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Approximation Methods in Optimal and Adaptive Control

J. H. WESTCOTT, J. J. FLORENTIN and J. D. PEARSON

Introduction

Both optimal and adaptive control problems can now be treated by the same decision theory approach[1]. Typical practical problems can be formulated in the required mathematical terms, but at present there is still difficulty in determining actual numerical solutions to problems of realistic size and complexity. It seems likely that a variety of approximate computation techniques will be developed, each with a restricted range of application.

Approximation is necessary due to the very extensive calculation called for, using multi-stage decision methods. These become sufficiently time-consuming, even when performed at the fast speeds of modern digital computers, for abbreviation to be necessary. Approximation becomes attractive due to the limited time allowable on the time scale of the dynamic process it is desired to control.

Approximation may be attempted either in the setting up of a particular problem or during the numerical process of solution. Whilst approximations in the setting up procedure are somewhat difficult to treat analytically they are likely to be important in practical applications of decision theory, and a worked example is presented to demonstrate some aspects of this. In numerical techniques two broad approaches can be distinguished—trajectory and function space methods. Certain approximation techniques in both these classes are discussed and illustrated with simple examples.

Review of Basic Equations

To introduce the approach and notation, a brief review of the basic equations will be given. More complete derivations can be found[1, 2]. The complete system including inputs must first be described by a set of state coordinates $\boldsymbol{x}(t)$. These are an absolute description of the system at the time instant t. The state coordinates may be obvious physical quantities such as position or velocity, or may be statistical quantities such as the mean and the variance. The motion of the system is conveniently described by a set of first order vector differential equations:

$$\boldsymbol{x}' = A(\boldsymbol{x}, \boldsymbol{u}, t) \tag{1}$$

where $\boldsymbol{u}$ is a vector of control variables. For reasons of simplicity no random components are included.

An optimal control function $\boldsymbol{u}(t)$ is to be found which maximizes or minimizes a given performance index, subject to constraints. Typically the performance index will be a path integral over a defined time interval t, T with the system starting at a given position $\boldsymbol{x}$. Here the performance index will be defined as

$$f(\boldsymbol{x}(t), t) = \min_{u} \int_t^T L(\boldsymbol{x}, y, t)\, \mathrm{d}\lambda \tag{2}$$

Isolating a small part of the time interval t, T, it can be seen[8] that the following problem is equivalent to (2)

$$f(\boldsymbol{x}(t), t) = \min_{u} \left[\int_t^{t+\Delta} L(\boldsymbol{x}, u, \lambda)\, \mathrm{d}\lambda + f(\boldsymbol{x}(t+\Delta), t+\Delta) \right] \tag{3}$$

Using a Taylor series expansion for the third term, a partial differential equation for the performance index can be found in the form

$$\frac{\partial f}{\partial t} = -\min_{u} \left[A_i \frac{\partial f}{\partial x_i} + L \right] \tag{4}$$

The solution of this partial differential equation reveals the performance index as a function of the initial position and time interval. The minimizing control law $\boldsymbol{u}(t)$ can be found as a function of the partial derivatives of f.

With appropriate modifications analogous equations can be developed for discrete time systems and for those containing random elements. When there are random elements the performance index must be changed to an ensemble average of path integrals[3].

For deterministic systems eqn (3) can be expressed in an alternative form by taking its characteristics. Define the function H as

$$H\left(\boldsymbol{x}, \frac{\partial f}{\partial \boldsymbol{x}}, t\right) = \sum_1^n A_i \frac{\partial f}{\partial x_i} + L \tag{5}$$

Employing the normal theory of partial differential equations, a new set of variables, p_i, is defined with a vector $\boldsymbol{p} = \partial f / \partial \boldsymbol{x}$. The set of equations for the characteristics is then

$$\boldsymbol{x} = \frac{\partial H}{\partial \boldsymbol{p}} \qquad \boldsymbol{p} = -\frac{\partial H}{\partial \boldsymbol{x}} \tag{6}$$

To solve these equations the boundary conditions of the $\boldsymbol{p}$ or co-state variables are required. As derived by Rozonoer[4], the problem can correspond either to a fixed end point variational problem in which

$$\boldsymbol{x}(T) \text{ known} \qquad \boldsymbol{p}(T) \text{ unknown} \tag{7}$$

or to a free end point variational problem in which

$$\boldsymbol{x}(T) \text{ may be chosen} \qquad \boldsymbol{p}(T) = 0 \tag{8}$$

In either case the problem reduces to a two-point boundary value one.

Computational Methods

Eqns (3) and (6) give rise to two families of computational techniques. The first is of wide application, it computes $\boldsymbol{f}(\boldsymbol{x}, t)$ over the $\boldsymbol{x}$ space at successive time intervals t_r. It will be termed the function space method. A predominant difficulty in the function space approach is that of storing a multi-dimensional function in a digital computer. The obvious methods become prohibitive, since for a three-dimensional function with a hundred points in each dimension a storage space of 10^6 words would be necessary.

The second method applies only to deterministic systems and is termed the trajectory method. Only half of the boundary conditions at each end of the trajectory are known and the usual difficulties are encountered.

Setting Up Problems

In practical situations the problem of control is usually not completely defined. In many cases it is possible to complete the specification by the selection of constraints which enable a simpler solution to be achieved than would otherwise be possible. The factors which can most usefully and easily be varied are the mathematical form of the performance index, the precise specification of the control constraints, and the selection of continuous or discrete time working.

To illustrate the effects of varying the formulation consider a typical non-linear problem. A vehicle, moving with constant velocity v in the x_1, x_2 phase plane, is to be guided from an initial point P to a final target O. Guidance is affected by adjusting the rate of turn of the vehicle. For practical reasons the maximum rate of turn is constrained. In accordance with *Figure 1* the dynamic equations of the problem are:

$$x_1' = v\cos x_3$$
$$x_2' = v\sin x_3 \qquad (9)$$
$$x_3' = u$$

where x_3 is the angle the velocity vector makes with the x_1 axis.

Minimum Time Trajectories

Take the performance index to be

$$f(x,t) = \min_u \left[\int_t^T h\cdot dt\right]$$

with the definition $h = 0$ in a region surrounding the origin and $h = 1$ elsewhere.

Following the usual formulation

$$H = \min_u [h + p_1 v\cos x_3 + p_2 v\sin x_3 + p_3 u]$$

leading to the optimal u being

$$u = -\operatorname{sign}(p_3) \qquad |u| \leq 2$$

The trajectory equations follow from the characteristics of H and reveal that since

$$p_1' = p_2' = 0$$
$$p_3' = p_1 v\sin x_3 - p_2 v\cos x_3 \qquad (10)$$

the optimal trajectories are composed of straight lines and circles depending on the boundary conditions of p_1 and p_2.

As a physical consideration the additional definition of

$$\operatorname{sign}(0) = 0$$

minimizes the number of switchings in u and the control is one commonly termed 'bang-bang'. For the chosen initial point P the optimal trajectories are at A in *Figures 2* and *3*. The vehicle reached the origin 0 in 3·24 sec.

Minimum Energy Trajectories

If the performance index is taken to be

$$f(x,t) = \min_u \left[\int_t^T \left(h + \frac{u^2}{2v}\right) d\lambda\right]$$

with v as a yet unchosen Lagrange multiplier, the optimum u is $u = -v p_3$. The trajectories equations are unchanged. At the terminal point 0, the x_3 boundary is unspecified and consequently p_3 and u are zero. To the first order of approximations along the trajectory through the origin

$$df = \frac{dx_1}{v}\cos x_3 + \frac{dx_2}{v}\sin x_3$$

and hence

$$\frac{\partial f}{\partial x_1} = p_1 = \frac{\cos x_3}{v} \qquad \frac{\partial f}{\partial x_2} = p_2 = \frac{\sin x_3}{v}$$

The problem of solving the trajectory equations reduces to the selection of a terminal x_3 and adjusting it until the initial boundary values are satisfied. For the given values of $v = 3$ the optimal trajectories are shown at B in *Figures 2* and *3*. The system reached the origin from P in 3·27 sec.

Minimum Squared Deviation and Control Energy

The performance index is taken to be

$$f(x,t) = \int_t^T \frac{1}{2}\left(x_1^2 + x_2^2 + \frac{u^2}{v}\right) d\lambda$$

from which the optimal control is unchanged at $u = v p_3$. However, the trajectory equations are now

$$p_1' = -x_1$$
$$p_2' = -x_2$$
$$p_3' \doteq p_1 V\sin x_3 - p_2 V\cos x_3$$

with free boundary value conditions.

The computational problem is not attractive in this case because at the terminal point, p_3' is very nearly always zero and tends to be influenced by rounding errors. The optimal trajectories are shown at C in *Figures 2* and *3* and the optimal time to reach the origin was 3·29 sec.

Minimum Squared Miss Distance

The performance index is taken to be

$$f(x,t) = \min_u \left[\int_t^T \frac{u^2}{2} d\right] + \frac{1}{2}x_1^2(T) + \frac{1}{2}x_2^2(T)$$

resulting in the same control signal $u = -v p_3$. The trajectory equations are those of (10) and (9) and the boundary conditions are clearly $p_1(T) = x_1$, $p_2(T) = x_2$, $p_3(T) = 0$. Since the target at 0 is the origin these values closely approximate those for the minimum energy trajectory presented earlier in this paper. The optimal solutions are those for C in *Figures 2* and *3* obtained for $v = 100$, and the minimum time achieved was 3·27 sec.

Summary of Formulations

Comparing the energy used in each of the four schemes reveals that apart from the minimum time trajectory there was little difference. The minimum time control scheme used 2·16 units as opposed to a minimum energy formulation using 1·23 units.

The insensitivity of the solution to the formulation suggests that the performance index producing the most tractible set of equations should be employed. In this example the minimum time solution is a geometric exercise, whilst the formulation of the minimum squared deviation and control energy generated a rather troublesome set of equations from the numerical point of view. Clearly, in general, there is room for experiment within the constraints of the problem.

Approximation Method for Boundary Value Problems

Since the trajectory methods lead to two point boundary value problems, methods for dealing with these are receiving a great deal of attention. Boundary value problems are far older than numerical variational methods, and many well established techniques exist in numerical practice. It is essential to distinguish between the time available for calculations which are of a design nature, and those that are undertaken during the control of the physical process. Real time work places a heavy penalty on inefficient computational techniques, and boundary value methods tend to be among the most inefficient.

Refining Techniques

Free boundary value problems of the type of eqn (8) lend themselves to reverse time computation from the guessed target position at time T. However, owing to the extreme sensitivity of the solutions and to some troublesome numerical details, it proves far simpler to work forwards from the initial time t. Consider the boundary value problem of eqn (8) and assume that it is possible to proceed by guessing the initial value of $\boldsymbol{p}(t)$ and adjusting this according to the error at the terminus. Represent the solution (6) for the terminal value of $\boldsymbol{p}$ by

$$\boldsymbol{p}(T)=\Psi[\boldsymbol{p}(t),\boldsymbol{x}(t);t,T] \qquad (11)$$

If the initial value is correctly chosen then, clearly, ψ will be zero. Furthermore, if T is extended, the change in the correct $\boldsymbol{p}(t)$ will satisfy the differential equation

$$\left[\frac{\partial \Psi_i}{\partial p_j}(t)\right]\cdot\frac{\mathrm{d}p(t)}{\mathrm{d}T}+\frac{\partial \Psi}{\partial T}=0 \qquad (12)$$

However, although boundary values for these equations are known the matrix $[\partial \Psi_i/\partial p_j]$ is not. In general $\boldsymbol{p}(T)$ will be non-zero at time T and the magnitude of its error can be evaluated by some arbitrary definite function $n[\boldsymbol{p}(t)]$ of the terminal boundary values. Small perturbations in the initial value $\boldsymbol{p}(t)$ will effect the value of n according to the expansion

$$n(p+\Delta p)=n(p)+\sum\frac{\partial n}{\partial p_i}\cdot\Delta p_i+0(\Delta^2 p_i) \qquad (13)$$

However, since n can be defined to have a minimum when the boundary value is satisfied, a correction scheme can be found by differentiation with respect to one of the p_i:

$$\Delta p_i=-\frac{\partial n}{\partial p_i}\bigg/\frac{\partial^2 n}{\partial p_1^2} \qquad (14)$$

and the corrected value of p_i will be $p_i+\Delta p_i$. The drawback of this simple approach is that the instability of the trajectory equations usually makes any function n extremely large and grossly sensitive to perturbations. It is, however, very simple and easy to code into a routine which could deal with all cases likely to arise in practice.

One way of overcoming this is to reduce substantially the time interval T–t. Clearly if this interval were zero the boundary values are known to be zero and consequently for a small interval a good estimate is available. Thus the procedure is to start with a small value T–t, perturb each of the p co-ordinates in turn, applying eqn (14) to reduce $p(t')$ to zero, t' being the temporary value of T. After one cycle of the perturbations it is essential to rotate the axis of the perturbation coordinates to make them lie along the direction of grad n. Having solved the first stage over the reduced interval, it can be extended and the process repeated. As the next guess for the boundary values either the previous value can be used or it can be up-dated with a crude solution of eqn (12). The matrix $[\partial \Psi_i/\partial p_i]$ is now available from the results of the previous perturbations and is invertible.

Proceeding in this way, the optimal trajectory can be progressively extended until it covers the given time interval. However, it may happen that some physical condition is satisfied before the full time, i.e. the target region, is entered, and then the computation can be terminated earlier with consequent economy. This procedure has the advantage that during real time computation a 'part-time' optimal solution is always available for the current position and this can be used as an approximation while its up-dated value is refined. It has the disadvantage that errors in the numerical integration tend to disguise the minimum sought for, and then increase with T–t.

A Boundary Value Example

The method can be illustrated by an example whose nonlinearities cause difficulties with the more conventional techniques. Consider a dynamic system of the pendulum type

$$x'-(x')^2+x=u \qquad (15)$$

Select the control law u to minimize the performance index

$$f(\boldsymbol{x},t,T)=\frac{1}{2}\int_t^T(x_1^2+x_2^2+u^2)\,\mathrm{d}t \qquad (16)$$

The dynamic system is controllable only in the absence of limits on u. Transforming to the usual phase coordinates and forming the optimal equations yields the set

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_2^2-x_1-p_2 \\ p_1' &= x_1+p_2 \\ p_2' &= -x_2-p_1-2x_2p_2 \end{aligned} \qquad (17)$$

If the end point is considered to be free, then the boundary values are $\boldsymbol{p}(T)$ zero. Direct computation of the trajectories backwards from the terminal region surrounding the origin reveals an optimum phase portrait of *Figure 5*. It shows that it is difficult to reach the chosen initial point ($x_1 = 1$, $x_2 = 0$) because of the sensitivity of the trajectories.

At the origin the non-linear system behaves as the linear one:

$$x' = A \cdot x$$

The solution of this system as t tends to infinity, is dominated by that of the largest positive eigenvalue of A, and lies parallel to the corresponding eigenvector. Thus all solutions tend to the one eigenvector through the origin and this implies extreme sensitivity when working in reverse time scale from the origin.

Using the technique described, which has been programmed into a series of short routines for the Ferranti Mercury computer, a trajectory can be found which satisfies any initial conditions and boundary values. *Figure 6* gives the final trajectory and also the variation of the end point $x(T)$ with the time interval $T-t$ given in *Figure 7*. *Figure 8* shows how the computed finite time boundary values tend to the Hamiltonian surface $\mathscr{H} = 0$ satisfied for the infinite interval.

Function Space Approximations

Function space methods are an alternative to trajectory methods in deterministic systems, but are the only approach in systems containing random components. In some systems the algebraic form of $f(x, t)$ is known in advance and the computational formulae may then be put into discrete or continuous time form, whichever is most convienient. However, when the form of $f(x, t)$ is unknown, only discrete time computing formulae are feasible. This is usually the situation in adaptive systems.

A Function Space Approximation for an Adaptive System

To illustrate the use of function space approximations consider the following example. A simple regulator contains a fixed unknown gain α, in the control path (*Figure 9*). The system is disturbed by a noise $\{\xi_n\}$ which is an independent gaussian sequence with zero mean and variance σ. In order to set up the problem to lead to a discrete time computing formulae it is assumed that the control is changed only at unit time intervals, when x is also observed. The dynamic equation for the regulator is

$$x_n = x_{n-1} + \alpha u_{n-1} + \xi_{n-1} \tag{18}$$

The unknown gain α is re-estimated after each observation of x, the estimation being made according to the Bayesien formula,

$$\text{posterior density} = \text{likelihood} \times \text{prior density} \tag{19}$$

Since $[\xi_n]$ is an independent gaussian sequence, the successive posterior densities can be made gaussian. The likelihood is given by

$$l(x_n|a) = \exp\left[-\frac{1}{2}\frac{(x_n - x_{n-1} - \alpha u_{n-1})^2}{\sigma}\right]$$

If the prior density is gaussian with mean m_{n-1} and variance v_{n-1}, then eqn (19) gives

$$m_n = \frac{u_{n-1}(x_n - x_{n-1}) + m_{n-1}\dfrac{\sigma}{v_{n-1}}}{u_{n-1}^2 + \dfrac{\sigma}{v_{n-1}}} \qquad v_n = \frac{\sigma}{u_{n-1}^2 + \dfrac{\sigma}{v_{n-1}}} \tag{20}$$

Eqns (20) can now be used to up-date the mean and variance after each observation. It is to be noted that they are non-linear. In order to compute the control at instant $n-1$, it is necessary to have ***a priori*** distribution of the mean at the next time instant n. This can be found by substituting for x_n as yet unknown, in eqn (20) from (18) yielding a stochastic equation

$$m_n = \frac{m_{n-1}\dfrac{\sigma}{v_{n-1}} + \alpha u_{n-1}^2 + u_{n-1}\xi_{n-1}}{u_{n-1}^2 + \dfrac{\sigma}{v_{n-1}}} \tag{21}$$

The performance index with r stages to go (note r indexes time backwards) is taken as

$$f_r(x_r, m_r, v_r) = \min E\left\{\sum_{k=r}^{1}(u_k^2 + x_{k-1}^2)\right\} \tag{22}$$

Note the n in formulae (18) and (20) will also index backwards when used in conjunction with (22). The authors regret this notational inconsistency.

A discrete time iteration for the performance index may now be set up by using Bellman's Principle or Optimality

$$f_r(x_r, m_r, v_r) = \min_{v_r} E\{x_{r-1}^2 + u_r^2 + f_{r-1}(x_{r-1}, m_{r-1}, v_{r-1})\}$$

$$f_1(x_1, m_1, v_1) = \min_{v_1} E\{x_0^2 + U_1^2\} \tag{23}$$

It is understood that the value of f_{r-1} to be used in this iteration is the one resulting from the application of u_r.

On substituting for x_0, and averaging over both ξ and the current density of α followed by minimization with respect to u_1; the analytic expression for u_1 and f_1 may readily be found as

$$u_1 = \frac{-m_1 x_1}{m_1^2 + v_1 + 1} \qquad f_1(x_1, m_1, v_1) = \frac{x_1^2(1 + v_1)}{m_1^2 + v_1 + 1} + \sigma \tag{24}$$

The expression for f_2, indexing n backwards, is now

$$f_2(x_2, m_2, v_2) = \min E\left\{u_2^2 + (x_2 + \alpha u_2^2 + \xi_2)^2 \frac{A}{B}\right\} + \sigma \tag{25}$$

where

$$A = 2 + \frac{2\sigma}{u_2^2 + \dfrac{\sigma}{v_2}} + \frac{\left(m_2\dfrac{\sigma}{v_2} + \alpha u_2^2 + u_2\xi_2\right)^2}{\left(u_2^2 + \dfrac{\sigma}{v_2}\right)^2}$$

$$B = 1 + \frac{\sigma}{u_2^2 + \dfrac{\sigma}{v_2}} + \frac{\left(m_2\dfrac{\sigma}{v_2} + \alpha u_2^2 + u_2\xi_2\right)^2}{\left(u_2^2 + \dfrac{\sigma}{v_2}\right)^2}$$

It is evident that no simple analytic expression can be found for f_2. The complexity of the performance index expression is seen to be a consequence of the highly non-linear nature of the estimation equations; this is a common occurrence in adaptive systems. Now it is simple to evaluate f_2 at any chosen point (x, m, v) using a digital computer, but this produces f_2 as a set of numerical values; to avoid storing all these points f_2 can be condensed into a set of three dimensional orthogonal polynomials. This was done on a small digital computer having 1,024 word working store using a programme developed by Cadwell and Williams[5].

Cadwell and Williams' programme is designed for a particular small computer. It uses a modification of Forsyth's method for generating orthogonal polynomials of successively higher order using only the previous two polynomials. Owing to machine limitations only 200 data points can be fitted in three variables. However, in this example, it was found that the mean square error could be made less than 0·5 per cent when tested over a large number of points. The polynomials were computed up to order 4, involving 35 coefficients of powers of x, m, and v.

Having approximated f_2 it is possible to compute f_3. Since the error of approximation is small the iteration for f_3 may be written

$$f_3(x_3, m_3, v_3) = \min_{U_3} E\{u_3^2 + x_2^2 + x_2^2 + f_2(x_2, m_2, v_2) + d_2(x_2, m_2, v_2)\} \quad (26)$$

where $d_2(x_2, m_2, v_2)$ is the error in f_2, and $f_3{}^*$ are the computed values of f_3. Now the minimum in f_3 will be close to that in f_3, so that

$$f_3 - f_3 + \min E\{d_2\} \quad (27)$$

Roughly, then, the error is additive at each stage.

To give an idea of the range of the coefficients of the powers of x, m, v, in f_3 the largest value was 11·2, then there were nine in the range 1–10, eight in the range 0·1–1, eleven in the range 0·01–0·1 and only seven below this. Numerical experiments showed that omission of some of the smaller coefficients had a serious effect in certain regions of the variables.

The optimal value of control was found by a gradient technique, the expression for f being evaluated for a sequence of u values. A defect of the method is that polynomial approximations tend to show ripples, especially near the end of the fitted range, these ripples acted as false minima during the gradient computations, and it was necessary to check each minimum by approaching it from two sides. However, even this check was a little uncertain when the minimum was very flat, in practical terms this could make great differences to the control and was important. These effects were greatly reduced by approximating the results so that the control was smooth function of x, m, v. Once f_3 had been found at a suitable set of points a further polynomial approximation could be found and the whole process repeated to find f_4. The polynomial approximation method required the same computational process at each stage, which is convenient.

In this example after three stages the control became a stationary function of state, some results are shown in *Figure 10*. By trial and error a simple approximation for the stationary control was found to be

$$u \simeq \frac{-mx}{m^2 + \frac{1}{2}v^2 + 1} \quad (28)$$

for x, m and v in the range 0–5·0.

On the particular computer used (180 μsec multiplication and 4 msec division time) each point involved about 1 min of computation and over four stages some 800 points were required, needing in all some 14 h of computation, it is therefore interesting to see what further method of approximation could be used so as to reduce this computation load.

Alternative possibility is to replace the system by one which, on physical grounds, would appear to have a similar control solution. The simplest alternative system is to regard α as a random variable with a fixed distribution at each stage, neglecting for the moment the transitions in mean and variance. Thus at each stage the only variable to be considered is x. The mean m, and the variance v, of the estimate of α, are then successively up-dated and used in the solution for u (30).

The functional iteration for the stochastic system is a function of x only

$$f_r(x_r) = \min_{U_r} E\{x_{r-1}^2 + u_r^2 + f_r(x_{r-1})\} \quad (29)$$

$$f_1(x_1) = \min_{U_1} E\{x_0^2 + u_1^2\}$$

This iteration may be carried out analytically very simply. The stationary solution is

$$u = \frac{-cmv}{1 + v + cv} \quad (30)$$

where c is the positive solution of

$$c = \frac{1}{2}\left(\frac{v}{m^2} - 1\right) \pm \frac{1+v}{m} \quad (31)$$

On comparing the resulting values of control from eqn (28) and (30) it will be found the stochastic control is 10–20 per cent smaller than the adaptive control. This is to be expected on physical grounds, since the adaptive control has exploratory element. However, the stochastic control value is a reasonable approximation to the adaptive one, considering the immense difference in the amount of computing involved.

At the present time the best approximation method for adaptive control known to the authors is the replacement of the fully adaptive system by the partial one as above. Since each unknown parameter often involves two, or more statistics, depending on the distribution used, the reduction in dimensionality can be substantial. It is usually apparent on physical grounds that the approximation will be a valid one. In many cases the accuracy of approximation will be better than in the example above.

A Function Space Approximation for Deterministic Systems

In non-linear deterministic systems a function space approximation can often be a useful alternative to the trajectory method. These methods depend on having an analytic form for the performance index of an approximated system. First it must be possible to rewrite the dynamic equation of the system in linearized form as

$$\boldsymbol{x} = A(\boldsymbol{x}, t)\,\boldsymbol{x} + B(\boldsymbol{x}, t)\,\boldsymbol{u} \quad (32)$$

To obtain an analytic form for the performance index choose

$$f(\boldsymbol{x}, t) = \min \int_0^\infty (\boldsymbol{x}^T Q \boldsymbol{x} + \boldsymbol{u}^T R \boldsymbol{u}) \cdot d\tau \quad (33)$$

where Q and R may be general time dependent matrices. However, for simplicity, take them to be constants in this description. The partial differential equation for $f(x, t)$ is

$$-\frac{\partial f}{\partial t}=\min_{u}\left[x^{T}Qx+u^{T}Ru+p^{T}(A(x,t)+B(x,t)u)\right] \quad (34)$$

On minimizing and regarding A, and B as constant it is found that

$$u=-R^{-1}B(x,t)^{T}p \quad (35)$$

where

$$p=[p_1, p_2 - - p_n]^{T} \quad \text{with} \quad p_i=\frac{\partial f}{\partial x_i}$$

After substituting for the optimal control it will be found that eqn (34) cn be solved by substituting $f(x, t) = \frac{1}{2} X^{T} P X$ where P is the solution of à matrix Ricatti equation:

$$P'+A^{T}(x,t)P+PA(x,t)+Q=PB(x,t)R^{-1}B^{T}(x,t)P$$

Utilizing the stability properties of the Ricatti equation[7] it can be shown that P is the positive definite solution of

$$PA(X,t)+A^{T}(X,t)P+Q=PB(X,t)R^{-1}B(X,t)^{T}P \quad (36)$$

The approximation scheme now uses eqn (36) to solve a state dependent matrix. The resulting matrix is substituted into eqn (35) to compute the control vector. The advantages of this method are (*a*) the solution requires only algebraic computation, and uses currently available quantities; (*b*) the precise nature of A and B is unimportant, thus the method is readily extendable to an adaptive scheme, where A and B vary as the result of measurement, and (*c*) it can be shown that an appropriate choise of the linearization will always result in a stable controlled system.

The stability and ease of realization of the resultant controlled system are probably the most important practical factors in favour of this technique. To show the stability, consider the general second order system

$$x_1'=a_{12}(x_2)x_2$$
$$x_2'=a_{21}(x_1)x_1+a_{22}(x_1,x_2)x_2+du \quad (37)$$

the factors $a_{21}(x_1)$ and $a_{22}(x_1, x_2)$ can be any bounded functions, but $a_{12}(x_2)$ must satisfy certain requirements given below. Take the performance index

$$f(x,t)=\min_{u}\left[\frac{1}{2}\int_{t}^{T}(x_1^2+x_2^2+u^2)\,d\tau\right] \quad (38)$$

Applying eqn (36) the elements p_{ij} of P satisfy

$$d^2p_{11}p_{12}-a_{12}p_{11}-(a_{21}+a_{22})p_{12}=0$$
$$d^2p_{12}^2-2a_{21}p_{12}-1=0 \quad (39)$$
$$d^2p_{22}^2-2a_{22}p_{22}-2u_{12}p_{12}-1=0$$

On finding the stable solution of eqns (39) it is possible to evaluate the control

$$u=-d(p_{12}x_1+p_{22}x_2)$$

where p_{12} and p_{22} are the positive solutions of eqns (39).

On substituting for u in the original equations the controlled system has the form:

$$x_1'=a_{12}(x_2)x_2$$
$$x_2'=-b_{21}(x_1)x_1-b_{22}(x_1,x_2)x_2 \quad (40)$$

where the functions b_{21} and b_{22} are the positive roots of

$$b_{21}(x_1)=[a_{21}^2(x_1)+d^2]^{\frac{1}{2}}$$
$$b_{22}(x_1,x_2)=[a_{22}^2(x_1,x_2)+d^2(2a_{12}(x_2)p_{12}+1)]^{\frac{1}{2}} \quad (41)$$

The stability of eqns (40) can now be established by application of the second method of Liapunov[8]. Consider the proposed Liapunov function

$$V(x)=\int_{0}^{x_1}wb_{21}(w)\,dw+\int_{0}^{x_2}wa_{12}(w)\,dw \quad (42)$$

Clearly $V(x)$ is a positive function which is bounded if $a_{12}(x_2)$ is positive and suitably restricted. Its time derivative is

$$V'=-b_{22}(x_1,x_2)a_{12}(x_2)x_2^2 \quad (43)$$

which is negative semi-definite. If the x_1 axis is not a permissible trajectory of the system of eqn (40) then the expression of eqn (42) is a Liapunov function of the system. This function defines a series of closed surfaces over the whole phase space about the origin which the trajectories enter. Thus the system is asymtotically stable about the origin.

A particular case which has been studies computationally is the Van der Pol equation

$$x''+a(1-x^2)x'+bx=du \quad (44)$$

This equation was linearized in phase space form by putting

$$x_1'=x_2, \quad a_{12}=1, \quad a_{21}=-b, \quad a_{22}=-a(1-x_1^2) \quad (45)$$

Using the performance index of eqn (38) some comparisons of the computed trajectories in the exact and approximated cases are shown in *Figure 11*, which gives the phase space trajectories and the necessary control signals for the exact and approximated cases, *Figure 12*. The closeness of the approximation occurs in many practical cases, and is an indication of the effectiveness of the method. To illustrate the implementation of the scheme an analogue computer arrangement is shown in *Figure 13*, which solves eqn (39).

The degree of approximation can be improved by varying the performance index slightly (i.e. Q and R). *Figure 4* shows the effects of such variations and *Figure 11* suggests that a system linearized and optimized with respect to

$$f(x,t)=\int_{t}^{\infty}\left(\frac{1}{4}x_1^2+\frac{1}{2}x_2^2+\frac{1}{2}u^2\right)d\tau$$

closely approximates the non-linear system optimized with respect to

$$f(x,t)=\frac{1}{2}\int_{t}^{\infty}(x_1^2+x_2^2+u^2)\,d\tau$$

Thus it appears possible to rescale the approximate phase plane to fit the optimal one by adjusting the performance index appropriately.

Conclusions

The general mathematical equations for optimal and adaptive control can now be set up, but comprehensive methods for solution are not known. Approximation methods are being developed, but they must be used according to the individual circumstances of each problem. A number of different possible techniques of approximation have been with appropriate examples.

The first point stressed is that the mathematical setting up of the problem can often be varied so as make the computation easier, whilst still giving a satisfactory physical solution. The second point is that the methods can be grouped into two classes, trajectory methods applicable only to deterministic systems, and the function space method which is of wide application. A systematic search technique for solving trajectory problems has been described. A function space method using orthogonal expansions and another using linearized equations has also been given.

Computing was done at the University of London Computer Unit.

Nomenclature

α	Unknown gain factor
A	Term in dynamic equation
B	Term in dynamic equation
d	Control coefficient
$f(x, t)$	Performance index
h	Hamiltonian type function
$l(x\ a)$	Likelihood function
L	Integrand in performance index
m	Estimated mean value
$n(p)$	Norm function
p	Co-state vector
P	Matrix in expansion of performance index
Q	Cost of state matrix
R	Cost of control matrix
$t, T,$	Time
u	Control vector
v	Estimated variance
$V(x)$	Liapunov function
x	State vector
σ	Variance
(ξ_n)	Noise process
ν	Lagrange multiplier

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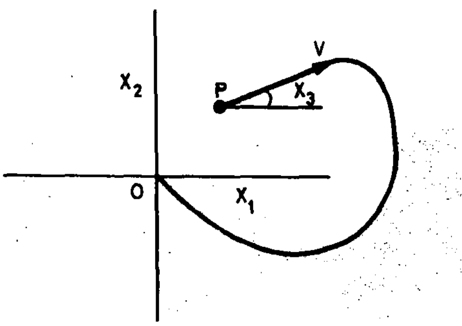

Figure 1.

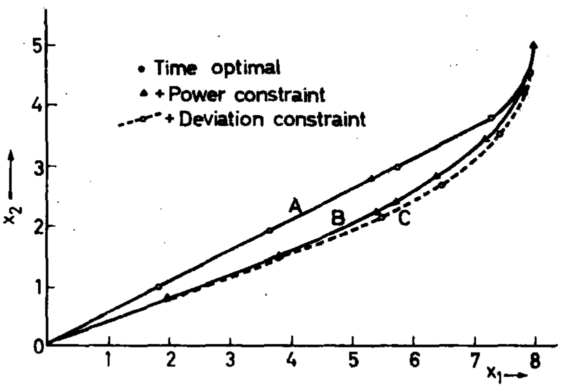

Figure 2.

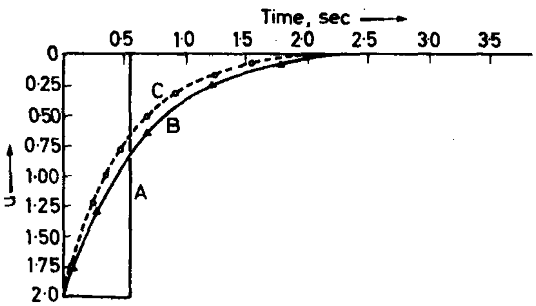

Figure 3. Control signals

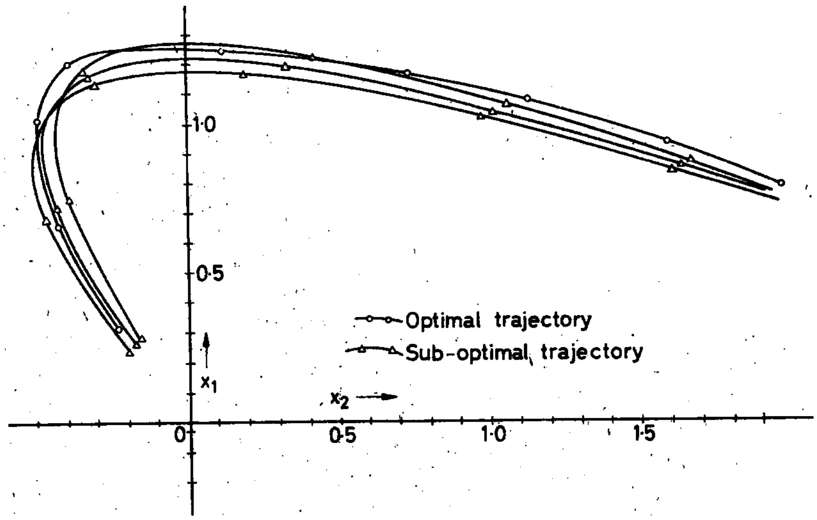

Figure 4. Varying performance index

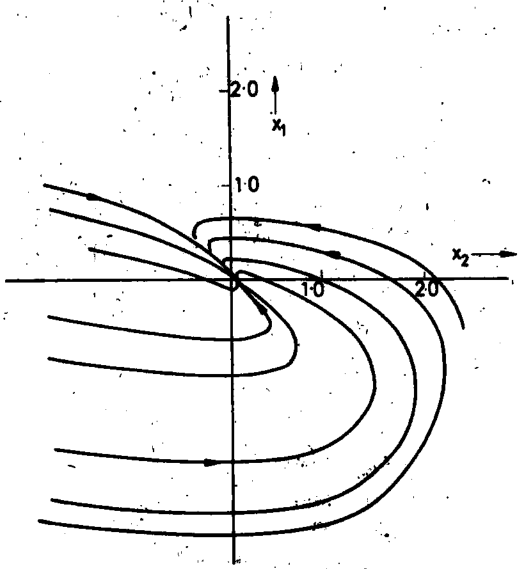

Figure 5. Optimal phase plane

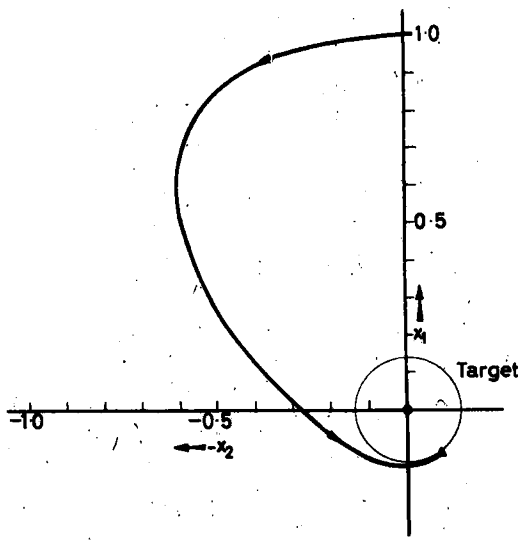

Figure 6. Optimal trajectory

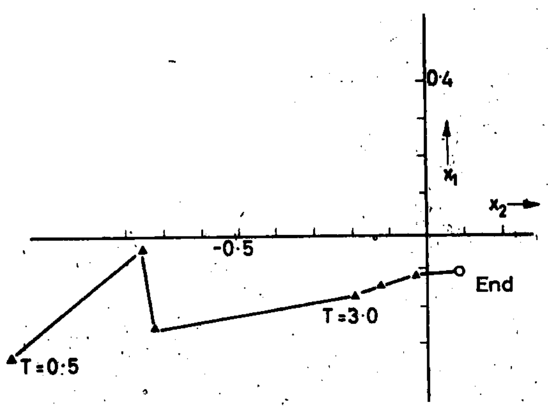

Figure 7. Variation of end point trajectory

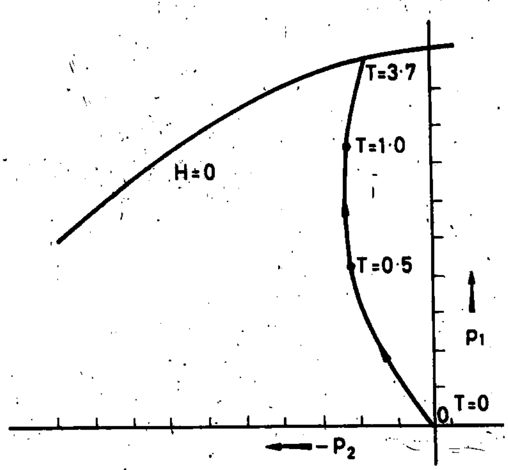

Figure 8. Variation of optimal boundary values

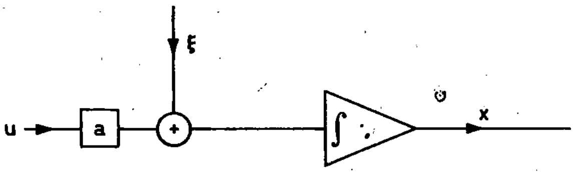

Figure 9. Regulator with unknown gain

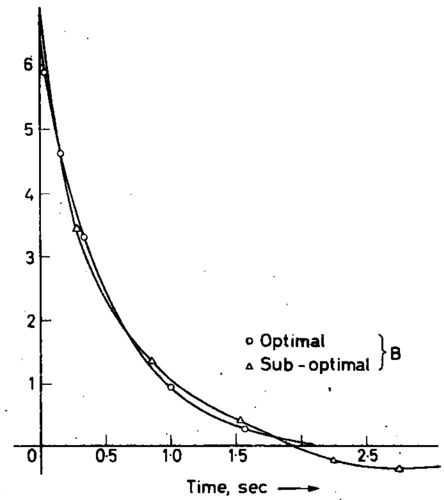

Figure 12. Optimal and sub-optimal control signals

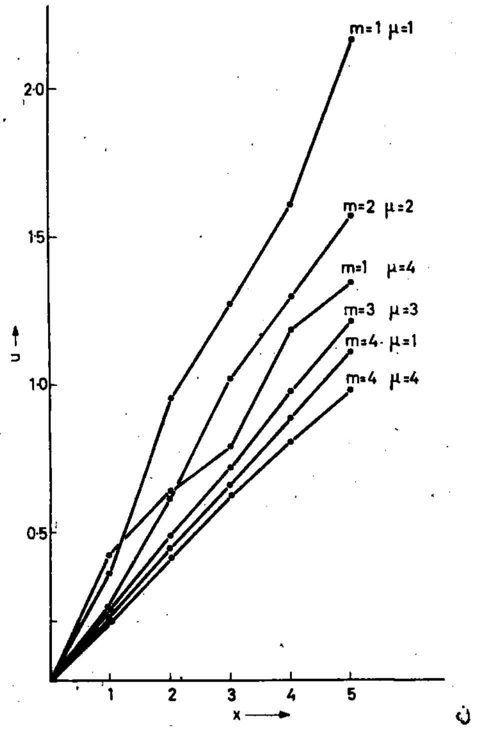

Figure 10. Stationary control law for adaptive regulator

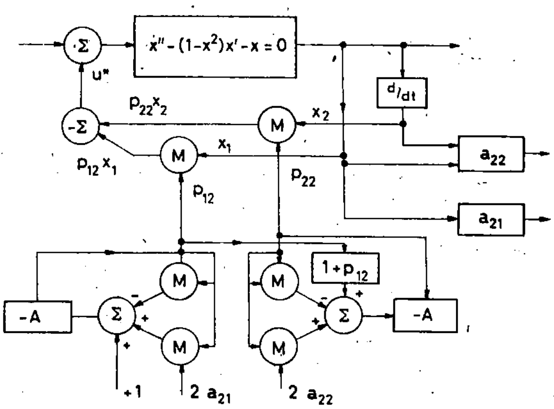

Figure 13. Analogue solution of sub-optimal system

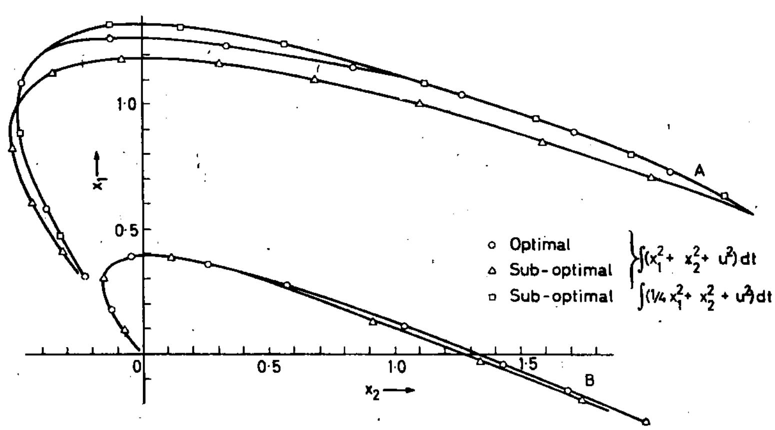

Figure 11. Optimal–sub-optimal trajectories

An Optimal Guidance Approximation for Quasi-circular Orbital Rendezvous

H. J. KELLEY and J.C. DUNN

Introduction

The second-order guidance approximation scheme employed in this paper has been developed in an earlier publication[1]. Essentially, the idea is to select a flight path, optimized in some appropriate sense, as a nominal trajectory, and then to base guidance upon a family, or 'field', of optimal trajectories approximated in the vicinity of the nominal.

The investigation, as is shown later, applies this scheme to the guidance problem for orbital rendezvous. The method is tractable for a fairly wide class of problems, although in general it must be carried through numerically. However, in the present case an analytical treatment becomes feasible because of the recent availability of a particularly simple optimal transfer manoeuvre suitable for use as a nominal trajectory. The nominal manoeuvre, as it appears in this paper, is a direct outgrowth of a co-planar circular orbit transfer analysis conducted by Hinz[2]. The problem posed by Hinz, although phrased somewhat different with regard to coordinate systems and assumptions employed in deriving the equations of motion, is mathematically equivalent to the nominal transfer manoeuvre problem investigated herein. Both cases yield to an analytical treatment of the boundary value problem whenever the manoeuvre duration is an integral multiple of a reference orbit's period.

The analysis of three-dimensional rendezvous guidance for the class of trajectories discussed above leads directly to a synthesis of a linear feedback control solution with time-varying gains given in closed form.

Some suggestions are also given for possible modifications which might enhance system accuracy and the range of operability during practical implementation of low-thrust rendezvous guidance.

The Differential Equations of Powered Flight

Considerations begin with the differential equations of three-dimensional powered flight in a central inverse-square force field (*Figure 1*):

$$\ddot{r} - r\cos^2\psi\dot{\theta}^2 - r\dot{\psi}^2 + \frac{k}{r^2} = \frac{F}{m}\cdot v\cos\beta\sin\alpha$$

$$\ddot{\theta} + 2\frac{\dot{r}\dot{\theta}}{r} - 2\tan\psi\dot{\psi}\dot{\theta} = \frac{F}{mr\cos\psi}\cdot v\cos\beta\cos\alpha$$

$$\ddot{\psi} + 2\frac{\dot{r}\dot{\psi}}{r} + \frac{1}{2}\sin 2\psi\dot{\theta}^2 = \frac{F}{mr}\cdot v\sin\beta \qquad (1)$$

$$\dot{m} = -\frac{vF}{C}; \quad -\frac{\pi}{2} \le \beta \le \frac{\pi}{2}, \quad 0 \le v \le 1$$

where F is the maximum thrust level of the reaction engine; C is the propellant exhaust velocity; v is a throttle variable; and m is the instantaneous vehicle mass. The difficulty in obtaining any sort of particular solution for these equations needs no comment here, except that it provides a motive for the simplifications which are now introduced. The object is to devise certain assumptions which will allow the replacement of eqn (1) by an approximate set of differential equations which are linear in the state variables r, θ, and ψ, and their time derivatives and, preferably, separable in the control variables v, α, and β. (Simplifications of this kind are required to make flight path optimization and guidance problems analytically tractable.) To be specific, it is preferable that these approximate differential equations describe low-thrust acceleration transfer trajectories between neighbouring circular orbits.

The following set of dependent and independent variable transformations will prove useful for our purposes. Let

$$r(t) = R_0[1+\eta(t)]$$

$$\theta(t) = \tau(t) + \varepsilon(t)$$

and

$$m(t) = m_0[1+\xi(t)]$$

where R_0 is the radius of a circular reference orbit situated in the $\psi = 0$ plane (*Figure 1*); m_0 is some reference mass; and τ is a fictitious angle defined by the differential expression,

$$\frac{d\tau}{dt} = \omega_0, \quad \tau(0) = 0$$

where ω_0 is the reference orbit's period.

Furthermore, since $\tau(t)$, as defined above, is a monotonic time-like parameter, it is permissible to change the independent variable in eqns (1) from t to τ. This can be accomplished by simply relating t derivatives to τ derivatives as follows:

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \omega_0\frac{d}{d\tau}$$

$$\frac{d^2}{dt^2} = \frac{d}{dt}\left(\frac{d}{dt}\right) = \omega_0^2\frac{d}{d\tau}$$

Finally, if note is made of the fact that $k/R_0^3\omega_0^2 = 1$, then eqns (1) can be put into the following reduced first-order form:

$$u' = (1+\eta)(1+v)^2\cos^2\psi - \frac{1}{(1+\eta)^2} + \frac{F^*}{(1+\xi)}v\cos\beta\sin\alpha$$

$$v' = -\frac{2u(1+v)}{1+\eta} + 2w(1+v)\tan\psi + \frac{F^*}{(1+\eta)(1+\xi)} \nu\cos\beta\cos\alpha$$

$$w' = -\frac{2uw}{1+\eta} - \frac{1}{2}(1+v)^2 \sin 2\psi + \frac{F^*}{(1+\eta)(1+\xi)} \nu \sin\beta$$

$$\eta' = u \qquad (2)$$

$$\epsilon' = v$$

$$\psi' = w$$

$$\xi' = -\frac{F^*}{C^*}\nu$$

where $F^* = F/m_0 R_0 w_0^2$ is the reduced maximum thrust acceleration; $C^* = C/R_0 w_0$ is the reduced exhaust velocity; and the superscribed prime denotes differentiation with respect to the reduced time, τ.

Now assume that F^*, ξ, u, v, w, η, and ψ are all terms of order δ or smaller ($\delta < < 1$). Under these circumstances, one would therefore anticipate that all terms of order δ in eqns (2) will become negligible with respect to terms of order δ. Thus the following simplified differential equations are arrived at:

$$u' = 2v + 3\eta + F^*\nu\cos\beta\sin\alpha$$

$$v' = -2u + F^*\nu\cos\beta\cos\alpha$$

$$w' = -\psi + F^*\nu\sin\beta \qquad (3)$$

$$\eta' = u$$

$$\varepsilon' = v$$

$$\psi' = w$$

Saying that, to the first order of small quantities, eqns (3)* are descriptive of quasi-circular flight for the following reason; then if, as has been assumed, u, v, η, etc. are of order δ, then it follows at once that

$$\left[\left(\frac{E-E_0}{E_0}\right)^2 + \left(\frac{h-h_0}{h_0}\right)^2\right]^{\frac{1}{2}} = 0(\delta)$$

where E and h are specific energy and angular momentum respectively. Consequently, the energy-momentum images of trajectories which are adequately described by eqns (3) should everywhere be close to the locus of circular orbits in the E–h phase plane (*Figure 2*).

Clearly, the validity of the quasi-circular differential equations will be compromised when the parameters F^*, F^*/C^*, and τ exceed certain critical values. Just precisely what these critical values are cannot be determined until the nature of the control schedule $\nu(\tau)$, $\alpha(\tau)$, and $\beta(\tau)$, is specified. The reader is advised to bear this in mind in the sequel.

* These equations are identical in form to the differential equations of Wheelon[3] and Anthony[4]. However, the dependent variables and thrust vector steering angles are not subject to the same interpretation. In particular, the quantity ε in eqns (3) is not required to be small—an important point in the subsequent development.

Optimal Transfer Between Neighbouring Circular Orbits

The optimal orbit transfer problem may be stated as follows: given two neighbouring circular orbits, find the steering angles $\alpha(\tau)$ and $\beta(\tau)$ and the throttle schedule $\nu(\tau)$ which produce a transfer between the two orbits in minimum time. For present purposes, the case for which the terminal or bitsare co-planar will be selected, a class of optimal transfer paths within the framework of the quasi-circular dynamics assumption derived, and later, these paths employed as nominal trajectories for the three-dimensional rendezvous guidance analysis.

To reiterate, if the subscripts 0 and f denote initial and final conditions respectively, search is made for a set of control functions $\alpha(\tau)$, $\beta(\tau)$, and $\nu(\tau)$ which minimize τ_f, produce a state transition which evolves in accord with eqns (3), and which satisfies the circular orbit boundary conditions, viz: at $\tau = 0$, $u = v = w = \eta = \varepsilon = \psi = 0$, $\xi = 0$; at $\tau = \tau_f$ $u = w = 2v + 3\eta = \psi = 0$, $\eta = K$ (const.). The problem so stated is a Mayer variational problem with bounded control variables. The necessary conditions to be satisfied by its solution are well known and are written here for the present application without further comment:

Let

x_i = state variables, u, v, w, η, ε, ψ, ξ

y_K = control variables ν, α, β

λ_i = multiplier functions

Λ_i = undetermined constant multipliers

P = function to be extremized = $\tau_F + \Lambda_1(u_f) + \Lambda_2$

$$(2v_f + 3\eta_f) + \Lambda_3(w_f) + \Lambda_4(\eta_f - K) + \Lambda_5(\psi) \qquad (4)$$

H = Hamiltonian function = $\sum_i \lambda_i x_i'$

Then the following equations and inequalities must be satisfied,

$$H(y) \geq H(\bar{y}) \qquad (5)$$

i.e., the optimal control y minimizes the function H.

$$\lambda_i' = -\frac{\partial H}{\partial x_i} \qquad (6)$$

$$x_i' = \frac{\partial H}{\partial \lambda_i} \quad \text{[eqns (3)]} \qquad (7)$$

together with the corresponding natural boundary and transversality conditions,

$$H_f = -\frac{\partial P}{\partial \tau_f} \qquad (5a)$$

$$\lambda_{i_f} = \frac{\partial P}{\partial x_{i_f}} \qquad (6a)$$

$$x_{i_0} = 0, i = 1, \ldots, 7$$

$$x_{1_f} = x_{3_f} = 2x_{2_f} + 3x_{4_f} = x_{6_f} = 0, x_{4_f} = K \qquad (7a)$$

Now, a minimum of H is attained at a minimum of H_1, where H_1 is that part of H which depends on the control variables, y. For the problem here,

$$H_1 = F^* \nu \left(\lambda_1 \cos\beta \sin\alpha + \lambda_2 \cos\beta \cos\alpha + \lambda_3 \sin\beta - \frac{\lambda_7}{C^*} \right) \quad (8)$$

However, note that the final mass, $m_0(1+\xi_f)$ does not appear in the pay-off, i.e., the final mass is left open. However, only those trajectories for which $|\xi_f|$ is of order δ are admissible because of assumptions implicit in eqns (3). Therefore, eqns (6) and (6a) imply that $\lambda_7 \equiv 0$. Consequently, eqn (8) simplifies to:

$$H_1 = F^* \nu (\lambda_1 \cos\beta \sin\alpha + \lambda_2 \cos\beta \cos\alpha + \lambda_3 \sin\beta) \quad (8a)$$

The requirements on the control variables α, β, and ν are determined by reasoning as follows.

Since α is unbounded, $\partial H_1/\partial\alpha = 0$ and $\partial^2 H_1/\partial\alpha^2 \geq 0$ at the minimum of H_1 and hence,

$$\sin\bar{\alpha} = -\lambda_1/(\lambda_1^2+\lambda_2^2)^{\frac{1}{2}}, \quad \cos\bar{\alpha} = -\lambda_2/(\lambda_1^2+\lambda_2^2)^{\frac{1}{2}} \quad (9)$$

H_1 reduces to:

$$H_1 = F^* \nu [-(\lambda_1^2+\lambda_2^2)^{\frac{1}{2}} \cos\beta + \lambda_3 \sin\beta] \quad (10)$$

which can be written in the form

$$H_1 = F^* \nu (\lambda_1^2+\lambda_2^2+\lambda_3^2)^{\frac{1}{2}} \sin(\beta - \varphi)$$

$$\varphi = \sin^{-1} \frac{(\lambda_1^2+\lambda_2^2)^{\frac{1}{2}}}{(\lambda_1^2+\lambda_2^2+\lambda_3^2)^{\frac{1}{2}}} \quad (11)$$

By virtue of the fact that $\sin\varphi \geq 0$, it follows that the principal value of φ lies between 0 and π. This last conclusion, together with eqn (11), permits the deduction that the minimum of H_1 occurs when $\beta = \varphi - \pi/2$*, i.e., when

$$\sin\bar{\beta} = -\frac{\lambda_3}{(\lambda_1^2+\lambda_2^2+\lambda_3^2)^{\frac{1}{2}}}, \quad \cos\bar{\beta} = \frac{(\lambda_1^2+\lambda_2^2)^{\frac{1}{2}}}{(\lambda_1^2+\lambda_2^2+\lambda_3^2)^{\frac{1}{2}}} \quad (12)$$

$$\left(-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2} \right)$$

and H_1 reduces still further to:

$$H_1 = -F^* \nu (\lambda_1^2+\lambda_2^2+\lambda_3^2)^{\frac{1}{2}} \quad (13)$$

From eqn (13) it follows immediately that $\nu = 1$ minimizes H_1 whenever

$$(\lambda_1^2+\lambda_2^2+\lambda_3^2)^{\frac{1}{2}} \neq 0 \quad (14)$$

* Notice that H_1 is also stationary with respect to this value of β, i.e., $\partial H_1/\partial\beta(\bar{\beta}) = 0$.

Furthermore, it can be verified [by solving eqns (6)] that $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \neq 0$ except at a finite number of points on any τ interval of length 2π. Thus the indeterminate values of ν corresponding to $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 0$ form a set of measure zero and our problem is therefore well behaved.

In summary, the optimal control variables depend upon the mutipliers λ_1, λ_2, and λ_3 in the following manner:

$$\sin\bar{\alpha} = -\frac{\lambda_1}{(\lambda_1^2+\lambda_2^2)^{\frac{1}{2}}}, \quad \cos\bar{\alpha} = -\frac{\lambda_2}{(\lambda_1^2+\lambda_2^2)^{\frac{1}{2}}}$$

$$\sin\bar{\beta} = -\frac{\lambda_3}{(\lambda_1^2+\lambda_2^2+\lambda_3^2)^{\frac{1}{2}}}, \quad \cos\bar{\beta} = \frac{(\lambda_1^2+\lambda_2^2)^{\frac{1}{2}}}{(\lambda_1^2+\lambda_2^2+\lambda_3^2)^{\frac{1}{2}}} \quad (15)$$

$$\nu = 1$$

The λ's in turn depend on the unknown Lagrange multipliers, Λ_i through eqn (6), and the solution of the boundary value problem turns upon one's capability to solve for these undetermined constants. However, before attention is directed to the boundary value problem, it is worth while to emphasize two points. First, it is easily demonstrated that the optimal control variables given by eqns (15) satisfy a strong form of eqn (5), i.e.,

$$H_1(y) > H(\bar{y}), \quad y \neq \bar{y} \quad (16)$$

But eqn (16) together with the linear character of eqns (3), are sufficient conditions for a strong relative minimum of P. Thus the assurance is that the control law of eqns (15) will provide a time-optimal transfer between two co-planar circular orbits which is unique whenever the boundary value equations possess a unique solution. Second, it should be noted that the result expressed by the last of eqns (15) is independent of the multiplier functions λ_i, and is therefore insensitive to the boundary conditions. Thus a full throttle operational mode is a characteristic of the entire extremal field of quasi-circular transfer trajectories for the problem of minimum time transfer with final mass open.

The Orbital Transfer Boundary Value Problem—a Special Class of Solutions for Rendezvous

The differential equations, eqns (3), when written in a symbolic matrix notation, have the following structure:

$$x' = Ax + g(y) \quad (17)$$

where A is a matrix of constant coefficients and g is a vector whose elements depend upon the control variables α, β, and ν. Solutions for eqn (17) can therefore usually be phrased in terms of a fundamental solution matrix $\Phi(\tau_f, \tau)$ and superposition integrals, e.g.:

$$x(\tau_f) = \Phi(\tau_f, 0)\, x(0) + \int_0^{\tau_f} \Phi(\tau_f, \tau)\, g[y(\tau)\, d\tau \quad (18)$$

$$\Phi(\tau_f, \tau) = \begin{bmatrix} \cos\hat{\tau} & 2\sin\hat{\tau} & 0 & 3\sin\hat{\tau} & 0 & 0 & 0 \\ -2\sin\hat{\tau} & -(3-4\cos\hat{\tau}) & 0 & -6(1-\cos\hat{\tau}) & 0 & 0 & 0 \\ 0 & 0 & \cos\hat{\tau} & 0 & 0 & -\sin\hat{\tau} & 0 \\ \sin\hat{\tau} & 2(1-\cos\hat{\tau}) & 0 & (4-3\cos\hat{\tau}) & 0 & 0 & 0 \\ -2(1-\cos\hat{\tau}) & -(3\hat{\tau}-4\sin\hat{\tau}) & 0 & -6(\hat{\tau}-\sin\hat{\tau}) & 1 & 0 & 0 \\ 0 & 0 & \sin\hat{\tau} & 0 & 0 & \cos\hat{\tau} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (19)$$

$$g = \{g_1, \ldots, g_7\}^T$$

where

$$g_1 = F^* v \cos\beta \sin\alpha \equiv -\frac{\lambda_1 F^*}{(\lambda_1^2+\lambda_2^2+\lambda_3^2)^{\frac{1}{2}}}$$

$$g_2 = F^* v \cos\beta \cos\alpha = -\frac{\lambda_2 F^*}{(\lambda_1^2+\lambda_2^2+\lambda_3^2)^{\frac{1}{2}}}$$

$$g_3 = F^* v \sin\beta = -\frac{\lambda_3 F^*}{(\lambda_1^2+\lambda_2^2+\lambda_3^2)^{\frac{1}{2}}} \tag{20}$$

$$g_4 = g_5 = g_6 = 0$$

$$g_7 = \frac{F^* v}{C^*} = -\frac{F^*}{C^*}$$

As already pointed out, the λ's depend on the undetermined constants Λ_i in accord with eqn (6a). Furthermore, it is noted that in matrix notation, eqns (6) have the form

$$\lambda' = -A^T \lambda \tag{21}$$

and thus are adjoint differential expressions for eqn (17) [i.e., eqns (3)]. Consequently, their solution is determined when $\Phi(\tau_f, \tau)$ is known, i.e.,

$$\lambda(\tau) = \Phi^T \lambda(\tau_f) \equiv \Phi^T \left\{\frac{\partial P}{\partial x_{i_f}}\right\} \tag{22}$$

where

$$\frac{\partial P}{\partial x_{1_f}} = \Lambda_1, \quad \frac{\partial P}{\partial x_{2_f}} = 2\Lambda_2 \text{ etc.}$$

In view of these considerations, the co-planar circle-to-circle transfer boundary conditions [see eqn (7)] are:

$$u_f = -F^* \int_0^{\tau_f} \frac{(\lambda_1 \cos\hat{\tau} + 2\lambda_2 \sin\hat{\tau})}{(\lambda_1^2+\lambda_2^2+\lambda_3^2)^{\frac{1}{2}}} d\tau = 0$$

$$2v_f + 3\eta_f = -F^* \int_0^{\tau_f} \frac{(2\lambda_2 \cos\hat{\tau} - \lambda_1 \sin\hat{\tau})}{(\lambda_1^2+\lambda_2^2+\lambda_3^2)^{\frac{1}{2}}} = 0$$

$$w_f = -F^* \int_0^{\tau_f} \frac{\lambda_3 \cos\hat{\tau}}{(\lambda_1^2+\lambda_2^2+\lambda_3^2)^{\frac{1}{2}}} d\tau = 0 \tag{23}$$

$$\psi_f = -F^* \int_0^{\tau_f} \frac{\lambda_3 \sin\hat{\tau}}{(\lambda_1^2+\lambda_2^2+\lambda_3^2)^{\frac{1}{2}}} d\tau = 0$$

$$\eta_f = -F^* \int_0^{\tau_f} \frac{[\lambda_1 \sin\hat{\tau} + 2\lambda_2(1-\cos\hat{\tau})]}{(\lambda_1^2+\lambda_2^2+\lambda_3^2)^{\frac{1}{2}}} d\tau = K$$

where

$$\lambda_1 = \Lambda_4 \left[\frac{\Lambda_1}{\Lambda_4}\cos\hat{\tau} + \left(1 - \frac{\Lambda_2}{\Lambda_4}\right)\sin\hat{\tau}\right]$$

$$\lambda_2 = 2\Lambda_4 \left[\frac{\Lambda_1}{\Lambda_4}\sin\hat{\tau} - \left(1 - \frac{\Lambda_2}{\Lambda_4}\right)\cos\hat{\tau} + 1\right]$$

and

$$\lambda_3 = \Lambda_4 \left[\frac{\Lambda_3}{\Lambda_4}\cos\hat{\tau} + \frac{\Lambda_5}{\Lambda_4}\sin\hat{\tau}\right] \tag{24}$$

The natural boundary condition on H [see eqn (5)] determines the magnitude of the scaling constant Λ_4 e.g.,

$$H_f = -\frac{\partial P}{\partial \tau_f} = -1$$

and therefore

$$|\Lambda_4| = \frac{1}{F^*\left(\left(\frac{\Lambda_1}{\Lambda_4}\right)^2 + \left(\frac{\Lambda_3}{\Lambda_4}\right)^2 + 4\left(\frac{\Lambda_2}{\Lambda_4}\right)^2\right)^{\frac{1}{2}}} \tag{25}$$

The sign of Λ_4 will depend upon the sign of K.

When eqns (24) and (25) are substituted in eqns (23), a set five boundary value conditions transcendental in five unknowns, Λ_1/Λ_4, Λ_2/Λ_4, Λ_3/Λ_4, Λ_5/Λ_4, and τ_f are obtained. In general, it is not possible to arrive at an analytical solution of the boundary value problem so stated for arbitrary values of K. However, the following observation is made: if one sets $\Lambda_1/\Lambda_4 = \Lambda_3/\Lambda_4 = \Lambda_5/\Lambda_4 = 0$, and $\Lambda_2/\Lambda_4 = 1$, then eqns (15), (24) and (25) indicate that:

$$|\Lambda_4| = 1/2\,F^*$$

$$\sin\bar{\alpha} = 0, \quad \cos\bar{\alpha} = -\text{sgn}(\Lambda_4)$$

$$\sin\bar{\beta} = 0, \quad \cos\bar{\beta} = 1 \tag{26}$$

which conditions in turn prescribe an in-plane circumferentially directed thrust vector. Furthermore, it should be noted that the first four of eqns (23) are identically satisfied by such a steering programme whenever $\tau_f = 2n\pi$, and the fifth is satisfied for this value of final time when $K = 4n\pi F^* \,\text{sgn}(\Lambda_4)$. Thus, in view of the assumptions, and the sufficiency argument connected with eqn (16), it is concluded that full-throttle in-plane, circumferentially directed thrust is time-optimal for ascending ($\text{sgn}\,\Lambda_4 = -1$) or decending ($\text{sgn}\,\Lambda_4 = +1$) thrust-limited transfer between two neighbouring co-planar circular orbits differing in altitude so that $|(R_f - R_A)/R_A| = 4n\pi F^*$. This result was first obtained by Hinz[2] in an analysis only slightly different from that presented here.

The relatively simple optimal orbit transfer manoeuvre just described can be used as a reference rendezvous trajectory for the guidance analysis, if one imagines that the transfer is initiated at the proper time. For example, if, as in *Figure 3*, a target at point B in orbit 2 leads the vehicle initially at point A in orbit 1 by an angle ε_{T_0}, then it is necessary that $\varepsilon_{T_0} = 3\eta_f^2/8F^*$ in order to effect a rendezvous at point C. The nature of the transfer is such that points B and C are symmetrically deployed with respect to the initial line $0A$.

The Optimal Rendezvous Guidance Approximation

With a class of time-optimal rendezvous trajectories available for reference flight paths, the position is now clear to consider optimal rendezvous guidance. Presumably, the guidance scheme selected should be preferred to be optimal in the same sense as the nominal trajectory. Small disturbance assumptions should also be invoked, thereby obtaining a relatively simple guidance law. However, because of the stationary character of an optimal flight path, an immediate conflict of interest arises. It is found that, to the first order of small control variations, all guidance schemes which satisfy the final boundary conditions of the original problem are equally attractive, i.e., performance is