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Algebraic Theories

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"Universal algebra has been looked on with some suspicion by many mathematicians as being comparatively useless as an engine of investigation."

Alfred North Whitehead
[Whitehead 1897, preface]

"General classifications of abstract systems are usually characterized by a wealth of terminology and illustration, and a scarcity of consequential deduction."

Garrett Birkhoff
[Birkhoff 1935, page 438]

"Since Hilbert and Dedekind, we have known very well that large parts of mathematics can develop logically and fruitfully from a small number of well-chosen axioms. That is to say, given the bases of a theory in an axiomatic form, we can develop the whole theory in a more comprehensible way than we could otherwise. This is what gave the general idea of the notion of mathematical structure. Let us say immediately that this notion has since been superseded by that of category and functor, which includes it under a more general and convenient form."

Jean Dieudonné
[Dieudonné 1970, page 138]

Preface

In the past decade, category theory has widened its scope and now interacts with many areas of mathematics. This book develops some of the interactions between universal algebra and category theory as well as some of the resulting applications.

We begin with an exposition of equationally defineable classes from the point of view of "algebraic theories," but without the use of category theory. This serves to motivate the general treatment of algebraic theories in a category, which is the central concern of the book. (No category theory is presumed; rather, an independent treatment is provided by the second chapter.) Applications abound throughout the text and exercises and in the final chapter in which we pursue problems originating in topological dynamics and in automata theory.

This book is a natural outgrowth of the ideas of a small group of mathematicians, many of whom were in residence at the Forschungsinstitut für Mathematik of the Eidgenössische Technische Hochschule in Zürich, Switzerland during the academic year 1966–67. It was in this stimulating atmosphere that the author wrote his doctoral dissertation. The "Zürich School," then, was Michael Barr, Jon Beck, John Gray, Bill Lawvere, Fred Linton, and Myles Tierney (who were there) and (at least) Harry Appelgate, Sammy Eilenberg, John Isbell, and Saunders Mac Lane (whose spiritual presence was tangible.)

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I wish to thank many of my colleagues, particularly Michael Arbib, Michael Barr, Jack Duskin, Hartmut Ehrig, Walter Felscher, John Isbell, Fred Linton, Saunders Mac Lane, Robert Paré, Michael Pfender, Walter Tholen, Donovan Van Osdol, and Oswald Wyler, whose criticisms and suggestions made it possible to improve many portions of this book; and Saunders Mac Lane, who provided encouragement on many occasions.

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Introduction

"Groups," "rings," and "lattices" are definable in the language of finitary operations and equations. "Compact Hausdorff spaces" are also equationally definable except that the requisite operations (of ultrafilter convergence) are quite infinitary. On the other hand, systems of structured sets such as "topological spaces" cannot be presented using only operations and equations. While "topological groups" is not equational when viewed as a system of sets with structure, when viewed as a system of "topological spaces with structure" the additional structure *is* equational; here we must say equational "over topological spaces."

The program of this book is to define for a "base category" $\mathscr{K}$—a system of mathematical discourse consisting of objects whose structure we "take for granted"—categories of $\mathscr{K}$-objects with "additional structure," to classify where the additional structure is "algebraic over $\mathscr{K}$," to prove general theorems about such algebraic situations, and to present examples and applications of the resulting theory in diverse areas of mathematics.

Consider the finitary equationally definable notion of a "semigroup," a set X equipped with a binary operation $x \cdot y$ which is associative:

$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

For any set A, the two "derived operations" or *terms*

$$a_1 \cdot [a_2 \cdot ((a_3 \cdot a_4) \cdot (a_5 \cdot a_6))], \qquad (a_1 \cdot a_2) \cdot [a_3 \cdot (a_4 \cdot (a_5 \cdot a_6))]$$

(with $a_1, \ldots, a_6$ in A) are "equivalent" in the sense that one can be derived from the other with (two) applications of associativity. The quotient set of all equivalence classes of terms with "variables" in A may be identified with the set of all parenthesis-free strings $a_1 \cdots a_n$ with $n > 0$; call this set AT. A function $\beta: B \longrightarrow CT$ extends to the function

$$\begin{array}{ccc} BT & \xrightarrow{\beta^{\#}} & CT \\ b_1 \cdots b_n & \longrightarrow & \beta_{b_1} \cdots \beta_{b_n} \end{array}$$

whose syntactic interpretation is performing "substitution" of terms with variables in C for variables of terms in BT. Thus, for each A, B, C, there is the composition

$$(A \xrightarrow{\alpha} BT, B \xrightarrow{\beta} CT) \longrightarrow A \xrightarrow{\alpha\beta} CT = A \xrightarrow{\alpha} BT \xrightarrow{\beta^{\#}} CT$$

There is also the map

$$A \xrightarrow{A\eta} AT, \qquad a \longmapsto a$$

which expresses "variables are terms." $\mathbf{T} = (T, \eta, \circ)$ is the "algebraic theory" corresponding to "semigroups."

In general, an *algebraic theory* (of sets) is any construction $\mathbf{T} = (T, \eta, \circ)$ of the above form such that $\circ$ is associative, η is a two-sided unit for $\circ$ and

$$(A \xrightarrow{f} B \xrightarrow{B\eta} BT) \circ (B \xrightarrow{\beta} CT) = A \xrightarrow{f} B \xrightarrow{\beta} CT$$

A **T**-*algebra* is then a pair (X, ξ) where $\xi : XT \longrightarrow X$ satisfies two axioms, and a **T**-*homomorphism* $f:(X, \xi) \longrightarrow (Y, \theta)$ is a function $f: X \longrightarrow Y$ which "preserves" the algebra structure; see section 1.4 for the details.

If **T** is the algebraic theory for semigroups then "semigroups" and "**T**-algebras" are isomorphic categories of sets with structure in the sense that for each set X the passage from semigroup structures $(X, \bullet)$ to **T**-algebra structures (X, ξ) defined by

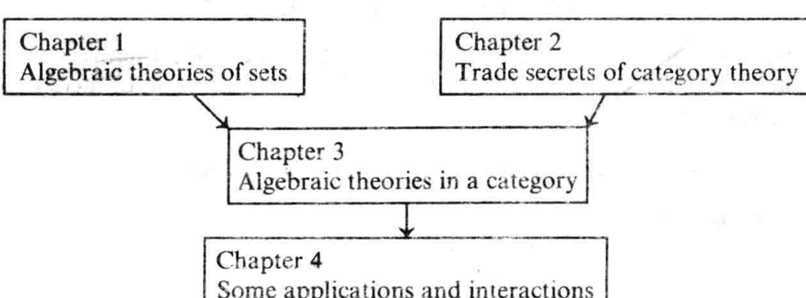
$$(x_1 \cdots x_n)\xi = x_1 \bullet \cdots \bullet x_n$$

is bijective in such a way that $f:(X, \bullet) \longrightarrow (Y, *)$ is a semigroup homomorphism if and only if it is a **T**-homomorphism between the corresponding **T**-algebras.

The situation "over sets," then, is as follows. Every finitary equational class induces its algebraic theory **T** via a terms modulo equations construction generalizing that for semigroups, and the **T**-algebras recover the original class. The "finitary" theories—those which are induced by a finitary equational class—are easily identified abstractly. More generally, any algebraic theory of sets corresponds to a (possibly infinitary) equationally-definable class. While the passage from finitary to infinitary increases the syntactic complexity of terms, there is no increase in complexity from the "algebraic theories" point of view. It is also true that many algebraic theories arise as natural set-theoretic constructions before it is clear what their algebras should be. Also, algebraic theories are interesting algebraic objects in their own right and are subject to other interpretations than the one we have used to motivate them (see section 4.3).

An examination of the definition of the algebraic theory **T** and its algebras and their homomorphisms reveals that only superficial aspects of the theory of sets and functions between them are required. Precisely what is needed is that "sets and functions" forms a category (as defined in the section on preliminaries). Generalization to the "base category" is immediate.

The relationship between the four chapters of the book is depicted below:



The first chapter is a selfcontained exposition (without the use of category theory) of the relationships between algebraic theories of sets and universal algebra, finitary and infinitary. The professional universal algebraist wishing to learn about algebraic theories will find this chapter very easy reading.

The second chapter may be read independently of the rest of the book, although some of the examples there relate to Chapter 1. We present enough category theory for our needs and at least as much as every pure mathematician should know! The section on "objects with structure" uses a less "puristic" approach than is currently fashionable in category theory; we hope that the reader will thereby be more able to generalize from previous knowledge of mathematical structures.

The third chapter, which develops the topics of central concern, draws heavily from the first two. The choice of applications in the fourth chapter has followed the author's personal tastes.

Why is the material of the third chapter useful? Well, to suggest an analogy, it is dramatic to announce that a concrete structure of interest (such as a plane cubic curve) is a group in a natural way. After all, many naturally-arising binary operations do not satisfy the group axioms; and, moreover, a lot is known about groups. In a similar vein, it is useful to to know that a category of objects with structure is algebraic because this is a special property with nice consequences and about which much is known.

Many exercises are provided, sometimes with extended hints. We have avoided the noisome practice of framing crucial lemmas used in the text as "starred" exercises of earlier sections. For lack of space we have, however, developed many important topics entirely in the exercises.

Reference a.b.c refers to item c of section b in Chapter a. Depending on context, d.e refers to section e of Chapter d or to item e of section d of the current chapter.

Preliminaries

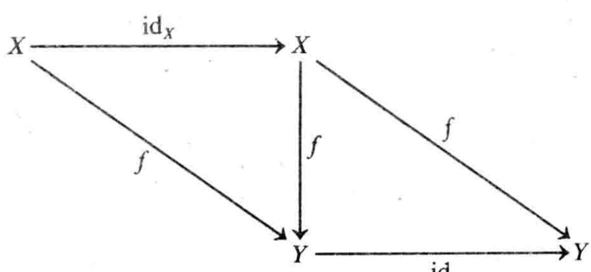
The reader is expected to have some background in set-theoretic pure mathematics. We assume familiarity with the concept of *function* $f: X \longrightarrow Y$ between sets and a minimum of experience with algebra and topology, e.g. the definitions of "topological space," "continuous mapping of topological spaces," "group," and "homomorphism of groups."

A variety of notations are employed for the evaluation of a function f on its argument x. Usually we write xf instead of fx or $f(x)$ (although $d(x, y)$, for the distance between two points in a metric space, is chosen over $(x, y)d$). Another notation for xf is $\langle x, f \rangle$. This notation is especially convenient when x or f is a long expression. We also employ the "passage arrow" $\mapsto$ and write $x \longmapsto xf$ which is read "x is sent to xf". This notation is useful when defining functions.

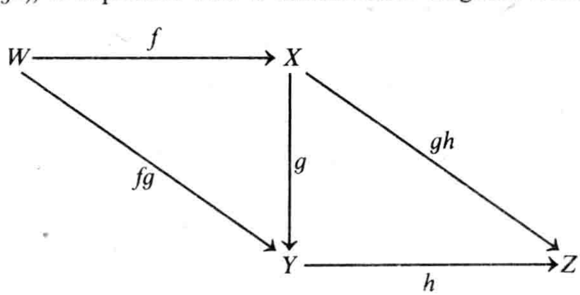
The composition of functions

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

will be written fg or $f.g$. Thus $x(fg) = x(f.g) = (xf)g$. For any set X, the *identity function of* X is the function $\mathrm{id}_X: X \longrightarrow X$ defined by $x(\mathrm{id}_X) = x$. It is clear that for any $f: X \longrightarrow Y$ we have $\mathrm{id}_X.f = f = f.\mathrm{id}_Y$. This may be expressed by the *commutative diagram*. We say the diagram *commutes*



because all composition paths between the same sets in the diagram are the same function. Similarly, the familiar associative law of composition, $(fg)h = f(gh)$, is expressed with a commutative diagram. Because of the



associative law, $fgh: W \longrightarrow Z$ is well defined; and it is this principle that allows commutative diagrams to display effectively the result of composing long chains of functions.

The theory of categories, functors between categories, and natural transformations between functors—to the extent that it is needed—is developed gradually beginning with Chapter 2. Since certain functors and natural transformations arise naturally in Chapter 1, these concepts are defined here. A *category* $\mathscr{K}$ is defined by the following data and axioms.

Datum 1. There is given a class $\text{Obj}(\mathscr{K})$ *of* $\mathscr{K}$*-objects.*

Datum 2. For each ordered pair (A, B) *of* $\mathscr{K}$*-objects there is given a class* $\mathscr{K}(A, B)$ *of* $\mathscr{K}$*-morphisms from* A *to* B. *If* $f \in \mathscr{K}(A, B)$, A *is the domain of* f *and* B *is the codomain of* f *(see Axiom 3).*

Datum 3. For each $\mathscr{K}$*-object* A *there is given a distinguished* $\mathscr{K}$*-morphism* $\text{id}_A \in \mathscr{K}(A, A)$ *called the identity of* A.

Datum 4. For each ordered triple (A, B, C) *of* $\mathscr{K}$*-objects there is given a composition law*

$$\mathscr{K}(A, B) \times \mathscr{K}(B, C) \longrightarrow \mathscr{K}(A, C)$$
$$(f, g) \longmapsto fg$$

Axiom 1. Composition is associative, that is given $f \in \mathscr{K}(A, B)$, $g \in \mathscr{K}(B, C)$ *and* $h \in \mathscr{K}(C, D)$ *then* $(fg)h = f(gh) \in \mathscr{K}(A, D)$.

Axiom 2. If $f \in \mathscr{K}(A, B)$ *then* $(\text{id}_A)f = f = f(\text{id}_B)$.

Axiom 3. If $(A, B) \neq (A', B')$ *then* $\mathscr{K}(A, B) \cap \mathscr{K}(A', B') = \varnothing$.

"Sets and functions" form a category which we will denote henceforth by **Set**. Thus a **Set**-object is an arbitrary set and **Set**(A, B) is the set of functions from A to B. Identities and composition are defined in the way already discussed. Axiom 3 asserts that for the purposes of category theory, a function is not properly defined unless the set it maps from and the set it maps to are included in the definition. Thus the polynomial x^2 thought of as mapping all the real numbers into itself is a different function from x^2 thought of as mapping all the nonzero real numbers into the set of all real numbers.

The reader should recognize at once that "topological spaces and continuous mappings" as well as "groups and group homomorphisms" are two further examples of categories.

If $\mathscr{K}$ is an arbitrary category we will write $f: A \longrightarrow B$ to denote $f \in \mathscr{K}(A, B)$. We will also use $f.g$ as an alternate notation to fg. Axioms 1, 2 can be expressed as commutative diagrams just as we did earlier for the category **Set**. Let $\mathscr{K}$ and $\mathscr{L}$ be two categories. A *functor, H, from* $\mathscr{K}$ *to* $\mathscr{L}$ is defined by the following data and axioms:

Datum 1. For each $\mathscr{K}$*-object* A, *there is given an* $\mathscr{L}$*-object* AH.

Datum 2. For each $\mathscr{K}$*-morphism of form* $f: A \longrightarrow B$ *there is given an* $\mathscr{L}$*-morphism of form* $fH: AH \longrightarrow BH$.

Axiom 1. *H preserves identities; that is, for every $\mathscr{K}$-object A,* $(\mathrm{id}_A)H = \mathrm{id}_{AH}$.

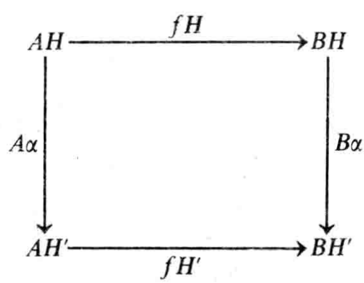
Axiom 2. *H preserves composition, that is, given $f:A \longrightarrow B$ and $g:B \longrightarrow C$ in $\mathscr{K}$, $(f.g)H = fH.gH:AH \longrightarrow CH$ in $\mathscr{L}$.*

We use the notation $H:\mathscr{K} \longrightarrow \mathscr{L}$ if H is a functor from $\mathscr{K}$ to $\mathscr{L}$.

Suppose now that H, $H':\mathscr{K} \longrightarrow \mathscr{L}$ are two functors between the same two categories. A *natural transformation* α *from H to H'* is defined by the following datum and axiom:

Datum. *For each $\mathscr{K}$-object A there is given an $\mathscr{L}$-morphism $A\alpha:AH \longrightarrow AH'$.*

Axiom. *For each $\mathscr{K}$-morphism $f:A \longrightarrow B$ the following square of $\mathscr{L}$-morphisms is commutative:*



i.e., $A\alpha.fH' = fH.B\alpha$.

We use the notation $\alpha:H \longrightarrow H'$ when α is a natural transformation from H to H'.

Chapter 1

Algebraic Theories of Sets

This chapter is a selfcontained introduction to algebraic theories of sets. Category theory is not used in the development. The motivating example of equationally-definable classes is eventually seen to be coextensive with algebraic theories with rank. Compact Hausdorff spaces and complete atomic Boolean algebras arise as algebras over theories (without rank) whereas complete Boolean algebras do not.

1. Finitary Universal Algebra

In this section we define (finitary) *equationally-definable classes*. Further systematic study of finitary universal algebra is referred to the literature (see the notes at the end of this section) but some of the standard examples are developed in the exercises.

There are a number of ways to define the concept of a group. Here are three of them:

1.1 Definition. *A group is a set X equipped with a binary operation $m: X \times X \longrightarrow X$ (multiplication), a unary operation $i: X \longrightarrow X$ (inversion) and a distinguished element $e \in X$ (the unit) subject to the equations*

$xymzm = xyzmm$	(m *is associative*)
$xem = x = exm$	(e *is a two-sided unit for* m)
$xixm = e = xxim$	(xi *is the multiplicative inverse of* x)

for all x, y, z in X.

(Notice the use, in 1.1, of parenthesis-free "Polish notation," e.g. $xymzm$ instead of $((x, y)m, z)m$. A formal proof that this notation works is given below in 1.11.)

1.2 Definition. *A group is a set X equipped with a binary operation $d: X \times X \longrightarrow X$ (division) subject to the single incredible equation*

$$xxxdydzdxxdxdzddd = y$$

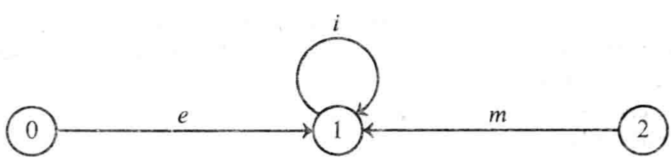
for all x, y, z in X. It is proved in [Higman & Neumann, '52] that a bijective passage from 1.1 to 1.2 is obtained by $xyd = xyim$. The structure of "$xxxdydzdxxdxdzddd$" is examined in 1.13 below.

1.3 Definition. *A group is a set X equipped with a binary operation m such that m is associative and admits unit and inverses, i.e., such that there exists* *a unary operation i and a distinguished element e of X subject to the equations of 1.1.*

Very roughly speaking, group theory is an algebraic theory and 1.1, 1.2, 1.3 are presentations of that theory. (Actually, the empty set is a group according to 1.2 but not according to 1.1 and 1.3; to remedy this one should modify 1.2 by requiring a distinguished element e satisfying $xed = x$.) The first two are equational presentations in that they take the form of a set of operations subject to a set of equations, whereas the third is not an equational presentation because existential quantification is not equationally expressible. We devote this section to setting down, in precise terms, the definition of a finitary equational presentation (Ω, E) and the resulting equationally-definable class (or variety) of all (Ω, E)-algebras.

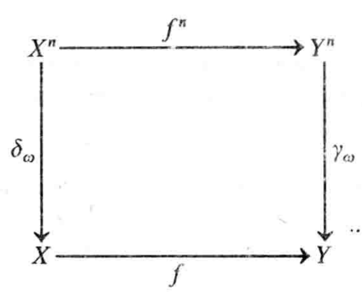
1.4 Definition. *An operator domain is a disjoint sequence of sets,* $\Omega = (\Omega_n : n = 0, 1, 2 \ldots)$. Ω_n *is the set of n-ary operator labels of* Ω.

We remark, as an aside, that an operator domain may be viewed as a directed graph whose nodes are natural numbers and whose edges terminate at 1. Thus a directed graph suitable for "groups" as in 1.1 is



This point of view is a natural precursor to viewing an operator domain as a category, an approach which receives only brief treatment in this book (see 1.5.35, the notes to section 3, Exercises 2.1.25–27 and Exercise 3.2.7).

An Ω*-algebra is a pair* (X, δ) *where* X *is a set and* δ *assigns to each* ω *in* Ω_n *an n-ary operation* $\delta_\omega : X^n \longrightarrow X$. *Given* Ω*-algebras* (X, δ) *and* (Y, γ) *an* Ω*-homomorphism from* (X, δ) *to* (Y, γ) *is a function* $f : X \longrightarrow Y$ *which commutes with the* Ω*-operations, that is, for all* $\omega \in \Omega_n$ *and n-tuples* $(x_1, \ldots, x_n)$ *of* X, *we have* $(x_1, \ldots, x_n)\delta_\omega f = (x_1 f, \ldots, x_n f)\gamma_\omega$. Denoting the passage of $(x_1, \ldots, x_n)$ to $(x_1 f, \ldots, x_n f)$ by $f^n : X^n \longrightarrow Y^n$, this may be equivalently written as the commutative square:



(1.5)

1.6 Example. Define $\Omega_0 = \{e\}$, $\Omega_1 = \{i\}$, $\Omega_2 = \{m\}$, $\Omega_n = \phi$ for all $n > 2$. Then every group (as in 1.1) is an Ω-algebra, but not conversely. The Ω-homomorphisms between groups are ordinary group homomorphisms.

An equational presentation, as is yet to be defined, should consist of a pair (Ω, E) where Ω is an operator domain and E is a set of Ω-equations. To properly formulate "Ω-equation" we must formalize the construction of expressions such as *xxmzm* and *xxim*.

1.7 Definition. *Let A be a set. A word in A is an n-tuple of elements of A with n an integer >0; n is the length of the word.* We will write $a_1a_2 \cdots a_n$ instead of $(a_1, \ldots, a_n)$ to convey the feeling of "word in the alphabet A." An expression such as a_1a_2m is a word in the appropriate "alphabet" A. In general, let Ω be an operator domain, set $|\Omega|$ to be the union of all Ω_n, and define an *Ω-word in A* to be a word in the disjoint union $A + |\Omega|$; (the *disjoint union* of the sets X, Y is the set $X + Y = (X \times \{0\}) \cup (Y \times \{1\})$). Notationally, we will use separate symbols for elements of A and elements of $|\Omega|$ and write Ω-words as words in $A \cup |\Omega|$. If Ω is as in 1.6, *abmcm*, *eam*, and *ei* are all Ω-words in A; unfortunately, so are nonsense words such as *mmamib*. An *Ω-term in A* is an Ω-word in A which can be derived by finitely many applications of 1.8 and 1.9 below:

(*1.8*) *a is an Ω-term in A for all $a \in A$.*

(*1.9*) *If $\omega \in \Omega_n$ and $p_1, \ldots, p_n$ are Ω-terms in A, then $p_1 \cdots p_n\omega$ is an Ω-term in A.*

The set of all Ω-terms in A will be denoted $A\Omega$.

Intuitively, an Ω-word in A is a term if and only if it has the appearance of a well-defined function in finitely-many variables of A. For example, if Ω is as in 1.6 and if A has at least three distinct elements a, b, c then the doubleton $\{abmcm, abcmm\}$ is the essence of the associative law; for if (X, δ) is any Ω-algebra and if (x_1, x_2, x_3) is any 3-tuple of elements of X then by virtue of the substitution "x_1 for a, x_2 for b, x_3 for c", *abmcm* induces the ternary operation $((x_1, x_2)\delta_m, x_3)\delta_m$ on X and *abcmm* similarly induces a ternary operation on X; (X, δ) satisfies the associative law if and only if these ternary operations are the same. This motivates

1.10 Definition. *Fix any convenient* (effectively enumerated, see, e.g., [Hermes '65, page 11]) *set* V *of abstract variables,* $V = \{v_1, v_2, \ldots, v_n \ldots\}$. *For example,* V *might be the set of positive integers. An Ω-equation is a doubleton $\{e_1, e_2\}$ of Ω-terms in* V. *An equational presentation is a pair (Ω, E) where Ω is an operator domain and E is a set of Ω-equations.*

The equational presentation corresponding to 1.1 is Ω as in 1.6 and $E = \{\{v_1v_2mv_3m, v_1v_2v_3mm\}, \{v_1em, v_1\}, \{ev_1m, v_1\}, \{v_1iv_1m, e\}, \{v_1v_1im, e\}\}$. This overly formal notation is difficult to read and in most situations we use the more colloquial "$e_1 = e_2$," use parenthetical notation instead of Polish notation, and write $x, y, z \ldots$ for $v_1, v_2, v_3 \ldots$. Thus, E as above is written:

$$((x, y)m, z)m = (x, (y, z)m)m$$
$$(x, e)m = x = (e, x)m$$
$$(xi, x)m = e = (x, xi)m$$

We now set forth to formalize the means which allowed us to make actual operations out of terms in the style that we accomplished this for *abmcm* in the preceding paragraph.

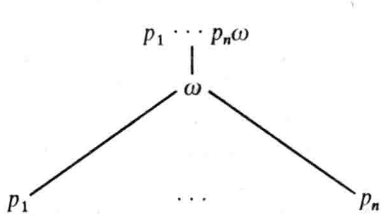
1.11 Uncoupling Lemma. Let A be a set and let Ω be an operator domain. Then for each $p \in A\Omega$ of word length greater than 1 there exists a unique integer n greater than 0 and unique $\omega \in \Omega_n$ and n-tuple $(p_1, \ldots, p_n) \in A\Omega^n$ such that $p = p_1 \cdots p_n\omega$.

Proof. Since p is constructed from (1.8) and (1.9) and has more than one symbol, it is clear that there exists a representation $p = p_1 \cdots p_n\omega$ as in the statement and that n and ω are unique. We must prove that if $p = q_1 \cdots q_n\omega$ is another such representation, then $p_i = q_i$ for all i. It is helpful to define the integer-valued *valency map*, val ([Cohn '65, p. 118]), on the set of all Ω-words in A by $\mathrm{val}(\tilde{\omega}) = 1 - m$ (for all $\tilde{\omega} \in \Omega_m$), $\mathrm{val}(a) = 1$ (for all $a \in A$), $\mathrm{val}(b_1 \cdots b_m) = \mathrm{val}(b_1) + \cdots + \mathrm{val}(b_m)$. Since an Ω-formula q can be constructed from (1.8) and (1.9), $\mathrm{val}(q) = 1$ and $\mathrm{val}(s) > 0$ for any left segment s of q (where, if $q = b_1 \cdots b_m$, the *left segments of* q are the m Ω-words $b_1 \cdots b_k$ for $1 \leqslant k \leqslant m$). The crucial observation is:

(1.12) If s is a proper left segment of $p_i \cdots p_n$ and if $s \in A\Omega$, then s is a left segment of p_i. (For otherwise, there exists $i \leqslant k < n$ and a left segment t of p_{k+1} such that $s = p_i \cdots p_k t$; it follows that $1 = \mathrm{val}(s) = \mathrm{val}(p_i \cdots p_k t) = k - i + 1 + \mathrm{val}(t)$ and $i - k = \mathrm{val}(t) \geqslant 0$ (i.e., if t is empty then $k > i$), the desired contradiction).

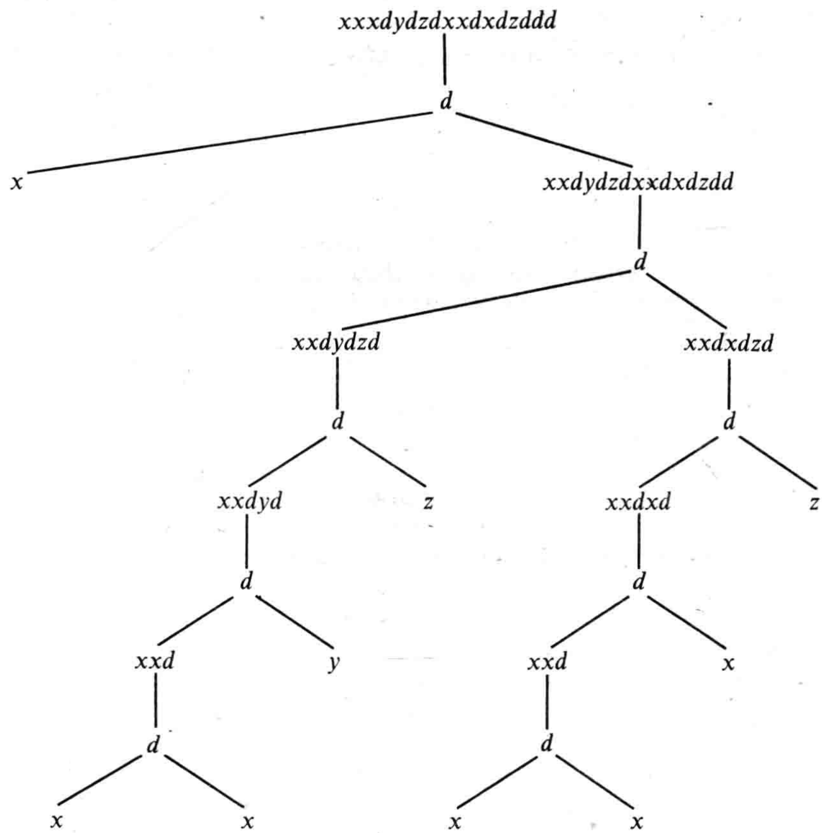
Applying 1.12 to $s = q_1$, we see that q_1 is a left segment of p_1. Symmetrically p_1 is a left segment of q_1, so $p_1 = q_1$. Therefore, $p_2 \cdots p_n\omega = q_2 \cdots q_n\omega$ and we can apply 1.12 to prove $p_2 = q_2$. Similarly, $p_3 = q_3, \ldots, p_n = q_n$. □

The uncoupling process of 1.11 can be geometrically depicted by the "tree"



Each p_i has shorter length than the original term. Each p_i of length greater than 1 can be similarly decoupled until we obtain the complete *derivation tree* of the term in which all terminal branches are terms of length 1, that is variables or 0-ary operations.

1.13 Example. The derivation tree of *xxxdydzdxxdxdzddd* as in 1.2 is



Since the derivation of Ω-terms is unique we have:

1.14 Principle of Finitary Algebraic General Recursion. Let Ω be an operator domain and let A be a set. To define a function ψ on $A\Omega$ it suffices to specify

(1.15) $a\psi$ for all $a \in A$.

(1.16) $(p_1 \cdots p_n\omega)\psi$ in terms of $p_i\psi$ and ω. □

1.17 Example (Substitution of Variables in Terms). Let $f: A \longrightarrow B$ be a function (substituting variables in B for variables in A). By algebraic general recursion we may define the function $f\Omega: A\Omega \longrightarrow B\Omega$ by

$$\langle a, f\Omega \rangle = af$$
$$\langle (p_1 \cdots p_n\omega), f\Omega \rangle = \langle p_1, f\Omega \rangle \cdots \langle p_n, f\Omega \rangle\omega$$

Thus, $\langle aaadbdcdaadadcddd, f\Omega \rangle = xxxdydzdxxdxdzddd$ if $af = x$, $bf = y$, $cf = z$. In the picture of 1.13, we plug in the appropriate terminal branches x, y, z and chase up the tree.

1.18 Example (The Total Description Map). Let (X, δ) be an Ω-algebra. The *total description map* $\delta^{@}: X\Omega \longrightarrow X$ is defined by algebraic general recursion:

$$x\delta^{@} = x$$
$$(p_1 \cdots p_n\omega)\delta^{@} = (p_1\delta^{@}, \ldots, p_n\delta^{@})\delta_\omega$$

Clearly, the total description map accomplishes what we wanted: it makes operations out of formulas, although we should note the role of 1.17 in interpreting variables as arguments. We are finally ready for:

1.19 Definition. Let Ω *be an operator domain, and let* (X, δ) *be an* Ω-*algebra. For each* V-*tuple* $r: V \longrightarrow X$ *there is an interpretation map* $r^\#$ *defined by* $r^\#: V\Omega \longrightarrow X = r\Omega.\delta^{@}$. Notice that $r^\#$ can be defined directly by algebraic general recursion: $vr^\# = vr$, $(p_1 \cdots p_n\omega)r^\# = (p_1r^\#, \cdots p_nr^\#)\delta_\omega$. If $\{e_1, e_2\}$ is an Ω-equation, say that (X, δ) *satisfies* $\{e_1, e_2\}$ if $e_1r^\# = e_2r^\#$ for all $r: V \longrightarrow X$. If (Ω, E) is an equational presentation, an (Ω, E)-*algebra* is an Ω-algebra which *satisfies E*, that is satisfies every equation in E. *The class of all* (Ω, E)-*algebras is said to be an equationally-definable class of algebras, or a variety of algebras.* For example, the equationally-definable class defined by the presentation in 1.10 is "groups" as in 1.1.

The above construction of interpretation maps is based on an important principle. Notice, first, that 1.9 defines an Ω-algebra structure on $A\Omega$ (and we will always regard $A\Omega$ as an algebra in this way). We can now state

1.20 Principle of Finitary Algebraic Simple Recursion. Let Ω be an operator domain, let (X, δ) be an Ω-algebra and let $f: A \longrightarrow X$ be a function. Then there exists a unique Ω-homomorphism $f^\#: A\Omega \longrightarrow (X, \delta)$ extending f.

Proof. By 1.14 there exists unique function $f^\#$ such that $af^\# = af$ and $(p_1 \cdots p_n\omega)f^\# = (p_1f^\#, \ldots, p_nf^\#)\delta_\omega$. □

To help explain the terminology of 1.20, recall that a sequence $x: \mathbf{N} \longrightarrow X$ (where $\mathbf{N} = \{0, 1, 2, 3 \ldots\}$) is defined by *simple recursion* if there exists an endomorphism $\delta: X \longrightarrow X$ such that $x_{n+1} = x_n\delta$. The general recursion of 1.14 amounts to "mathematical induction" (see the notes at the end of this section). Observe that if $X = \{a, b\}$ and if x is defined by $x_0 = x_1 = a$, $x_n = b$ for $n > 1$, then x is not definable by simple recursion. This situation is an instance of 1.20 and 1.14, corresponding to the operator domain