Lecture Notes in Mathematics 1667

Jesús M.F. Castillo   Manuel González

Three-space Problems in Banach Space Theory

Springer

Jesús M.F. Castillo   Manuel González

Three-space Problems in Banach Space Theory

Springer

Authors

Jesús M. F. Castillo
Departamento de Matemáticas
Universidad de Extremadura
Avda. de Elvas s/n
E-06071 Badajoz, Spain
E-mail: castillo@ba.unex.es

Manuel González
Departamento de Matemáticas
Universidad de Cantabria
Avda. de los Castros s/n
E-39071 Santander, Spain
E-mail: gonzalem@ccaix3.unican.es

Library of Congress Cataloging-in-Publication Data

Castillo, Jesús M. F.
Three-space problems in Banach space theory / Jesús M.F. Castillo, Manuel González.
p. cm. -- (Lecture notes in mathematics ; 1667)
Includes bibliographical references (p. -) and index.
ISBN 3-540-63344-8 (softcover : alk. paper)
1. Banach spaces. 2. Ultraproducts. 3. Duality theory (Mathematics) I. González, Manuel. II. Title. III. Series: Lecture notes in mathematics (Springer-Verlag) ; 1667.
QA3.L28 no. 1667
[QA332.2]
510 s--dc21
[515'.732]

Mathematics Subject Classification (1991): 46B03, 46B20, 46B08, 46B10

ISSN 0075-8434
ISBN 3-540-63344-8 Springer-Verlag Berlin Heidelberg New York

Typesetting: Camera-ready output (Word Perfect 5.1) by the authors
SPIN: 10553314 46/3142-543210 - Printed on acid-free paper

Editorial Policy

§ 1. Lecture Notes aim to report new developments - quickly, informally, and at a high level. The texts should be reasonably self-contained and rounded off. Thus they may, and often will, present not only results of the author but also related work by other people. Furthermore, the manuscripts should provide sufficient motivation, examples and applications. This clearly distinguishes Lecture Notes manuscripts from journal articles which normally are very concise. Articles intended for a journal but too long to be accepted by most journals, usually do not have this "lecture notes" character. For similar reasons it is unusual for Ph. D. theses to be accepted for the Lecture Notes series.

§ 2. Manuscripts or plans for Lecture Notes volumes should be submitted (preferably in duplicate) either to one of the series editors or to Springer- Verlag, Heidelberg . These proposals are then refereed. A final decision concerning publication can only be made on the basis of the complete manuscript, but a preliminary decision can often be based on partial information: a fairly detailed outline describing the planned contents of each chapter, and an indication of the estimated length, a bibliography, and one or two sample chapters - or a first draft of the manuscript. The editors will try to make the preliminary decision as definite as they can on the basis of the available information.

§ 3. Final manuscripts should preferably be in English. They should contain at least 100 pages of scientific text and should include

- a table of contents;
- an informative introduction, perhaps with some historical remarks: it should be accessible to a reader not particularly familiar with the topic treated;
- a subject index: as a rule this is genuinely helpful for the reader.

Further remarks and relevant addresses at the back of this book.

Lecture Notes in Mathematics 1667

Springer
Berlin
Heidelberg
New York
Barcelona
Budapest
Hong Kong
London
Milan
Paris
Santa Clara
Singapore
Tokyo

Foreword

A three-space problem, in the Banach space setting, has the form: Let P be a Banach space property, let Y be a subspace of X and let X/Y be the corresponding quotient space. Is it true that X has property P when Y and X/Y have property P ? If the answer is positive then P is said to be a *three-space property*. We shall often shorten "three-space" to $3SP$ (see below).

Three-space questions face the general problem of the structure of arbitrary subspaces and quotients of Banach spaces. Nowadays, it is clear that only Hilbert spaces can be labelled as simple: all subspaces are complemented Hilbert spaces and all quotients are Hilbert spaces. Other spaces may contain (and usually do) bizarre uncomplemented subspaces and/or quotients. The point of view of $3SP$ problems is to consider the structure of a Banach space contemplating *as a whole* the structure of subspaces and the quotients they produce. Nevertheless, as it is noted in [136], "it may happen that a Banach space with quite complicated structure may possess nice factors through nice subspaces." When a property turns out to be a $3SP$ property, it means that, despite the complexity that the combination of subspaces and quotients may have, the structure related with P is maintained. Incidentally, this gives a way to prove that a space possesses property P.

In the study of a $3SP$ question it is interesting, when the answer is affirmative, to uncover the structure behind the proof; for instance, *being finite-dimensional, reflexive,* or *no containing* l_1 are all $3SP$ properties for essentially the same reason: the possibility of a lifting for the different types of sequences considered (resp. convergent, weakly convergent or weakly Cauchy). When it is not, the construction of a counterexample is more often than not a rewarding task. Also, there is the general question of how to construct the "middle" space starting with two spaces having P: one space Y that plays the role of the subspace and other Z that plays the role of the quotient. The simplest form to do this is making their direct sum $Y \oplus Z$. In direct sums the factor spaces appear complemented. Spaces admitting Y as a non-necessarily complemented subspace and Z as the corresponding quotient are called "twisted sums." The interesting general problem of how to construct twisted sums is considered, at an elementary level, in Chapters 1 and 3. This problem spreads into many ramifications: categorical methods, semi L-summands, inductive limits, construction of $C(K)$ spaces, interpolation theory, etc, each of them contemplating a different aspect of the problem.

Conversely, a negative answer to a $3SP$ question is also of interest because it must involve a new construction of a space or an operator (usually the quotient map). It is then no surprise that some 3*SP* examples can be translated into examples of pathological operators; this topic has not been sufficiently exploited in the literature, reason for which we present an overall view in an appendix to Chapter 2.

The origin of 3*SP* problems can be traced back to a problem of Palais: If Y and X/Y are Hilbert spaces, has X to be isomorphic to a Hilbert space? Chronologically, 3*SP* results did not formally appear until middle seventies, although Krein and Smullian's proof that reflexivity is a 3*SP* property came around 1940. Counterexamples appeared with Enflo, Lindenstrauss and Pisier's solution to Palais problem (1975), while methods can be seriously considered after Jarchow's paper (1984). In this case, however, methods and counterexamples can be thought of as two different approaches to the same problem: some counterexamples provide a method, and methods sometimes provide a counterexample. Examples of counterexamples that provide methods could be the Kalton and Peck solution to Palais problem, which gave birth to the beautiful theory of quasi-linear maps; or Lusky's proofs that every Banach space X containing c_0 admits a subspace Y such that both Y and X/Y have a basis.

Currently it is a standard question to ask for the 3*SP* character of a new property. For this reason we devoted Chapter 2 to explain methods to obtain 3*SP* properties. These methods include (and, we hope, cast some light on) some classical topics such as lifting results or factorization of operators. The counterexamples can be found in chapters 3 to 7 collected in what could be seen as something like a zoo.

Thus, we propose the reader a guided tour through that zoo. These notes exhibit alive all (all?) specimens of 3*SP* problems that have been treated in the literature and which freely lived disseminated in many research papers, their natural habitat since they scarcely appeared in books. They have been patiently hunted and captured, carefully carried over (the common features of counterexamples have been emphasized), neatly polished (we often give simpler proofs) and classified following their nature (the proofs were unified through several general methods). There are also many new results and open problems.

The notation we follow is rather standard except in one point: the abbreviation 3*SP* for "three-space." Although it is not standard, it is clear, clean and direct; started with our e-mail messages and has achieved some success.

We have intended to give an essentially self-contained exposition of 3*SP* problems in the context of Banach spaces. There is a vast unexplored land of 3*SP* problems in other contexts (locally convex spaces, Banach algebras,...) that we shall not consider. All properties appearing in the book are defined and their basic relationships stated with appropriate references. Thus, some bounds about where to stop had to be imposed: things directly involved with 3*SP* questions appear in detail; other results shall be just cited when they have already appeared in books. The word **Theorem** is reserved for results having the form: property *P* is (or is not) a 3*SP* property. We made an effort to keep the reader informed about who did exactly what; this information is often given during the introductory comments although, in some cases, even conversations with the authors could not completely clarify where the ideas came from, and so we give our interpretation. About the problems and questions scattered through the text, we can only say that, to the best of our knowledge, they are open; some appear certainly to be difficult, some could even be easy, all seem interesting.

The background required to read these notes is a course in functional analysis and some familiarity with modern Banach space methods, as can be obtained from, e.g., J. Diestel's *Sequences and Series in Banach Spaces*. In order that the book can be used as a reference text we have included a summary, in alphabetical order, containing four entries for each property: 1) name; 2) yes, not, or open, for the corresponding 3*SP* problem; 3) general method of proof, counterexample or additional information; 4) location of the result in the book.

Many people provided us with a constant flow of preprints, information, questions or advice that kept the project alive. To all of them our gratitude flows back. Working friends Félix Cabello, Rafael Payá, Fernando Sánchez and David Yost wrote several new proofs for this book, tried to re-do some sections ... and we are sure they would have written the entire book ... had we left them the opportunity. Susan Dierolf, Ricardo García, Hans Jarchow, Mar Jiménez, Antonio Martínez-Abejón, Mikhail Ostrovskii and Cristina Pérez made useful suggestions. Moreover, thanks should go to the colleagues of the Mathematics Department of the Universidad de Extremadura and of the University of Cantabria for providing the natural conditions for working. The topic of the book was discussed with colleagues and lectured on at several meetings during these three long years of elaboration: the Curso de Verano of the Universidad de Cantabria en Laredo; the Analysis Seminar at the Universitá di Bologna, organized by Piero Papini; the Conference on Functional Analysis at Camigliatello, organized by Antonio Carbone and Giuseppe Marino; the Conference on Function Spaces at the Universidad Complutense de Madrid, organized by José Luis LLavona and Angeles Prieto; and the 2nd Congress on Banach spaces at Badajoz. Thanks are due in each case to the organizers and supporting institutions.

Beloved Boojums and the Bellman. The first author thanks the students of the Analysis seminar who attended a course on 3*SP* problems, and especially to F. Arranz "Curro" who arranged a place for some invention. During this time, what started as a harmless photographic safari became a frumious hunting, sometimes galumphing. Perhaps no one has described it better than Lewis Carroll in *The Hunting of the Snark*; we suggest a careful reading of it to anyone interested in mathematical research. For instance, take the last but one stanza of the third fit.

```
I engage with the Snark - every night after dark -
        In a dreamy delirious fight:
 I serve it with greens in those shadowy scenes,
      And use it for striking a light.
```

On a higher ground, it seems to me that my mother and my sister have been working almost as much as I did to carry this book through. Without their supporting hand and their endless patience I would probably have softly and suddenly vanished away. This not being so, the real 3*SP* problem for me has been then to combine the space of personal life, that of my University duties and the space required to make the book. Luna was the (precious) kernel that made this sequence exact. And one cannot leave unmentioned the exact sequence of our dogs: Suerte, Burbujas and Schwarz —which is my sister's dog, but anyway. Luna's mother and father, Javier Blanco, Antonio Hinchado (as the Banker) and Paco Pons (as the Booker) made a nice crew that helped to mix the bowsprit with the rudder sometimes.

The manuscript was patiently typed by Luna Blanco. Financial support came in part from the DGICYT (Spain) Project PB94-1052.

Although Bellman [51] affirms *"everything I say three times is true"* that does not necessarily apply to everything *we* say about three-spaces.

Contents

Chapter 1

Three-space constructions

This chapter is about the structure of a three-space (in short, 3*SP*) construction. We first show the tools of the theory of categories that shall be needed: short exact sequences, pull-back and push-out constructions and the three-lemma; some knowledge of functors and natural transformations and the functor *Ext*. Given spaces *Y* and *Z*, an *extension of Y by Z* is a Banach space *X* such that *Y* is a closed subspace of *X* and *X*/*Y* is isomorphic to *Z*. For a given couple (*Y*, *Z*) of spaces, *Ext*(*Y*, *Z*) denotes the set of all extensions of *Y* by *Z* modulo a suitable equivalence relation. If **Ban** denotes the category of Banach spaces and **Set** that of sets, the correspondence is done in such a way that *Ext*: **Ban**×**Ban** → **Set** is a functor in each component. Category theory provides two methods to obtain the elements of *Ext*(*Y*, *Z*): one using projective elements ($l_1(I)$-spaces) and the other using injective elements ($l_\infty(I)$-spaces). We shall describe the two methods in detail translating the elements of the general theory to the Banach space setting.

For quasi-Banach spaces (in general, not locally convex spaces), categories do not provide with effective ways to calculate *Ext*(*Y*, *Z*) since projective or injective spaces do not necessarily exist. A new theory was developed by Kalton [203] and Kalton and Peck [213] to work with quasi-Banach spaces: it is based on the so-called quasi-linear maps, and consists in proving that extensions of two quasi-Banach spaces correspond to quasi-linear maps. We shall describe this approach in correspondence, when the spaces are locally convex, with the categorical point of view .

Chronologically, the seminal breakthrough was Enflo-Lindenstrauss-Pisier negative solution to Palais problem: *if a Banach space X admits a closed subspace isomorphic to a Hilbert space and such that the corresponding quotient is also isomorphic to a Hilbert space, must X be isomorphic to a Hilbert space?* (or, *is to be isomorphic to a Hilbert space a 3SP property?*). Their construction, although related to finite-dimensional properties of Hilbert spaces, relied on the quasi-linear behaviour of some functions. Indeed, two were the ingredients of their counterexample: (1) construction of (in the terminology of this book) 0-linear finite-dimensional maps such that (2) are increasingly far from linear as the dimension increases. Then Ribe [306] and Kalton [203], independent and almost simultaneously, obtained solutions to the 3*SP* problem for Banach spaces, showing that there exists a quasi-Banach space having an uncomplemented copy of $\mathbb{R}$ such that the corresponding quotient space is isomorphic to l_1. Kalton's paper included many other results, such as the proof that an example replacing l_1 by l_p cannot exist. Moreover, an embryo of what shall become the general theory of twisted sums appears there, with a proof that the only way to obtain a nontrivial twisted sum (i.e., not a direct sum) is that the quasi-linear map be at infinite distance of linear maps. Then it came Kalton and Peck' shaking ground paper [213] with a new counterexample to Palais problem and the whole theory about twisted sums, showing that the way in which Enflo, Lindenstrauss and Pisier had done it was essentially the only way (with the freedom to choose the quasi-linear map).

While this chapter contains the general theory, solutions to Palais problem (i.e., concrete ways to obtain nontrivial quasi-linear maps) shall occur in Chapter 3.

1.1 Short exact sequences

For all basic elements of category theory and homological algebra we suggest [169].

The category **Ban** has Banach spaces as objects and continuous linear mappings as arrows. A short exact sequence in **Ban** is a diagram

$$0 \to Y \overset{i}{\to} X \overset{q}{\to} Z \to 0$$

where the image of each arrow coincides with the kernel of the following one. When no further explanation appears, i denotes the injection and q the quotient map. The open mapping theorem ensures that, given a short exact sequence as before, Y is a subspace of X and Z is the corresponding quotient X/Y. In this context, a 3*SP* problem is: let $0 \to Y \to X \to Z \to 0$ be a short exact sequence in which the spaces Y and Z have a given property P. Must the space X also have P?

In the category **Ban** the *kernel* of a (linear continuous) map α: $X \to Y$ is the canonical inclusion i: $Ker\ \alpha \to X$ (and the *cokernel* is the canonical quotient map q: $Y \to Y/\overline{Im\ \alpha}$). Some basic results about short exact sequences (valid in arbitrary categories where kernels and cokernels exist) follow:

Two short exact sequences $0 \to Y \to X \to Z \to 0$ and $0 \to Y \to U \to Z \to 0$ are said to be *equivalent* if some arrow $X \to U$ exists making commutative the squares of the diagram

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & X & \to & Z & \to & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \to & Y & \to & U & \to & Z & \to & 0 \end{array}$$

This definition makes sense thanks to "the 3-lemma" :

The 3-lemma. *Given a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\ 0 & \to & D & \to & E & \to & F & \to & 0 \end{array}$$

with exact rows, if α *and* γ *are injective then* β *is injective; also, if* α *and* γ *are surjective then* β *is surjective. Consequently, if* α *and* γ *are isomorphisms, then also* β *is an isomorphism.*

Proof. Since there is no place for confusion, we call i the injections in the two short exact sequences and q the quotient maps.

(INJECTIVITY) If $\beta b=0$ with $b \in B$ then $\gamma q b=0$ and $qb=0$. Thus $b=ia$ for some $a \in A$, and since $i\alpha a = \beta i a = \beta b = 0$, hence $a=0$ and thus $b=ia=0$.

Observe that the exactness of the sequence $0 \to D \to E \to F \to 0$ at E or F has not been used. This shall be important later.

(SURJECTIVITY) Let $e \in E$. Since γ is surjective, for some $c \in C$, $\gamma c=qe$. Thus, for some $b \in B$ it must happen $\gamma q b=qe$. Since $q(\beta b-e)=0$, $\beta b-e = id$ for some $d \in D$; and since α is surjective, $\alpha a = d$, which implies $\beta i a = i\alpha a = id = \beta b - e$ and then $e = \beta(b-ia)$. □

Observe that the exactness of the sequence $0 \to A \to B \to C \to 0$ at A or B has not been used. This shall be important later.

A short exact sequence $0 \to Y \to X \to Z \to 0$ is said to *split* if it is equivalent to the sequence $0 \to Y \to Y \times Z \to Z \to 0$. Given an arrow α in a category **Cat**, a *section* for α is a right inverse; i.e., an arrow s in **Cat** such that $\alpha s=1$. A *retraction* for α is a left inverse, i.e., an arrow r in the category such that $r\alpha=1$. When no other comment is made, we understand that sections or retractions are so in **Ban**. If they simply are sections or retractions in **Set** then we shall usually refer to them as "selection maps."