\Phi g = \big((\det Q_1)g, (\det Q_1)(1,1)\big)$ specifies a group homomorphism of $\Pi_1$ onto $\Pi \oplus H$, giving the isomorphism. The induced action of the factor group then may be identified with the action of $\Pi$ on M given by (2.9). ■