Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Algebroid Curves in Positive Characteristic

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INTRODUCTION

A number of definitions of equisingularity have appeared, since Zariski published his "Studies in Equisingularity". Those definitions are equivalent in the particular case of plane algebroid curves over an algebraically closed field of characteristic zero (the situation that Zariski considered initially). However the case of characteristic $p > 0$ has not received extensive attention and only a few papers are available: Lejeune [15], Moh [18], and more recently Angermüller [3].

These notes intend to give a systematic development of the theory of equisingularity of irreducible algebroid curves over an algebraically closed field of arbitrary characteristic, using as main tool the Hamburger-Noether expansion instead of the Puiseux expansion which is usually employed in characteristic zero.

The so called Hamburger-Noether expansion first appeared as an attempt to obtain parametrizations of plane algebraic curves in any characteristic. It was completely developed in a work by G. Ancochea, published in Acta Salamanticiensis (Universidad de Salamanca) and not available any longer. Essentially it is based on a parametrization of an irreducible algebroid curve $\square = k[[x,y]]$ over k of the type

$$x = x(z_r)$$
$$y = y(z_r) \quad ,$$

z_r being an element of the quotient field of $\square$, obtained from x, y by a chain of relations

$$y = a_{01}x + a_{02}x^2 + \dots + a_{0h}x^h + x^h z_1$$
$$x = a_{12}z_1^2 + \dots + a_{1h_1}z_1^{h_1} + z_1^{h_1} z_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$z_{r-1} = a_{r2}z_r^2 + \dots\dots\dots\dots$$

where $a_{ji} \in k$.

This expansion enables us to define a system of characteristic exponents of a plane curve which is equivalent to that derived from the Puiseux expansion in characteristic zero. These exponents determine and are determined by the resolution process for the singularity of the curve, by the semigroup of values of its local ring, etc...

Chapter I contains well known definitions and results on algebroid curves, existence of parametrizations, and resolution of singularities of an irreducible algebroid curve. Chapter II is devoted to the Hamburger-Noether expansion and comparison of it with the Puiseux expansion in characteristic zero.

In chapter III, by using a complex model for the singularity of a curve over an algebraically closed field of any characteristic, we introduce the characteristic exponents. From this model we compare these exponents with the usual ones, and compute them in terms of Hamburger-Noether expansions and Newton polygons.

The semigroup of values of the local ring of the curve is calculated from the values of the maximal contact or from the characterisitic exponents in chapter IV. We also find the relationship between the characteristic exponents and the Newton coefficients given by Lejeune.

Finally, in chapter V we study several criteria for equisingularity of irreducible twisted curves.

I would like to express my sincere thanks to Professor Aroca for his comments and suggestions.

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CHAPTER I

PARAMETRIZATIONS OF ALGEBROID CURVES

This chapter is devoted to the systematization of the concept of local parametrization of irreducible algebroid curves over an algebraically closed ground field of any characteristic. Although there are not essential differences with the case of characteristic zero, we have thought it useful to treat this case in detail.

1. PRELIMINARY CONCEPTS.

Let k be an algebraically closed field of arbitrary characteristic. If $\underline{X} = \{X_i\}_{1 \leq i \leq N}$ is a set of indeterminates over k, we shall denote by $k((\underline{X})) = k((X_1, \ldots, X_N))$ the formal power series ring in the indeterminates $\underline{X}$ with coefficients in k. The function order on $k((\underline{X}))$ will be denoted by $\underline{\upsilon}$.

The Weierstrass Preparation Theorem (W.P.T.) will be used frecuently in this work. It is stated down and its proof and direct consequences may be found in Zariski-Samuel, (29).

Theorem 1.1.1.(W.P.T.).- Let $f(\underline{X}) \in k((\underline{X}))$ be a series which is regular in X_N of order s, i.e., $s = \underline{\upsilon}(f(0, \ldots, 0, X_N))$. Then, there exist a unique unit $U(\underline{X})$ in $k((\underline{X}))$ and a unique degree s monic polynomial $P((\underline{X}'), X_N)$ (where $\underline{X}' = (X_1, \ldots, X_{N-1})$) in X_N with coefficients in $k((\underline{X}'))$ such that

$$f(\underline{X}) = U(\underline{X}) \cdot P((\underline{X}'), X_N).$$

Definition 1.1.2.- An irreducible algebroid curve (or simply a curve if there is no confusion) over k is a noetherian local domain $\square$ such that:

1) $\square$ is complete.

2) $\square$ has Krull dimension 1.

3) k is a coefficient field for $\square$.

If $\underline{m}$ is the maximal ideal of $\square$, the property 3) means that k is contained in $\square$ and is isomorphic to the field $\square/\underline{m}$ by the canonical epimorphism $\square \longrightarrow \square/\underline{m}$.

Remark 1.1.3.- Since $\square$ is noetherian, the vector space $\underline{m}/\underline{m}^2$ over k is finite dimensional. The number $\mathrm{Emb}(\square) = \dim_k(\underline{m}/\underline{m}^2)$ is called the embedding dimension of $\square$.

For every basis $B = \{x_i\}_{1 \leq i \leq N}$ of the maximal ideal $\underline{m}$,

$$S = \{x_i + \underline{m}^2\}_{1 \leq i \leq N}$$

is a set of generators of the k- vector space $\underline{m}/\underline{m}^2$. The set S becomes a basis of this vector space if and only if B is a minimal basis of the ideal $\underline{m}$.

Let $B = \{x_i\}_{1 \leq i \leq N}$ a basis of $\underline{m}$. Using 3) and the completeness of $\square$ we find a natural surjective k-homomorphism

$$(1) \qquad \begin{array}{ccc} k((\underline{X})) & \longrightarrow & \square \\ X_i & \longmapsto & x_i \end{array} \qquad , \quad \underline{X} = \{X_i\}_{1 \leq i \leq N} .$$

Thus, there exists an isomorphism $k((\underline{X}))/\underline{p} \cong \square$, where $\underline{p}$ is a prime ideal of $k((\underline{X}))$. The condition 2) means that the depth of $\underline{p}$ is 1.

We may identify $\square$ with the ring $k((\underline{X}))/\underline{p}$. In fact, if we set $x_i = X_i + \underline{p}$, we can write $\square = k((x_1, \ldots, x_N))$. The minimun N such that these isomorphisms exist, is exactly $\mathrm{Emb}(\square)$. When an identification as above is done, we shall say that the curve is embedded in an N-space. Thus, to give an embedded curve C is to give a prime ideal $p \subset k[[\underline{X}]]$ of depth 1. Then, □ is called the local ring of C. The word "embedding" has a precise meaning in scheme theory: The N-space is by definition the affine scheme $\mathrm{Spec}(k[[\underline{X}]])$, the curve □ is identified with Spec(□) and the homomorphism (1) induces a closed embedding of schemes $\mathrm{Spec}(\square) \longrightarrow \mathrm{Spec}(k[[\underline{X}]])$.

Now we shall give the following normalization theorem which allows to simplify the form of the ideal $\underline{p}$. Notice the assumptions that $\underline{p}$ is prime and of depth 1, which do not affect the proof. Therefore, we shall prove it for any ideal $\underline{a}$ of a formal power series ring.

Theorem 1.1.4.- Let $\underline{Y} = \{Y_i\}_{1 \leqslant i \leqslant N}$, $\underline{X} = \{X_i\}_{1 \leqslant i \leqslant N}$ be two sets of indeterminates over k and $\underline{a}'$ an ideal of $k[[\underline{Y}]]$, $\underline{a}' \neq (0),(1)$. There exists an integer m, $0 \leqslant m \leqslant N-1$, and an isomorphism Φ from $k[[\underline{Y}]]$ onto $k[[\underline{X}]]$ defined by linear relations $\Phi(Y_i) = L_i(\underline{X})$, $1 \leqslant i \leqslant N$, independent over k, such that the ideal $\underline{a} = \Phi(\underline{a}')$ has the following properties:

1) $\underline{a} \cap A_m = (0)$, $\underline{a} \cap A_i \neq (0)$, $m+1 \leqslant i \leqslant N$;

where $A_i = k[[X_1, \ldots, X_i]]$, $0 \leqslant i \leqslant N$.

2) There are $N-m$ non zero series $f_i(X_1, \ldots, X_i) \in \underline{a} \cap A_i$, $m+1 \leqslant i \leqslant N$, such that

$$\underline{\upsilon}(f_i(0, \ldots, 0, X_i)) = \underline{\upsilon}(f_i(X_1, \ldots, X_i)) .$$

Proof: We shall construct by induction the isomorphism Φ .

First, let us construct an isomorphism $\Phi^{(1)}$ from $k[[\underline{Y}]]$ onto $k[[\underline{Y}^{(1)}]]$, with $\underline{Y}^{(1)} = \{Y_i^{(1)}\}_{1 \leqslant i \leqslant N}$ new variables, such that if $\underline{a}^{(1)} = \Phi^{(1)}(\underline{a}')$, then there exists $f_N^{(1)} \in \underline{a}^{(1)}$ verifying $\underline{\upsilon}(f_N^{(1)}(0, \ldots, 0, Y_N^{(1)})) = \underline{\upsilon}(f_N^{(1)}(Y_1^{(1)}, \ldots, Y_N^{(1)}))$.

To prove this, take $f'_N \in \underline{a}'$, $f'_N \neq 0$. Let $f_{N,q}$ the leading form of f'_N . If $f_{N,q}(0, \ldots, 0, Y_N) \neq 0$ we set $Y_i^{(1)} = Y_i$, $1 \leqslant i \leqslant N$,

$\Phi^{(1)}$ the identity in $k[[\underline{Y}]]$, and $f_N = f'_N$. If $f_{N,q}(0,\ldots,0,Y_N) = 0$ we pick out $a_i^{(1)} \in k$, $1 \leq i \leq N-1$, such that $f_{N,q^{(1)}}(a_1^{(1)},\ldots,a_{N-1}^{(1)},1) \neq 0$ (Hilbert's Nullstellensatz). Then, the linear forms

$$L_i^{(1)} = Y_i^{(1)} + a_i^{(1)} Y_N^{(1)} \qquad , \quad 1 \leq i \leq N-1$$
$$L_N^{(1)} = Y_N^{(1)}$$

are lineary independent over k and the isomorphism

$$\Phi^{(1)} : k[[\underline{Y}]] \longrightarrow k[[\underline{Y}^{(1)}]]$$
$$Y_i \longmapsto L_i^{(1)}(\underline{Y}^{(1)})$$

and the series $f_N^{(1)} = \Phi^{(1)}(f'_N) \in \underline{a}^{(1)}$ verify our conditions.

Now, let p be an integer, $1 \leq p < N$, and $\underline{Y}^{(p)} = \{Y_i^{(p)}\}_{1 \leq i \leq N}$ indeterminates over k. Assume that there exists an isomorphism $\Phi^{(p)} : k[[\underline{Y}]] \longrightarrow k[[\underline{Y}^{(p)}]]$ given by lineary independent forms such that, if $\underline{a}^{(p)} = \Phi^{(p)}(\underline{a}')$, there exist non zero series $f_i^{(p)} \in \underline{a}^{(p)} \cap k[[Y_1^{(p)},\ldots,Y_i^{(p)}]]$, $N-(p-1) \leq i \leq N$, such that

$$\underline{\upsilon}(f_i^{(p)}(Y_1^{(p)},\ldots,Y_i^{(p)})) = \underline{\upsilon}(f_i^{(p)}(0,\ldots,0,Y_i^{(p)})).$$

Assume that

$$(1) \qquad \underline{a}^{(p)} \cap k[[Y_1^{(p)},\ldots,Y_{N-p}^{(p)}]] \neq (0).$$

Then, take a non zero series in $\underline{a}^{(p)} \cap k[[Y_1^{(p)},\ldots,Y_{N-p}^{(p)}]]$ and use the above procedure to find an appropriate linear change which gives rise to an isomorphism $\Phi^{(p+1)} : k[[\underline{Y}]] \longrightarrow k[[\underline{Y}^{(p+1)}]]$ such that, if $\underline{a}^{(p+1)} = \Phi^{(p+1)}(\underline{a}')$, then there exist non zero series $f_i^{(p+1)} \in \underline{a}^{(p+1)} \cap k[[Y_1^{(p+1)},\ldots,Y_i^{(p+1)}]]$, $N-p \leq i \leq N$, verifying

$$\underline{\upsilon}(f_i^{(p+1)}(Y_1^{(p+1)},\ldots,Y_i^{(p+1)})) = \underline{\upsilon}(f_i^{(p+1)}(0,\ldots,0,Y_i^{(p+1)})).$$

Since $\underline{a}' \neq (1)$ we have $\underline{a}' \cap k = (0)$, hence there exists an integer p such that (1) does not hold. Setting $m=N-p$, $\underline{X} = \underline{Y}^{(p)}$, $\Phi = \Phi^{(p)}$ and $f_i = f_i^{(p)}$ the conditions stated in the theorem are trivially true.

Remark 1.1.5.- In the above theorem if $g_i \in A_i$ denotes an irreducible series which divides f_i, then

$$\underline{\upsilon}\,(g_i(0,\ldots,0,X_i)) = \underline{\upsilon}\,(g_i(X_1,\ldots,X_i)).$$

Thus if $\underline{a}'$ is prime, the series f_i may be chosen to be irreducible.

Let $\square$ be a complete local ring (for its $\underline{m}$-adic topology, where $\underline{m}$ is its maximal ideal). Suppose that k is a coefficient field of $\square$. For any finite set $\{z_i\}_{1 \leqslant i \leqslant N} \subset \underline{m}$ and indeterminates $\underline{Z} = \{Z_i\}_{1 \leqslant i \leqslant N}$ there is a homomorphism

$$\phi : k((\underline{Z})) \longrightarrow \square$$

given by $\phi(Z_i) = z_i$, $1 \leqslant i \leqslant N$, which is continuous for their respective Krull topologies.

Definition 1.1.6.- We say that $\{z_i\}_{1 \leqslant i \leqslant N}$ are formally independent over k if the above homomorphism is injective.

Theorem 1.1.7.- Let $\underline{X} = \{X_i\}_{1 \leqslant i \leqslant N}$ be indeterminates over k, $\underline{a}$ an ideal of $k((\underline{X}))$ and m an integer, $0 \leqslant m \leqslant N-1$, such that:

(a) $\underline{a} \cap A_m = (0)$, $\underline{a} \cap A_i \neq (0)$, $m+1 \leqslant i \leqslant N$.

(b) There exist non zero series $f_i \in \underline{a} \cap A_i$, $m+1 \leqslant i \leqslant N$, such that

$$\underline{\upsilon}\,(f_i(0,\ldots,0,X_i)) = \underline{\upsilon}(f_i(X_1,\ldots,X_i)) .$$

Set $\square = k((\underline{X}))/_{\underline{a}}$, $x_i = X_i + \underline{a}$, and denote by $\underline{M}$ (resp. $\underline{m}$) the maximal ideal of $k((\underline{X}))$ (resp. $\square$). Then the following statements are true:

1) If $m > 0$,

i) $\{x_i\}_{1 \leq i \leq m}$ are formally independent over k.

ii) $\square = k((x_1, \ldots, x_m))\,[x_{m+1}, \ldots, x_N]$.

iii) $\square$ is an integral extension of $k((x_1, \ldots, x_m))$.

iv) The height of $\underline{a}$ is $N-m$, and hence its depth is m. Particulary $m = \dim(\square)$.

2) $m = 0$ if and only if $\underline{a}$ is a M-primary ideal.

Proof:

1) Case $m > 0$.

i) The canonical homomorphism $\phi : k((X_1, \ldots, X_m)) \longrightarrow \square$ given by $\phi(X_i) = x_i$ is injective because $\underline{a} \cap A_m = (0)$.

ii) As $k((x_1, \ldots, x_m))$ is a subring of $\square$, we have

$$k((x_1, \ldots, x_m))\,[x_{m+1}, \ldots, x_N] \subset \square .$$

Conversely, if $f(\underline{X}) \in k((\underline{X}))$, by using the properties of the series f_i we may apply the division algorithm (Zariski-Samuel, [29]), and write

$$f(X_1, \ldots, X_N) = \sum_{i=m+1}^{N} U_i(X_1, \ldots, X_N)\, f_i(X_1, \ldots, X_i) +$$

$$+ \sum_{i_N=0}^{q_N-1} \cdots \sum_{i_{m+1}}^{q_{m+1}-1} R_{i_N, \ldots, i_{m+1}}(X_1, \ldots, X_m)\, X_{m+1}^{i_{m+1}} \cdots X_N^{i_N} ,$$

where $U_i(X_1, \ldots, X_N) \in k((\underline{X}))$; $R_{i_N; \ldots, i_{m+1}}(X_1, \ldots, X_m) \in k((X_1, \ldots, X_m))$ and $q_i = \underline{v}(f_i(X_1, \ldots, X_i))$. We obtain

$$f(x_1, \ldots, x_N) = \sum_{i_N=0}^{q_N-1} \cdots \sum_{i_{m+1}=0}^{q_{m+1}-1} R_{i_N, \ldots, i_{m+1}}(x_1, \ldots, x_m)\, x_{m+1}^{i_{m+1}} \cdots x_N^{i_N} \in$$

$$k((x_1, \ldots, x_m))\,[x_{m+1}, \ldots, x_N] .$$

iii) By using an argument as in (ii) we may conclude that

$$k[[x_1,\ldots,x_i]] = k[[x_1,\ldots,x_m]]\,[x_{m+1},\ldots,x_i], \quad m+1 \leqslant i \leqslant N.$$

Now, the W.P.T. applied to f_{i+1}, $m+1 \leqslant i \leqslant N$, give us

$$f_{i+1}(X_1,\ldots,X_{i+1}) = V_i(X_1,\ldots,X_{i+1})\; g_{i+1}((X_1,\ldots,X_i),X_{i+1})$$

where V_i is a unit in $k[[X_1,\ldots,X_{i+1}]]$ and g_{i+1} a monic polynomial in X_{i+1} with coefficients in $k[[X_1,\ldots,X_i]]$. It follows that x_{i+1} is integral over $k[[x_1,\ldots,x_i]] = k[[x_1,\ldots,x_m]]\,[x_{m+1},\ldots,x_i]$ (since $g_{i+1}((x_1,\ldots,x_i),x_{i+1}) = 0$). Hence, each x_i is integral over $k[[x_1,\ldots,x_m]]$, $m+1 \leqslant i \leqslant N$.

iv) By (i) and (iii)

$$\text{depth}(\underline{a}) = \dim(\square) = \dim(k[[x_1,\ldots,x_m]]) = m.$$

Therefore $\text{height}(\underline{a}) = N - \text{depth}(\underline{a}) = N-m$.

2) If $m = \text{depth}(\underline{a}) > 0$, $\underline{a}$ is clearly not M-primary. Conversely if $m=0$, we have $\underline{a} \cap k[[X_1]] = (X_1^r)$, $r > 0$. Thus

$$\text{depth}(\underline{a}) = \dim(\square) = \dim(k[[x_1]]) = \dim(k[[X_1]]/(X_1^r)) = 0$$

implies that $\underline{a}$ is M-primary.

<u>Corollary 1.1.8.</u>- Let $\square$ be an irreducible algebroid curve over k. Then, if $N \geqslant \text{Emb}(\square)$ there exists a prime ideal $\underline{p} \subset k[[\underline{X}]]$ such that $\square \cong k[[\underline{X}]]/\underline{p}$ and if we set $x_i = X_i + \underline{p}$, the following properties hold:

1) $\underline{p} \cap k[[X_1,\ldots,X_i]] \neq (0)$, $2 \leqslant i \leqslant N$; $\underline{p} \cap k[[X_1]] = (0)$.

2) There exist non zero (irreducible) series $f_i \in \underline{p} \cap k[[X_1,\ldots,X_i]]$, $2 \leqslant i \leqslant N$, such that

$$\underline{v}(f_i(X_1,\dots,X_i)) = \underline{v}(f_i(0,\dots,0,X_i)).$$

3) x_1 is formally independent over k.

4) $k[[x_1,\dots,x_N]] = k[[x_1]]\,[x_2,\dots,x_N]$

5) $k[[x_1,\dots,x_N]]$ is an integral extension of $k[[x_1]]$.

Remark 1.1.9.- The five above properties will be assumed in the sequel. Therefore we shall write $\square = k[[x_1]]\,[x_2,\dots,x_N]$, x_1 being formally independent over k, and each x_i integral over $k[[x_1]]$, $2 \leqslant i \leqslant N$.

By (5) $\square$ is integral over $k[[x_1]]$, and hence algebraic over $k((x_1))$. Every $z \in \underline{m}$ is a zero of an irreducible polynomial over $k((x_1))$ having its coefficients in $k[[x_1]]$,

$$g((x_1),Z) = Z^s + b_{s-1}(x_1)\,Z^{s-1} + \dots + b_o(x_1).$$

This polynomial is actually distinguished, i.e., $b_j(0) = 0$ for $0 \leqslant j \leqslant s-1$. Indeed, if i is the smallest integer for which $b_i(0) \neq 0$, we may use the W.P.T., applied to the two variable series $g((X_1),Z)$, in order to find a new polynomial $g^*((X_1),Z)$ with coefficients in $k[[X_1]]$ and degree $i < s$ (note that $i > 0$ since g is not a unit) and a unit $U(X_1,Z) \in k[[X_1,Z]]$ such that

$$g((x_1),Z) = U(x_1,Z).\, g^*((x_1),Z) \quad .$$

Hence $g^*((x_1),z) = 0$ which is a contradiction, because $g((x_1),Z)$ was the polynomial with z as a zero which had the minimum degree.

Since $$k[[X_1]]\,[Z] /_{(g((X_1),Z))} \cong k[[X_1,Z]] /_{(g(X_1,Z))}$$

(Zariski-Samuel, [29], p. 146), the polynomial $g((X_1),Z)$ is also irreducible as two variable series.

Remark 1.1.10.- If $\mathrm{Emb}(\square) \leqslant N$ the curve $\square$ can be embedded in an N-space. When $N=2$ the curve is said to be plane. In this case, for an embedding in a 2-space (=algebroid plane or plane if there is no confusion), the ideal $\underline{p}$ is actually principal, $\underline{p} = (f(X,Y))$. Furthermore if $X=X_1$ does not divide the leading form of f, the five properties in 1.1.8. are trivially satisfied.

Proposition 1.1.11.- □ is a regular domain if and only if Emb(□) is one.

This is a well known result in commutative algebra, but it can also be obtained from the normalization theorem as a corollary. Moreover, □ is regular if and only if □ is isomorphic as k-algebra to a formal power series ring in one indeterminate over k.

2. THE TANGENT CONE.

In this section we shall study the tangent cone of an irreducible algebroid curve from an algebraic and geometric view-point.

Let □ be an irreducible algebroid curve over the algebraically closed field k. Consider the graded ring

$$gr_{\underline{m}}(\square) = \bigoplus_{n=0}^{\infty} \underline{m}^i / \underline{m}^{i+1} .$$

Definition 1.2.1.- The tangent cone to the curve □ is defined to be the affine algebraic variety $\mathrm{Spec}(gr_{\underline{m}}(\square))$.

If a basis $\{x_i\}_{1 \leqslant i \leqslant N}$ of $\underline{m}$ is given, the graded ring $gr_{\underline{m}}(\square)$ is generated as k-algebra by the classes $\{x_i + \underline{m}^2\}_{1 \leqslant i \leqslant N}$. Then there is a canonical epimorphism

$$\begin{array}{ccc} k(X_1, \ldots, X_N) & \longrightarrow & gr_{\underline{m}}(\square) \\ X_i & \longmapsto & x_i + \underline{m}^2 \end{array}$$

($\{X_i\}_{1 \leq i \leq N}$ being indeterminates over k), and therefore an isomorphism

$$gr_{\underline{m}}(\square) \cong k[X_1, \ldots, X_N] / \underline{a}$$

where $\underline{a}$ is a homogeneous ideal of $k[X_1, \ldots, X_N]$. This gives rise to an embedding of the tangent cone in k^N : It is the affine algebraic variety in k^N defined by the ideal $\underline{a}$.

Remark 1.2.2.- If $\square = k[[\underline{X}]] / \underline{p}$ is the curve defined by the ideal $\underline{p}$ and if $\underline{a}$ is the homogeneous ideal of $k[\underline{X}]$ defining its tangent cone as above, then $\underline{a}$ is generated by the leading forms of all the series in $\underline{p}$.

Proposition 1.2.3.- There is no series in $\underline{p}$ with a leading form of type $a X_1^m$, $a \in k$, $a \neq 0$.

Proof: Otherwise $X_1^m \in \underline{a}$, $m > 0$, we would have $X_1 \in \sqrt{\underline{a}}$. Now, as the series $f_2 \in \underline{p}$ (corollary 1.1.8.) has a leading form of type $b X_2^q + X_1 g(X_1, X_2)$, we would conclude that $X_2 \in \sqrt{\underline{a}}$. In the same way, by replacing f_i instead of f_2, $i > 2$, and using induction, we would obtain $X_i \in \sqrt{\underline{a}}$. Then,

$$\dim gr_{\underline{m}}(\square) = \dim k[X_1, \ldots, X_N] / \underline{a} = \dim k[X_1, \ldots, X_N] / (X_1, \ldots, X_N) = 0$$

would be a contradiction, since $\dim gr_{\underline{m}}(\square) = 1$ (see Zariski-Samuel, [29], p. 235).

Corollary 1.2.4.- Let $g((x_1), Z) = Z^s + b_{s-1}(x_1) Z^{s-1} + \ldots + b_o(x_1)$ $b_j(x_1) \in k[[x_1]]$, the irreducible polynomial over $k((x_1))$ of an element $z \in m$. Consider g as a two variable series. Then, the leading form of g is a power of a linear form and $\underline{\upsilon}(g((x_1), Z)) = \underline{\upsilon}(g((0), Z)) = s$. In particular, the series f_i in 1.1.8. may be taken to be $g((X_1), X_i)$.

Proof: First, we prove that if a series $f(X,Y)$ is irreducible then its leading form $f_r(X,Y)$ is a power of a linear form. In fact, making a linear change of variables, f may be considered to be regular in Y of order r and hence, by the W.P.T., we may assume that it is a polynomial of $k[[X]][Y]$ of degree r. Then $f'(X,Y')=f(X,XY')/_{Y'^r}$ is an irreducible monic polynomial of $k[[X]][Y']$, and hence by the Hensel's lemma $f_r(1,Y')= f'(0,Y')=(Y'+a)^r$ with $a\in k$, and so $f_r(X,Y)=(Y+aX)^r$. Now, $g((x_1),Z)$ is irreducible as two variable series (see 1.1.9.) and so its leading form is $(ax_1+bZ)^r$ with $a,b\in k$. By the previous proposition $b\neq 0$, and since g is distinguished we have $r=s$. Hence the proof follows easily.

Lemma 1.2.5.- The tangent cone to a curve is a straight line.

Proof: Choose the series $g_i((X_1),X_i)$ in the above corollary instead of $f_i(X_1,\dots,X_i)$. The leading form of g_i is of type $(X_i+\alpha_i X_1)^{r_i}$. Then, since $\dim k[X_1,\dots,X_N]/\sqrt{\underline{a}}=1$, we must have

$$\sqrt{\underline{a}} = (X_2+\alpha_2 X_1,\dots, X_N+\alpha_N X_1).$$

It follows that the tangent cone is the straight line defined by

$$X_2+\alpha_2 X_1 = \dots = X_N+\alpha_N X_1 = 0.$$

3. LOCAL PARAMETRIZATION.

Let $\square$ be an irreducible algebroid curve over k. Let F be the quotient field of $\square$. Choose a normalized basis $\{x_i\}_{1\leqslant i\leqslant N}$ (i.e., a basis for which the conditions of 1.1.8. hold) of the maximal ideal $\underline{m}$.

Since $k((x_1))[x_2,\dots,x_N]$ is a subfield of F containing $\square = k[[x_1]][x_2,\dots,x_N]$, we have: