Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Springer-Verlag
Berlin Heidelberg New York London Paris Tokyo

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Mathematics Subject Classification (1980): 58F08, 54H20, 26A18

ISBN 3-540-50329-3 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-50329-3 Springer-Verlag New York Berlin Heidelberg

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

Lecture Notes in Mathematics

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Vol. 1172: Algebraic Topology, Göttingen 1984. Proceedings. Edited by L. Smith. VI, 209 pages. 1985.

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Piecewise monotone mappings on an interval provide simple examples of discrete dynamical systems whose behaviour can be very complicated. These notes are concerned with some of the properties of such mappings. It is hoped that the material presented can be understood by anyone who has had a basic course in (one-dimensional) real analysis. This account is self-contained, but it can be regarded as a sequel to *Iterates of maps on an interval* (Springer Lecture Notes in Mathematics, Vol. 999).

I would like to thank Lai-Sang Young, Richard Hohmann-Damaschke and Jürgen Willms for their suggestions and comments during the writing of these notes. My thanks go also to the staff of the *FSP Mathematisierung* at the University of Bielefeld for technical assistance in the preparation of the text.

Bielefeld
May 1987

Chris Preston

ITERATES OF PIECEWISE MONOTONE MAPPINGS ON AN INTERVAL - CONTENTS

1. INTRODUCTION

Let $I = [a,b]$ be a closed, bounded interval and let $\mathbf{C}(I)$ denote the set of continuous functions $f : I \to I$ which map the interval I back into itself. For $f \in \mathbf{C}(I)$ we define $f^n \in \mathbf{C}(I)$ inductively by $f^0(x) = x$, $f^1(x) = f(x)$ and (for $n > 1$) $f^n(x) = f(f^{n-1}(x))$. f^n is called the **n th. iterate** of f. The set of iterates $\{f^n\}_{n\geq 0}$ of a mapping $f \in \mathbf{C}(I)$ provides us with a very simple example of a dynamical system. This system can be thought of as describing some process, whose states are represented by the points of the interval I, and which is observed at discrete time intervals (say once a year or every ten minutes); the process evolves in such a way that, if at some observation time the process is in the state x, then at the next observation time the process will be in the state $f(x)$. Hence if the process is originally in the state x (at time 0) then it will be in the state $f^n(x)$ at time n. The sequence $\{f^n(x)\}_{n\geq 0}$ is called the **orbit** of x (under f), and it describes the successive states of the process, given that x was the starting state.

Dynamical systems of this type have been used as models in the biological sciences (see, for example, May (1976), May and Oster (1976) and Guckenheimer, Oster and Ipaktchi (1977)), as well as in the physical sciences (see, for example, Lorenz (1963), Collet and Eckmann (1980) and Gumowski and Mira (1980)). They are also ideally suited for making numerical "experiments" using a computer (see, for example, Feigenbaum (1978) and (1979)).

In these notes we study the iterates of a special class of mappings in $\mathbf{C}(I)$, namely the iterates of piecewise monotone mappings. A mapping $f \in \mathbf{C}(I)$ is called **piecewise monotone** if there exists $N \geq 0$ and $a = d_0 < d_1 < \cdots < d_N < d_{N+1} = b$ such that f is strictly monotone on $[d_k, d_{k+1}]$ for each $k = 0,\ldots, N$. The set of piecewise monotone mappings in $\mathbf{C}(I)$ will be denoted by $\mathbf{M}(I)$. If $f \in \mathbf{M}(I)$ then $w \in (a,b)$ is called a **turning point** of f if f is not monotone in any neighbourhood of w. $\mathbf{M}(I)$ includes all of the mappings in $\mathbf{C}(I)$ whose iterates have been used as models of "real" processes; in fact, most such models use mappings having one turning point, for example the elements in the family $p_\mu \in \mathbf{M}([0,1])$, $0 < \mu \leq 4$, where $p_\mu(x) = \mu x(1-x)$.

We will analyse the iterates $\{f^n\}_{n\geq 0}$ for a general mapping $f \in \mathbf{M}(I)$. The analysis is carried out in two stages. In Sections 2 to 6 we will be concerned with following question for an element f from $\mathbf{M}(I)$: What does the asymptotic behaviour of the orbit $\{f^n(x)\}_{n\geq 0}$ look like for a "typical" point $x \in I$? Here "typical" is meant in a topological (rather than in a measure-theoretical) sense: we want to make statements about the asymptotic behaviour of $\{f^n(x)\}_{n\geq 0}$ which hold for all points x lying in a "large" subset of I , where a "large" subset is one which contains a dense open subset of I , or at least is a residual subset of I . (A residual subset is one which contains a countable intersection of dense open subsets of I .) The main result of the first stage of the analysis is Theorem 2.4; this is a generalization of Theorem 5.2 in Preston (1983) , which dealt with the case of mappings having only one turning point and which was based on results of Guckenheimer (1979) and Misiurewicz (1981). Theorem 2.4 says roughly that if $f \in \mathbf{M}(I)$ then one of three things happens to the orbit $\{f^n(x)\}_{n\geq 0}$ of a "typical" point $x \in I$:

(1.1) The orbit eventually ends up in an f-invariant subset $C \subset I$, C consisting of finitely many closed intervals, on which f acts topologically transitively (which means that the orbit of some point in C is dense in C).

(1.2) The orbit is attracted to an f-invariant Cantor-like set $R \subset I$, on which f acts minimally (which means that the orbit of each point in R is dense in R).

(1.3) The orbit is contained in an f-invariant open set $Z \subset I$, which is such that on each of its connected components f^n is monotone for each $n \geq 0$.

(A subset $A \subset I$ is called **f-invariant** if $f(A) \subset A$.)

The second stage of the analysis is carried out in Sections 7 to 12. This can be seen as a study of the structure of the set of points $x \in I$ for which none of (1.1), (1.2) and (1.3) hold. By Theorem 2.4 this set is "small"; however, it turns out that the global complexity of the iterates of f can be strongly influenced by the behaviour of f on this set.

The aim of these notes is to analyse the topological structure of the iterates of a mapping $f \in \mathbf{M}(I)$. This means that we are only interested in results which are invariant under topological equivalence. To make this more precise we need a definition: Mappings f, g $\in$ **M**(I) are said to be **conjugate** if there exists a homeomorphism $\psi : I \to I$ (i.e. ψ is a continuous and strictly monotone mapping of I onto itself) such that $\psi \circ f = g \circ \psi$. The results which we present are all invariant under conjugacy; i.e. if f, g $\in$ **M**(I) are conjugate then any statement which occurs in a theorem is either satisfied by both of f and g or it is satisfied by neither of them.

The topological structure of the iterates of a mapping f $\in$ **M**(I) is also analysed in Milnor and Thurston (1977) and in Nitecki (1982). The present notes have little in common with the first of these papers, except for Section 6, the main part of which is only a slight modification of material in Milnor and Thurston. Nitecki not only considers mappings from **M**(I) , but also the iterates of a general mapping from **C**(I) . Section 4 of Nitecki's paper deals with a "spectral decomposition" theorem for the non-wandering set of a mapping f $\in$ **M**(I) , a result which is due to Hofbauer (1981) (and for mappings with only one turning point to Jonker and Rand (1981) and van Strien (1981)). The second stage of our analysis is likewise concerned with a kind of "spectral decomposition" (though not for the non-wandering set), and in a certain sense our approach parallels that taken by Nitecki. We strongly recommend the reader to study Nitecki's paper, and, if it can be got hold of, that of Milnor and Thurston. The book by Collet and Eckmann (Collet and Eckmann (1980)), which mainly treats mappings having a single turning point, is also highly recommended.

Some of the results in these notes can be extended to mappings which are allowed discontinuities at their turning points, (i.e. to the set of mappings $f : I \to I$ for which there exists $N \geq 0$ and $a = d_0 < d_1 < \cdots < d_N < d_{N+1} = b$ such that f is strictly monotone and continuous on each of the open intervals (d_k, d_{k+1}) , $k = 0, \ldots, N$). Results of this type corresponding to the first stage of the analysis given here can be found in Willms (1987). Furthermore, Hofbauer's "spectral decomposition" theorem (Hofbauer (1981), (1986)) remains valid in this extended set-up.

As we have already mentioned, we hope that the material in these notes can understood by anyone who has had a basic course in real analysis. The most important prerequisite is a familiarity with the standard topological properties of the real line (as are covered, for example, in the first five chapters of Rudin (1964)). There are, however, a couple of results which we need to use, and which probably do not occur in a typical introductory real analysis course (for example, the Baire category theorem). These results are stated and proved in Section 13.

Throughout these notes I denotes the closed bounded interval $[a,b]$, on which most of the mappings are defined; the symbols a and b are only ever used for the end-points of this interval. We consider I as our basic topological space; hence "open" always means with respect to the topology on I , and so, for example, $[a,c)$ is an open subset of I for each $c \in (a,b]$. If $A \subset I$ then $\bar{A}$ denotes the closure of A and $int(A)$ the interior of A (again with respect to the topology on I). If $J \subset I$ is an interval then $|J|$ will denote the length of J . (We also use $|A|$ to denote the cardinality of the set A ; however, this should not create any problems.)

We now give an outline of what is contained in the various sections of these notes.

Section 2: Piecewise monotone mappings This section introduces the basic definitions and facts about piecewise monotone mappings. For $f \in \mathbf{M}(I)$ let $T(f)$ denote the set of turning points of f and let

$$Z(f) = \{ \ x \in (a,b) : \text{there exists } \varepsilon > 0 \text{ such that } f^n \text{ is monotone on } (x-\varepsilon,x+\varepsilon) \text{ for all } n \geq 0 \ \} ;$$

then $Z(f)$ is open, and in fact $Z(f)$ is the largest open set $G \subset (a,b)$ such that $f^n(G) \cap T(f) = \emptyset$ for all $n \geq 0$. Moreover, it is not hard to see that $Z(f)$ is f-invariant. ($Z(f)$ is the set which occurs in (1.3).)

Let $m \geq 1$; we call a closed set $C \subset I$ an f-**cycle** with period m if C is the disjoint union of non-trivial closed intervals $B_0,\ldots, B_{m-1}$ such that $f(B_{k-1}) \subset B_k$ for $k = 1,\ldots, m-1$ and $f(B_{m-1}) \subset B_0$; (in particular we then have $f(C) \subset C$). We call C **proper** if $f(B_{k-1}) = B_k$ for $k = 1,\ldots, m-1$ and $f(B_{m-1}) = B_0$. An f-cycle C is said to be **topologically transitive** if whenever F is a closed subset of C with $f(F) \subset F$ then either $F = C$ or $int(F) = \emptyset$. There are several other definitions of "topologically transitive" which are equivalent to the one we have used, and some of these are given in Proposition 2.8; for example, an f-cycle C is topologically transitive if and only if the orbit of x is dense in C for some $x \in C$. (The subset C which occurs in (1.1) is a topologically transitive f-cycle.)

Let $\{C_n\}_{n\geq 1}$ be a decreasing sequence of f-cycles, C_n having period m_n ; it is then easy to see that $m_n | m_{n+1}$ for each $n \geq 1$. We call the sequence $\{C_n\}_{n\geq 1}$ **splitting** if $m_{n+1} > m_n$ for each $n \geq 1$. We say that $R \subset I$ is an **f-register-shift** if there exists a splitting sequence of proper f-cycles $\{K_n\}_{n\geq 1}$ with $K_1 \cap Z(f) = \emptyset$ such that $R = \bigcap_{n\geq 1} K_n$, and we then call $\{K_n\}_{n\geq 1}$ a **generator** for R . If R is an f-register-shift then Propositions 2.2 and 2.10 show that $\operatorname{int}(R) = \emptyset$ and that f maps R homeomorphically back onto itself; moreover, the orbit of x is dense in R for each $x \in R$. (The subset R which occurs in (1.2) is an f-register-shift.)

If C is an f-cycle then let

$$A(C,f) = \{ x \in I : f^n(x) \in \operatorname{int}(C) \text{ for some } n \geq 0 \} .$$

If R is an f-register-shift and $\{K_n\}_{n\geq 1}$, $\{K'_n\}_{n\geq 1}$ are two generators for R then it follows from Proposition 2.2 that $\bigcap_{n\geq 1} A(K_n,f) = \bigcap_{n\geq 1} A(K'_n,f)$; we can thus define $A(R,f) = \bigcap_{n\geq 1} A(K_n,f)$. (In Proposition 2.9 we will see that

$$A(R,f) = \{ x \in I : \lim_{n\to\infty} \min_{z\in R} |f^n(x)-z| = 0 \} .)$$

Now let C , C' be topologically transitive f-cycles and R , R' be f-register-shifts. Proposition 2.2 will show that either $C = C'$ or $A(C,f) \cap A(C',f) = \emptyset$, and that either $R = R'$ or $A(R,f) \cap A(R',f) = \emptyset$; also $A(C,f)$, $A(R,f)$ and $Z(f)$ are all disjoint. Furthermore, each of $A(C,f)$ and $A(R,f)$ contains a turning point of f . Therefore if $C_1,\dots, C_r$ are the topologically transitive f-cycles and $R_1,\dots, R_\ell$ are the f-register-shifts, then $\ell+r \leq |T(f)|$ and the sets $A(C_1,f),\dots, A(C_r,f)$, $A(R_1,f),\dots, A(R_\ell,f)$ and $Z(f)$ are disjoint. The first main result of Section 2 is Theorem 2.4, which says that

$$\Lambda(f) = A(C_1,f) \cup \cdots \cup A(C_r,f) \cup A(R_1,f) \cup \cdots \cup A(R_\ell,f) \cup Z(f)$$

is a dense subset of I , (and thus $\Lambda(f)$ is residual, since it can clearly be written as a countable intersection of open subsets of I). Theorem 2.4 is the precise formulation of the statement that for the orbit of a "typical" point in I one of (1.1), (1.2) and (1.3) holds.

The second main result of Section 2 (Theorem 2.5) gives us more information about the topologically transitive f-cycles. We say that $f \in \mathbf{M}(I)$ is **(topologically) exact** if for each non-trivial interval $J \subset I$ there exists $n \geq 0$ such that $f^n(J) = I$. We call $f \in \mathbf{M}(I)$ **semi-exact** if there exists $c \in (a,b)$ such that $f([a,c]) = [c,b]$, $f([c,b]) = [a,c]$, and the restriction of f^2 to $[a,c]$ is exact. Let $f \in \mathbf{M}(I)$ and C be a proper f-cycle with period m ; let B be one of the m components of C . We say that C is **exact** (resp. **semi-exact**) if the restriction of f^m to B is exact (resp. semi-exact). (It is easy to see that these definitions do not depend on which of the components of C is used.) If C is either exact or semi-exact then C is clearly topologically transitive. Theorem 2.5 states that the converse of this is true, namely that each topologically transitive f-cycle is either exact or semi-exact.

Let $f \in \mathbf{M}(I)$ be topologically transitive, (by which we mean that the whole interval I is a topologically transitive f-cycle). Then Theorem 2.5 gives us that f is either exact or semi-exact. An important property of such mappings is provided by a result of Parry (Corollary 3 in Parry (1966)). This implies that if $f \in \mathbf{M}(I)$ is either exact or semi-exact then f is conjugate to a uniformly piecewise linear mapping $g \in \mathbf{M}(I)$, where a mapping $g \in \mathbf{M}(I)$ is said to be **uniformly piecewise linear** with **slope** $\beta > 0$ if, on each of the intervals where it is monotone, g is linear with slope either β or $-\beta$. Thus if $f \in \mathbf{M}(I)$ is topologically transitive then f is conjugate to a uniformly piecewise linear mapping. We give a proof of this result in Section 6.

Section 3: Theorems 2.4 and 2.5 are proved in this section.

Section 4: Sinks and homtervals Here we analyse the set $Z(f)$. For $f \in \mathbf{M}(I)$ let

$$Z_*(f) = \{ x \in I : f^n(x) \in Z(f) \text{ for some } n \geq 0 \} ;$$

then $Z_*(f)$ is open, $Z(f) \subset Z_*(f)$ and $f(Z_*(f)) \subset Z_*(f)$. In fact there is not much difference between $Z(f)$ and $Z_*(f)$, since in Proposition 4.1 we show that each point in $Z_*(f)-Z(f)$ is an isolated point of $I-Z(f)$, and so in particular $Z_*(f)-Z(f)$ is countable. $Z_*(f)$ can be described in terms of sinks and homtervals. A non-empty open interval $J \subset (a,b)$ is called a **sink** of f if there exists $m \geq 1$ such that f^m is monotone on J and $f^m(J) \subset J$. (If J is a sink of f then it follows that f^n is monotone on J for all $n \geq 0$.) A non-empty open interval $J \subset (a,b)$ is called a **homterval** of f if for each $n \geq 0$ we have f^n is monotone on J and $f^n(J)$ is not contained in any sink. Let

$$\text{Sink}(f) = \{ x \in I : f^n(x) \in J \text{ for some sink } J \text{ and some } n \geq 0 \},$$

$$\text{Homt}(f) = \{ x \in I : f^n(x) \in L \text{ for some homterval } L \text{ and some } n \geq 0 \}.$$

Sink(f) and Homt(f) are clearly both open. We will see that both these sets are f-invariant, $\text{Sink}(f) \cap \text{Homt}(f) = \emptyset$ and $\text{Sink}(f) \cup \text{Homt}(f) = Z_*(f)$.

For $f \in \mathbf{M}(I)$ and $n \geq 1$ let $\text{Per}(n,f)$ denote the set of **periodic points** of f with **period** n, i.e.

$$\text{Per}(n,f) = \{ x \in I : f^n(x) = x,\ f^k(x) \neq x \text{ for } k = 1,\dots, n-1 \}.$$

There is a strong connection between sinks and "attracting" periodic points. To see this, we need a couple of definitions. Let $f \in \mathbf{M}(I)$; for $x \in \text{Per}(m,f)$ put $[x] = \{x, f(x), \dots, f^{m-1}(x)\}$, so $[x]$ is the periodic orbit containing x. Let $\alpha([x],f)$ denote the set of points in I which are attracted to the orbit $[x]$, i.e. $\alpha([x],f) = \{ y \in I : \lim_{n\to\infty} f^{mn}(y) = f^k(x) \text{ for some } 0 \leq k < m \}$. Also let $\delta([x],f) = \{ y \in I : f^n(y) = x \text{ for some } n \geq 0 \}$; thus $\delta([x],f)$ consists of those points in I which eventually hit the orbit $[x]$. Clearly $\delta([x],f) \subset \alpha([x],f)$, and $\delta([x],f)$ is countable (because $f^{-1}(\{z\})$ is finite for each $z \in I$). Now let $f \in \mathbf{M}(I)$ and $x \in \text{Per}(m,f)$; then Proposition 4.3 shows that either

(1) $\alpha([x],f) = \delta([x],f)$, (in which case $\alpha([x],f)$ is countable), or

(2) there exists a non-trivial interval J with $x \in J \subset \alpha([x],f)$ such that $\text{int}(J)$ is a sink of f.

Moreover, if *(2)* holds then $\alpha([x],f) - \delta([x],f)$ is a non-empty open subset of Sink(f). We say that x is **attracting** if *(2)* holds.

Let $f \in \mathbf{M}(I)$ and M be the set of attracting periodic points of f (and note that M is countable). Put $\alpha(f) = \bigcup_{x \in M} \alpha([x],f)$. Proposition 4.4 states that if $Per(m,f)$ is finite for each $m \geq 1$ then $Sink(f) - \alpha(f)$ and $\alpha(f) - Sink(f)$ are both countable, and so in this case there is not much difference between the sets $Sink(f)$ and $\alpha(f)$. The assumption in Proposition 4.4 (that $Per(m,f)$ be finite for each $m \geq 1$) will clearly be satisfied if f is the restriction to I of an analytic function defined in some (complex) neighbourhood of I . Mappings which have been used in applications are usually of this type (for example, polynomial mappings).

Most of Section 4 is involved with a general analysis of the sinks of a mapping $f \in \mathbf{M}(I)$. This is very straighforward, and the material is taken, with only minor modifications, from Section 5 of Preston (1983), which in turn was based on results and ideas from Guckenheimer (1979), Collet and Eckmann (1980) and Misiurewicz (1981).

Section 5: Examples of register-shifts Here we give a couple of simple examples to demonstrate that register-shifts actually occur. The mappings we consider are not smooth, but they have the advantage of being defined explicitly, and they are very easy to analyse.

Our first example of a mapping having a register-shift is similar to one occurring in Milnor and Thurston (1977). Let $I = [0,1]$, and for each $n \geq 0$ let $a_n = \frac{1}{2}(1-3^{-n})$ and put $f(a_n) = \frac{4}{5}(1-6^{-n})$. Now define f to be linear on each interval $[a_n,a_{n+1}]$, $n \geq 0$ (and so f has slope 2^{-n+1} on $[a_n,a_{n+1}]$). This defines f as a strictly increasing, continuous function on $[0,\frac{1}{2})$. Put $f(\frac{1}{2}) = \frac{4}{5}$ and let $f(x) = f(1-x)$ for $x \in (\frac{1}{2},1]$. Then $f \in \mathbf{M}([0,1])$, $f(0) = f(1) = 0$ and $\frac{1}{2}$ is the single turning point of f .

The reason for defining f in this way is that we then have the following "self-similarity" property (which is proved in Lemma 5.2): $f^2([\frac{1}{3},\frac{2}{3}]) \subset [\frac{1}{3},\frac{2}{3}]$, and if g is the restriction of f^2 to $[\frac{1}{3},\frac{2}{3}]$ then $g(x) = \frac{1}{3}(2 - f(2-3x))$ for all $x \in [\frac{1}{3},\frac{2}{3}]$, i.e. $g = \psi \circ f \circ \psi^{-1}$, where $\psi : [0,1] \to [\frac{1}{3},\frac{2}{3}]$ is the linear change of variables given by $\psi(t) = \frac{1}{3}(2-t)$. (This means that g is just f turned upside down and scaled down by a factor of 3 .) Using this property and induction we construct a decreasing sequence of proper f-cycles $\{K_n\}_{n\geq 1}$ with $per(K_n) = 2^n$ for each $n \geq 1$. In Lemma 5.4 we show that $Z(f) = \emptyset$, and hence $R = \bigcap_{n\geq 1} K_n$ is an f-register-shift. Moreover, it follows from Lemma 5.3 that $[0,1] - A(R,f)$ is countable.

Let $f \in \mathbf{M}(I)$ and R be an f-register-shift; we say that R is **tame** if there exists an f-cycle K with $R \subset K$ such that $K - A(R,f)$ is countable. (In Section 9 we show that the structure of a tame register-shift is somewhat special.) The example given above is thus tame. However, it is easy to modify this example to obtain a non-tame register-shift. In fact, again let $I = [0,1]$, and for $n \geq 0$ let $a_n = \frac{1}{2}(1-7^{-n})$ and $f(a_n) = \frac{24}{27}(1-28^{-n})$. Define f to be linear on $[a_n, a_{n+1}]$, $n \geq 0$ (and so f has slope $2\cdot 4^{-n}$ on $[a_n, a_{n+1}]$). As before this defines f as a strictly increasing, continuous function on $[0,\frac{1}{2})$. Put $f(\frac{1}{2}) = \frac{24}{27}$ and for $x \in (\frac{1}{2},1]$ let $f(x) = f(1-x)$. This mapping f also has a "self-similarity" property, namely: $f^3([\frac{3}{7},\frac{4}{7}]) \subset [\frac{3}{7},\frac{4}{7}]$, and if g is the restriction of f^3 to $[\frac{3}{7},\frac{4}{7}]$ then $g(x) = \frac{1}{7}(4 - f(4-7x))$. (This means that g is just f turned upside down and scaled down by a factor of 7.) As in the first example, this property allows us to construct a decreasing sequence of proper f-cycles $\{K_n\}_{n\geq 1}$, this time with $per(K_n) = 3^n$ for each $n \geq 1$. Again we have $Z(f) = \emptyset$, and so $R = \bigcap_{n\geq 1} K_n$ is an f-register-shift. However, we show that in this example R is not tame.

The reason that the register-shift in the second example is not tame is because the interval $[\frac{3}{7},\frac{4}{7}]$ sits in $[0,1]$ in a complicated way; more precisely, the set $\{ x \in [0,1] : f^n(x) \notin [\frac{3}{7},\frac{4}{7}]$ for all $n \geq 0 \}$ contains a Cantor-like set, and is thus uncountable. This is in contrast to the first example, where the corresponding set (i.e. $\{ x \in [0,1] : f^n(x) \notin [\frac{1}{3},\frac{2}{3}]$ for all $n \geq 0 \}$) consists only of the two points 0 and 1.

Section 6: A proof of Parry's theorem In this section we give a proof of the following result in Parry (1966):

Theorem 6.1 If $f \in \mathbf{M}(I)$ is topologically transitive then f is conjugate to a uniformly piecewise linear mapping.

We do not use Parry's proof, but instead one taken, with a few minor modifications, from Milnor and Thurston (1977).

Let $\mathbf{V}(I) = \{ \psi \in \mathbf{C}(I) : \psi$ is increasing and onto $\}$ (where by increasing we mean only that $\psi(x) \geq \psi(y)$ whenever $x \geq y$). If $\psi \in \mathbf{V}(I)$ and $g \in \mathbf{M}(I)$ then we say that (ψ,g) is a **reduction** (or semi-conjugacy) of $f \in \mathbf{M}(I)$ if $\psi \circ f = g \circ \psi$. Lemma 6.2 states that if $f \in \mathbf{M}(I)$ is topologically transitive and (ψ,g) is a reduction of f then ψ is automatically a homeomorphism, and so in particular f and g are conjugate. Lemma 6.2 reduces the proof of Theorem 6.1 to showing that if $f \in \mathbf{M}(I)$ is topologically transitive then there exists a reduction (ψ,g) of f with g uniformly piecewise linear.

For $f \in \mathbf{M}(I)$ let $\ell(f) = |T(f)| + 1$, and also let $h(f) = \inf_{n\geq 1} \frac{1}{n} \log \ell(f^n)$; thus $h(f) \geq 0$. Lemma 6.3 shows that in fact $h(f) = \lim_{n\to\infty} \frac{1}{n} \log \ell(f^n)$ for each $f \in \mathbf{M}(I)$, and in Lemma 6.4 we show that $h(f) > 0$ whenever f is topologically transitive. Theorem 6.1 therefore follows from Theorem 6.5, which says that if $f \in \mathbf{M}(f)$ with $h(f) > 0$ then there exists a reduction (ψ,g) of f such that g is uniformly piecewise linear with slope β , where $\beta = \exp(h(f))$.

Theorem 6.5 and its proof are due to Milnor and Thurston. The proof goes roughly as follows: Fix $f \in \mathbf{M}(I)$ with $h(f) > 0$ and put $r = \exp(-h(f))$; thus $r = 1/\beta$ and $0 < r < 1$. Note that by Lemma 6.3 we have $\beta = \lim_{n\to\infty} \ell(f^n)^{1/n}$, and hence r is the radius of convergence of the power series $\sum_{n\geq 0} \ell(f^n)t^n$; in particular the series $L(t) = \sum_{n\geq 0} \ell(f^n)t^n$ converges for all $t \in (0,r)$. Now let $J \subset I$ be a non-trivial closed interval, and for $n \geq 0$ let $\ell(f^n|J) = |T(f^n) \cap \text{int}(J)| + 1$. Then $\ell(f^n|J) \leq \ell(f^n|I) = \ell(f^n)$, and therefore the series $L(J,t) = \sum_{n\geq 0} \ell(f^n|J)t^n$ also converges for all $t \in (0,r)$. Hence we can define $\Lambda(J,t) = L(J,t)/L(I,t)$ for each $t \in (0,r)$ (since $L(I,t) = L(t) \neq 0$), and we have $0 \leq \Lambda(J,t) \leq 1$, because $L(J,t) \leq L(I,t)$. In Lemma 6.10 it is shown that there exists a sequence $\{t_n\}_{n\geq 1}$ from $(0,r)$ with $\lim_{n\to\infty} t_n = r$, and such that $\{\Lambda(J,t_n)\}_{n\geq 1}$ converges for each non-trivial closed interval $J \subset I$. For each such interval J we can thus define $\Lambda(J) = \lim_{n\to\infty} \Lambda(J,t_n)$, and this gives us a mapping $\pi : I \to [0,1]$ obtained by letting $\pi(a) = 0$ and $\pi(x) = \Lambda([a,x])$ for $x \in (a,b]$.

Lemmas 6.11, 6.12 and 6.13 show that the mapping $\pi : I \to [0,1]$ is continuous, increasing and onto, that there exists a unique mapping $\alpha : [0,1] \to [0,1]$ with $\pi \circ f = \alpha \circ \pi$, and that this mapping α is uniformly piecewise linear with slope β. Thus, by a simple linear rescaling of α and π, we get a reduction (ψ, g) of f such that $g \in \mathbf{M}(I)$ is also uniformly piecewise linear with slope β.

At the end of Section 6 we give a result from Misiurewicz and Szlenk (1980), which provides an alternative method of calculating $h(f)$ for a mapping $f \in \mathbf{M}(I)$. For $f \in \mathbf{C}(I)$ let

$$\mathrm{Var}(f) = \sup \left\{ \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \ : \ a = a_0 < x_1 < \cdots < x_n = b \right\} .$$

In particular, if $f \in \mathbf{M}(I)$ has turning points $d_1, \ldots, d_N$, where $a = d_0 < d_1 < \cdots < d_N < d_{N+1} = b$, then clearly $\mathrm{Var}(f) = \sum_{k=0}^{N} |f(d_{k+1}) - f(d_k)|$. The result of Misiurewicz and Szlenk says that if $f \in \mathbf{M}(I)$ then $h(f) > 0$ if and only if $\limsup_{n\to\infty} \frac{1}{n} \log \mathrm{Var}(f^n) > 0$; moreover, $h(f) = \lim_{n\to\infty} \frac{1}{n} \log \mathrm{Var}(f^n)$ whenever $h(f) > 0$. Consider the special case of a uniformly piecewise linear mapping $g \in \mathbf{M}(I)$ with slope $\beta \geq 1$; then this result shows that $h(g) = \log \beta$, (since g^n is uniformly piecewise linear with slope β^n, and so $\mathrm{Var}(g^n) = (b-a)\beta^n$ for each $n \geq 1$).

Section 7: Reductions In this section we start the second stage in the analysis of the iterates of a mapping $f \in \mathbf{M}(I)$. Up to now the main result has been Theorem 2.4, which gives us information about the asymptotic behaviour of the orbits $\{f^n(x)\}_{n\geq 0}$ for all points x lying in a residual subset $\Lambda(f)$ of I. The set $I - \Lambda(f)$, being the complement of a residual set, is topologically "small"; however, the behaviour of f on this set can strongly influence the global complexity of the iterates of f. In order to study this we look at the reductions of f: Recall from Section 6 that if $\psi \in \mathbf{V}(I)$ and $g \in \mathbf{M}(I)$ then (ψ, g) is a reduction of f if $\psi \circ f = g \circ \psi$. Let (ψ, g) be a reduction of $f \in \mathbf{M}(I)$; then in a certain sense g describes the behaviour of f on $\mathrm{supp}(\psi)$, where

$$\mathrm{supp}(\psi) = \{ x \in I : \psi(J) \text{ is non-trivial for each open interval } J \subset I \text{ with } x \in J \} .$$