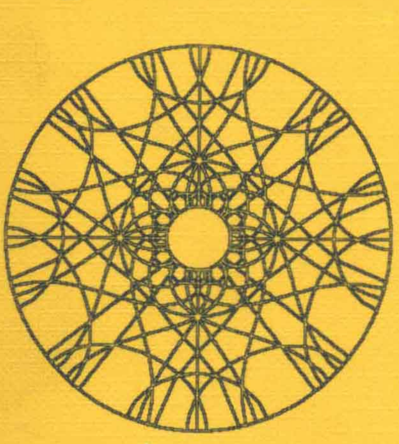
Lecture Notes in Mathematics
1591

Marco Abate Giorgio Patrizio

Finsler Metrics – A Global Approach

with applications to geometric function theory



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Preface

When, some years ago, we started working on a differential geometric study of the structure of strongly convex domains in $\mathbb{C}^n$, we did not expect to end up writing a book on global Finsler geometry. But, along the way, we found ourselves needing several basic results on real and complex Finsler metrics that we were unable to find in the literature (or at least not in the form necessary to us). So we felt compelled to provide proofs — and this is the final result of our work.

Our exposition is very much in the vein of the work of Cartan [C], Chern [Ch1, 2], Bao-Chern [BC] and Kobayashi [K]; in particular the latter gave us the preliminary idea for our approach to smooth complex Finsler metrics. We would also like to say that we would have been very happy to know earlier of [BC], which, although only marginally related to our work, would have been of great help in solving questions which we treated independently.

Our starting point was the study of the existence and global behavior of complex geodesics for intrinsic metrics in complex manifolds. Our goal was to look at this question from a differential geometric point of view, with the hope of possibly reproducing in a wider class of complex manifolds what Lempert [L] was able to prove for strongly convex domains in $\mathbb{C}^n$. The idea was to treat complex geodesics through a point as images of disks through the origin in the tangent space at the point via the exponential map of a complex Finsler metric; thus we were led to study the local and global theory of geodesics of a Finsler metric. As in Hermitian (and Riemannian) geometry, the local theory of geodesics means the study of the first variation of the length integral, and of the associated Euler-Lagrange equation. The global theory, on the other hand, involves the accurate control of the second variation and hence of the curvature, together with Jacobi fields, conjugate points, the Morse index form and the like. In particular, we needed a Finsler version of the Cartan-Hadamard theorem (originally proved by Auslander [Au1, 2]), and a way to apply it in a complex situation.

The main difficulty at this point was that the problems we were interested in involved complex Finsler metrics, and whereas there is a clear understanding of the relationship between the complex geometry and the underlying real geometry of a Hermitian manifold, nothing of the kind was available to us in Finsler geometry. We then started following the tradition of "linearizing" the questions by passing from the study of Finsler metrics on the tangent bundle (real or complex) to the study of the associated Hermitian structure on the tangent bundle of the tangent bundle. At this level it is also possible to describe the correct relationship between the complex and the corresponding real structure of objects like connections and curvatures.

But our approach is different from the traditional one for two main reasons. First of all, we everywhere stress global objects and global definitions (in fact, we are interested in global results), using local coordinates almost uniquely as a computational tool (in a way not too far from the first chapter of Bejancu [B]). But the main difference is another one. Possibly because of our motivations, working in this area we discovered that there might be a danger of carrying out the linearization program previously described too far. In fact, the formal setting naturally leads to very general definitions which make proofs of theorems easier, but do not give much geometrical insight: we had the feeling that working only at the tangent-tangent level was too restrictive, too formal, too far away from the actual geometry of the manifold. For this reason, our point of view now is to stick to notions which really provide informations about the geometry of geodesics on the manifold, and about the curvature of the manifold. This approach leads to "minimal" definitions, which are probably more complicated to state and surely more difficult to handle, but nevertheless more effective and really conveying the geometry of the manifold. For instance, there are many ways of generalizing the notion of Kählerianity to Finsler metrics, but not all of them have non-trivial examples and applications. We shall show how the notions we singled out can be effectively used by illustrating their applications in complex geometric function theory.

The first two chapters of this book are devoted to the exposition of our approach to real and complex Finsler geometry. In the first chapter, after setting the stage introducing the necessary general definitions and objects, we define in a global way the classical Cartan connection, and we discuss the variation formulas of the length integral, the exponential map, Jacobi fields, conjugate points and the Morse index form up to provide a proof of the Cartan-Hadamard and Bonnet theorems for Finsler metrics suitable for our needs in complex geometry. In the exposition we stress the similarities with the standard Riemannian treatment of the subject, as naturally suggested by our global approach.

In the second chapter we study the geometry of complex Finsler metrics. After having adapted the general definitions of chapter 1 to the complex setting, we introduce (following Kobayashi [K]) the Chern-Finsler connection, which is our main tool. We discuss at some length several Kähler conditions, and we introduce the notion of holomorphic curvature of a complex Finsler manifold, showing the equivalence of the differential geometric definition with a variational definition previously used in function theory.

Finally the third chapter contains the results and applications that motivated our work. From a differential geometric point of view, it is devoted to the study of the function theory on Kähler Finsler manifolds with constant nonpositive holomorphic curvature; from a complex analysis point of view, it is devoted to the study of manifolds where there is a Monge-Ampère foliation with exactly the same properties as the one discovered by Lempert in strongly convex domains. In particular we prove the existence of nice foliations and strictly plurisubharmonic exhaustions satisfying the Monge-Ampère equation on Kähler Finsler manifolds with constant nonpositive holomorphic curvature. Furthermore we prove that the only complex manifold admitting such a metric with zero holomorphic curvature is $\mathbb{C}^n$, and we describe a characterization of strongly convex circular domains in terms of differential geometric properties of the Kobayashi metric.

Of course, this book is not intended as a definitive treatise on the subject; on the contrary, it is just the description of an approach to Finsler metrics that we found reasonable and fruitful, but still leaving a lot of open problems. Just to mention a couple of them: the comparison between the complex Finsler geometry and the underlying real one carried out in section 2.6 seems to suggest that the Cartan connection contains terms which have no direct influence on the geometry of the manifold — and so maybe it is not the correct connection to use even in real Finsler geometry. Or: in the third chapter we give a fairly complete description of the complex structure of Kähler Finsler manifolds of constant nonpositive holomorphic curvature, which is satisfying from a geometric function theory point of view, but it still leaves completely open the problem of classifying the metrics with these properties (we remark that there are many more such manifolds and metrics than in the Hermitian case: there are at least all the strongly convex domains in $\mathbb{C}^n$ endowed with the Kobayashi metric, thanks to Lempert's work [L]) — and in fact it is even still far from being completed the classification of simply connected real Finsler manifolds with constant (horizontal flag) real curvature. Or: it follows from chapter 3 that the only part of Lempert's results actually depending on the strong convexity of the domain is the smoothness of the Kobayashi metric. It would be then interesting to construct directly a smooth weakly Kähler Finsler metric of constant holomorphic curvature -4 on any strongly convex domain; then this metric would automatically be the Kobayashi metric of the domain, and we would have recovered the full extent of Lempert's work.

So we hope that the possibly new perspectives on Finsler geometry introduced in this book will eventually lead to new results in this field; and in particular in geometric function theory of complex Finsler manifolds, where all this work started.

Contents

CHAPTER 1

Real Finsler geometry

1.0. Introduction

As already discussed in the preface, this book is mainly devoted to the study of complex Finsler geometry; but of course such a study cannot leave out of consideration real Finsler manifolds. So this first chapter is devoted to a discussion of real Finsler geometry, starting from the very basics and ending with a proof of the appropriate versions of the Cartan-Hadamard and Bonnet theorems for Finsler manifolds, obtained using global Riemannian-style techniques.

Let M be a real manifold endowed with a Finsler metric, that is with a positively homogeneous function $F: TM \to \mathbb{R}^+$ smooth outside the zero section of TM and strongly convex on each tangent space. Roughly speaking, our main idea is to replace the given Finsler metric on TM by a Riemannian metric on a suitable sub-bundle of $T(TM)$ — in a certain sense we linearize the Finsler metric going one step upstairs — and then use the standard tools of Riemannian geometry there. A canonically defined isometric embedding of TM (outside the zero section, actually) into this bundle will then allow us to transfer information back and forth, thus giving geometrical results about the original manifold. For instance, we shall be able to recover for Finsler manifolds more or less all the results described in the first chapter of [CE] for Riemannian manifolds. We also refer to [C], [Rd1], [M], [Ch1] and [B] for a description of the standard theory of real Finsler metrics, and to [Ch2] and [BC] for a recent approach akin in spirit to ours.

To be more precise, let M be a manifold, and let $\pi: TM \to M$ denote the tangent bundle of M; $\tilde{M}$ will stand for $TM \setminus \{\text{zero section}\}$. The *vertical bundle* is $\mathcal{V} = \ker d\pi$, a sub-bundle of $T\tilde{M}$. Take a Finsler metric $F: TM \to \mathbb{R}^+$ on M, and set $G = F^2$. Then it is easy to see (section 1.4) that using the Hessian of G it is possible to define a Riemannian metric on $\mathcal{V}$ in such a way that a canonically defined section $\iota: \tilde{M} \to \mathcal{V}$ of $\mathcal{V}$ (see section 1.1) turns out to be an isometric embedding of $\tilde{M}$ into $\mathcal{V}$.

But this is not yet the setting mentioned before. The point is that to such a Riemannian metric on the vertical bundle it is possible to associate two objects: a linear connection D on $\mathcal{V}$ with respect to which the given Riemannian metric is parallel; and a horizontal bundle, that is a sub-bundle $\mathcal{H}$ of $T\tilde{M}$ such that $T\tilde{M} = \mathcal{H} \oplus \mathcal{V}$. The general theory of horizontal bundles yields a bundle isomorphism $\Theta: \mathcal{V} \to \mathcal{H}$; using Θ we can define a Riemannian metric and a connection on $\mathcal{H}$ — and hence on $T\tilde{M}$. This is the *Cartan connection*, the exact analogue in the Finsler setting of the Levi-Civita connection (e.g., its torsion is almost zero — in a very definite sense; furthermore, the torsion is identically zero if and only if the Finsler metric actually is Riemannian, and in this case the Cartan connection coincides with the Levi-Civita connection). The bundle $\mathcal{H}$ with this structure (and with its section $\chi = \Theta \circ \iota$) provides the aforementioned setting, where one can use Riemannian tools to get Finsler statements.

The main examples of this assertion are provided by the first and second variations of the length integral, derived in section 1.5; we get formulas formally identical to the Riemannian ones, just replacing the curvature of the Levi-Civita connection by (a suitable contraction of the horizontal part of) the curvature of the Cartan connection. Then we shall be able to recover the Hopf-Rinow theorem for Finsler manifolds (this is not too surprising, since it holds in much more general settings; see [Ri]) and the theory of Jacobi fields and of the Morse index form, in a way exactly parallel to the one presented in standard Riemannian geometry texts. In particular, in section 1.7 we shall be able to prove the generalizations (originally due to Auslander [Au1, 2]) to Finsler manifolds of the classical Cartan-Hadamard and Bonnet theorems.

In detail, the content of this chapter is the following. In section 1.1 we discuss at some length the general theory of horizontal bundles, horizontal maps (i.e., maps like our Θ above) and non-linear connections on M. In section 1.2 we introduce the concept of vertical connection (i.e., of linear connection on the vertical bundle), and we show how to associate to certain vertical connections (we call them the *good* ones) a horizontal bundle, and hence a non-linear connection on M and a linear connection on $\tilde{M}$. In section 1.3 we define and discuss the torsion and the curvature of a good vertical connection. In section 1.4 we define Finsler metrics, and we show that to any Finsler metric F is canonically associated a good vertical connection, the Cartan connection mentioned before. Section 1.5 is devoted to prove the first and second variation of the length integral for Finsler metrics; section 1.6 to parallel transport, geodesics, the exponential map and the Hopf-Rinow theorem for Finsler metrics. Finally in section 1.7 we shall define Jacobi fields and the Morse index form in this setting, and we shall prove the Finsler versions of the Cartan-Hadamard and Bonnet theorems.

1.1. Non-linear connections

1.1.1. Preliminaries

In this subsection we fix our notations and collect a few formulas concerning change of coordinates. We choose symbols and notations so to be compatible with the complex case we shall discuss in chapters 2 and 3; this is the reason behind some apparently slightly unusual choices (u instead of v to denote tangent vectors, and the like).

Let M be a real manifold of dimension m; we shall denote by TM its tangent bundle, and by $\pi: TM \to M$ the canonical projection, as usual. The cotangent bundle will be denoted by T^*M.

If $(x^1, \dots, x^m)$ are local coordinates on M about a point $p_0 \in M$, a vector $u \in T_pM$ (with p close to p_0) is represented by

$$u = u^a \left.\frac{\partial}{\partial x^a}\right|_p ,$$

where we are using the Einstein convention, and lowercase roman letters run from 1 to m. In particular, local coordinates on TM are given by $(x^1, \dots, x^m, u^1, \dots, u^m)$, and so a local frame of $T(TM)$ is given by $\{\partial_1, \dots, \partial_m, \dot{\partial}_1, \dots, \dot{\partial}_m\}$, where

$$\partial_a = \frac{\partial}{\partial x^a} \qquad \text{and} \qquad \dot{\partial}_b = \frac{\partial}{\partial u^b}.$$

We shall denote by $o: M \to TM$ the zero section of TM, that is $o(p) = o_p$ is the origin of T_pM, and we set $\tilde{M} = TM \setminus o(M)$, the tangent bundle minus the zero section. $\tilde{M}$ is naturally equipped with the projection $\pi: \tilde{M} \to M$, the restriction of the canonical projection of TM. Correspondingly, $T\tilde{M} \subset T(TM)$ comes equipped with a natural projection $\tilde{\pi}: T\tilde{M} \to \tilde{M}$, the restriction of the natural projection $\tilde{\pi}: T(TM) \to TM$.

We shall use uppercase roman letters to denote different coordinate patches. A coordinate patch (U_A, φ_A) in M determines a coordinate patch $(\tilde{U}_A, \tilde{\varphi}_A)$ in TM (and $\tilde{M}$) setting $\tilde{U}_A = \pi^{-1}(U_A)$ and

$$\forall u \in \tilde{U}_A \qquad\qquad \tilde{\varphi}_A(u) = d\varphi_A(u).$$

If $\varphi_A = (x^1_A, \dots, x^m_A)$, then $\{(\partial/\partial x^j_A)|_p\}$ is a basis of T_pM for any $p \in U_A$. Writing $u = u^a_A(\partial/\partial x^a_A)$, then

$$\tilde{\varphi}_A(u) = (x^1_A, \dots, x^m_A, u^1_A, \dots, u^m_A).$$

On $U_A \cap U_B$ we have

$$dx^i_B = \frac{\partial x^i_B}{\partial x^j_A}\, dx^j_A, \qquad \frac{\partial}{\partial x^i_B} = \frac{\partial x^j_A}{\partial x^i_B} \frac{\partial}{\partial x^j_A},$$

where $\partial x^i_B/\partial x^j_A = \partial(\varphi_B \circ \varphi_A^{-1})^i/\partial x^j_A$. By the way, we set

$$\mathcal{J}_{BA} = \left(\frac{\partial x^i_B}{\partial x^j_A}\right);$$

clearly,

$$\mathcal{J}_{AB} = \mathcal{J}_{BA}^{-1} \circ (\varphi_A \circ \varphi_B^{-1}).$$

Taking $u \in \widetilde{U}_A \cap \widetilde{U}_B$ and expressing it in local coordinates, we find

$$u^i_B = dx^i_B(u) = \frac{\partial x^i_B}{\partial x^j_A} dx^j_A \left(u^k_A \frac{\partial}{\partial x^k_A}\right) = \frac{\partial x^i_B}{\partial x^j_A} u^j_A,$$

that is

$$u_B = \mathcal{J}_{BA}\, u_A.$$

Therefore

$$(x_B, u_B) = \tilde{\varphi}_B \circ \tilde{\varphi}_A^{-1}(x_A, u_A) = (\varphi_B \circ \varphi_A^{-1}(x_A), \mathcal{J}_{BA} u_A). \tag{1.1.1}$$

Up to now everything was quite standard. But now something different: change of coordinates in $T(TM)$. Define $(\widetilde{\widetilde{U}}_A, \tilde{\tilde{\varphi}}_A)$ by setting

$$\widetilde{\widetilde{U}}_A = \tilde{\pi}^{-1}(\widetilde{U}_A) = (\pi \circ \tilde{\pi})^{-1}(U_A)$$

and $\tilde{\tilde{\varphi}}_A(X) = d\tilde{\varphi}_A(X)$ for any $X \in \widetilde{\widetilde{U}}_A$.

A vector $X \in T(TM)$ in local coordinates is expressed by

$$X_A = X^i_A(\partial_i)_A + \dot{X}^j_A(\dot{\partial}_j)_A = h(X_A) + v(X_A).$$

Taking derivatives of (1.1.1), we find the Jacobian matrix for $T(TM)$:

$$\widetilde{\mathcal{J}}_{BA} = \left(\begin{array}{c|c} \dfrac{\partial x^i_B}{\partial x^j_A} & \dfrac{\partial x^i_B}{\partial u^k_A} \\ \hline \dfrac{\partial u^h_B}{\partial x^j_A} & \dfrac{\partial u^h_B}{\partial u^k_A} \end{array}\right) = \left(\begin{array}{c|c} (\mathcal{J}_{BA})^i_j & 0 \\ \hline \dfrac{\partial^2 x^h_B}{\partial x^j_A \partial x^l_A} u^l_A & (\mathcal{J}_{BA})^h_k \end{array}\right).$$

Setting

$$(H_{BA})^h_{kl} = \frac{\partial^2 x^h_B}{\partial x^k_A \partial x^l_A},$$

we find

$$\begin{aligned}(x_B, u_B, h(X_B), v(X_B)) &= \tilde{\tilde{\varphi}}_B \circ \tilde{\tilde{\varphi}}_A^{-1}(x_A, u_A, h(X_A), v(X_A)) \\ &= \Big(\varphi_B \circ \varphi_A^{-1}(x_A), \mathcal{J}_{BA}\, u_A, \mathcal{J}_{BA}\, h(X_A), (H_{BA})^{\bullet}_{kl}\, u^l_A\, h(X_A)^k + \mathcal{J}_{BA}\, v(X_A)\Big).\end{aligned} \tag{1.1.2}$$

Now $\{\partial_i, \dot\partial_j\}$ is a local frame for $T(TM)$; let $\{dx^i, du^j\}$ be the dual coframe (note that $dx^i|_v$ is not the same as $dx^i|_p$). First of all, (1.1.2) yields

$$
\begin{aligned}
dx^i_B &= \frac{\partial x^i_B}{\partial x^j_A}\, dx^j_A = (\mathcal{J}_{BA})^i_j\, dx^j_A \\
du^h_B &= \frac{\partial x^h_B}{\partial x^k_A}\, du^k_A + \frac{\partial^2 x^h_B}{\partial x^k_A \partial x^l_A}\, u^l_A\, dx^k_A = (\mathcal{J}_{BA})^h_k\, du^k_A + (H_{BA})^h_{kl}\, u^l_A\, dx^k_A.
\end{aligned}
\tag{1.1.3}
$$

Recalling that $\{dx^i, du^j\}$ is the dual frame of $\{\partial_i, \dot\partial_j\}$, we get

$$
\begin{aligned}
(\partial_i)_B &= \frac{\partial x^j_A}{\partial x^i_B}\,(\partial_j)_A - \frac{\partial x^r_A}{\partial x^h_B}\,\frac{\partial^2 x^h_B}{\partial x^k_A \partial x^l_A}\, u^l_A\, \frac{\partial x^k_A}{\partial x^i_B}\,(\dot\partial_r)_A \\
&= (\mathcal{J}^{-1}_{BA})^j_i\,(\partial_j)_A - (\mathcal{J}^{-1}_{BA})^k_i (H_{BA})^h_{kl} (\mathcal{J}^{-1}_{BA})^r_h\, u^l_A\,(\dot\partial_r)_A, \\
(\dot\partial_h)_B &= \frac{\partial x^k_A}{\partial x^h_B}\,(\dot\partial_k)_A = (\mathcal{J}^{-1}_{BA})^k_h\,(\dot\partial_k)_A.
\end{aligned}
\tag{1.1.4}
$$

1.1.2. Horizontal and vertical bundles

Now we may introduce our first main actor.

DEFINITION 1.1.1: The *vertical bundle* of a manifold M is the vector bundle $\tilde\pi\colon \mathcal{V} \to TM$ of rank $m = \dim M$ given by

$$\mathcal{V} = \ker d\pi \subset T(TM).$$

In local coordinates,

$$\varphi_A \circ \pi \circ \tilde\varphi_A^{-1}(x_A, u_A) = x_A,$$

and so

$$
\begin{aligned}
\tilde\varphi_A \circ d\pi \circ \tilde{\tilde\varphi}_A^{-1}(x_A, u_A, X_A) &= d\varphi_A \circ d\pi \circ (d\tilde\varphi_A)^{-1}(x_A, u_A, X_A) \\
&= d(\varphi_A \circ \pi \circ \tilde\varphi_A^{-1})(x_A, u_A, h(X_A), v(X_A)) \\
&= (x_A, h(X_A)).
\end{aligned}
$$

This means that $\{\dot\partial_h\}$ is a local frame for $\mathcal{V}$. We get charts restricting $\tilde{\tilde\varphi}_A$, and in particular (1.1.2) yields

$$(x_B, u_B, V_B) = \tilde{\tilde\varphi}_B \circ \tilde{\tilde\varphi}_A^{-1}(x_A, u_A, V_A) = (\varphi_B \circ \varphi_A^{-1}(x_A), \mathcal{J}_{BA}\, u_A, \mathcal{J}_{BA}\, V_A).$$

Let $j_p\colon T_pM \to TM$ be the inclusion and, for $u \in T_pM$, let $k_u\colon T_pM \to T_u(T_pM)$ denote the usual identification. Then we get a natural isomorphism

$$\iota_u = d(j_{\pi(u)})_u \circ k_u\colon T_{\pi(u)}M \to \mathcal{V}_u. \tag{1.1.5}$$

DEFINITION 1.1.2: The *radial vertical vector field* is the natural section $\iota: TM \to \mathcal{V}$ given by

$$\iota(u) = \iota_u(u);$$

clearly, $\iota(u) \in \mathcal{V}_u$.

In local coordinates,

$$\iota_u\left(\left.\frac{\partial}{\partial x^j}\right|_{\pi(u)}\right) = \dot{\partial}_j|_u;$$

in particular, if $u = u^a(\partial/\partial x^a)$ then

$$\iota(u) = \iota\left(u^a \frac{\partial}{\partial x^a}\right) = u^a \dot{\partial}_a|_u.$$

Note that the derivatives with respect to x (coordinates in M) become derivatives with respect to u (coordinates in TM).

The vertical bundle is canonically defined; this is not the case for a horizontal bundle. We may describe horizontal bundles using three different points of view, each with its own advantages and disadvantages. The first two are easily introduced:

DEFINITION 1.1.3: A *horizontal bundle* is a subbundle $\mathcal{H}$ of $T(TM)$ such that

$$T(TM) = \mathcal{H} \oplus \mathcal{V}.$$

DEFINITION 1.1.4: A *horizontal map* is a bundle map $\Theta: \mathcal{V} \to T(TM)$ such that

$$\forall u \in TM \qquad (d\pi \circ \Theta)_u = \iota_u^{-1}. \tag{1.1.6}$$

We defined horizontal bundles (and horizontal maps) on TM, but it turns out that they are interesting only over $\tilde{M}$. In fact, let $o: M \to TM$ denote the zero section. It is easy to check that $do_p(\partial/\partial x^j) = \partial_j$; therefore we have the natural splitting

$$T_{o_p}(TM) = \mathcal{H}_{o_p} \oplus \mathcal{V}_{o_p},$$

where $\mathcal{H}_{o_p} = do_p(T_pM)$. *We shall then assume* that all our horizontal bundles coincide with $do_p(T_pM)$ over the zero section, and, analogously, that all our horizontal maps satisfy

$$\Theta_{o_p}(\dot{\partial}_h|_{o_p}) = \partial_h|_{o_p}.$$

Clearly, this may cause problems with the smoothness at the origin. We shall henceforth assume that our horizontal bundles and maps will be *smooth over* $\tilde{M}$, but they may be not smooth over the zero section. The reasons behind this approach will become clear in section 1.4, when we shall define the concept of Finsler metric.

As mentioned before, there is a third approach to horizontal bundles, via the notion of non-linear connection. But to describe it we need a digression on linear connections.

If $p: E \to M$ is any bundle over M, we shall denote by $\mathcal{X}(E)$ the space of sections of E.

Definition 1.1.5: A *linear connection* on a manifold M is a $\mathbb{R}$-linear map

$$\tilde{D}: \mathcal{X}(TM) \to \mathcal{X}(T^*M \otimes TM)$$

satisfying the derivation property

$$\forall \xi \in \mathcal{X}(TM)\, \forall f \in C^\infty(M)\ \tilde{D}(f\xi) = df \otimes \xi + f\tilde{D}\xi. \tag{1.1.7}$$

As a consequence, $\tilde{D}\xi$ at a point $p \in M$ depends only on the value of ξ and $d\xi$ at the point p. Indeed, let $\xi' \in \mathcal{X}(TM)$ be another vector field with $\xi(p) = \xi'(p)$ and $d\xi_p = d\xi'_p$. Then $\xi' = \xi + f\eta$ for suitable $\eta \in \mathcal{X}(TM)$ and $f \in C^\infty(M)$ with $f(p) = 0$ and $df_p = 0$. So

$$\tilde{D}\xi' = \tilde{D}\xi + \tilde{D}(f\eta) = \tilde{D}\xi + df \otimes \eta + f\tilde{D}\eta,$$

and $\tilde{D}\xi'_p = \tilde{D}\xi_p$.

There is another way of expressing this. Let ξ, $\xi' \in \mathcal{X}(TM)$ be such that $\xi(p) = \xi'(p) = u$; then $\xi' = \xi + f\eta$ with $f(p) = 0$. In particular,

$$\tilde{D}\xi'_p = \tilde{D}\xi_p + df_p \otimes \eta(p).$$

Now, for any $v \in T_pM$, writing $v = v^a(\partial/\partial x^a)$ and $\xi = \xi^b(\partial/\partial x^b)$, one has

$$d\xi_p(v) = v^a \partial_a|_u + v^a \frac{\partial \xi^b}{\partial x^a}(p)\dot{\partial}_b|_u. \tag{1.1.8}$$

So $d\xi'_p - d\xi_p$ maps T_pM into $\mathcal{V}_u$; furthermore, (1.1.8) also yields

$$\forall v \in T_pM\ \ d\xi'_p(v) - d\xi_p(v) = v(f)\iota_u\big(\eta(p)\big) = \iota_u\big(v(f)\eta(p)\big),$$

and thus

$$\iota_u^{-1} \circ (d\xi'_p - d\xi_p) = df_p \otimes \eta(p). \tag{1.1.9}$$

Summing up, if $\xi(p) = \xi'(p) = u$ we get

$$\tilde{D}\xi'_p - \tilde{D}\xi_p = \iota_u^{-1} \circ (d\xi'_p - d\xi_p), \tag{1.1.10}$$

which we may consider as an intrinsic way of saying that $\tilde{D}\xi_p$ depends only on $\xi(p)$ and $d\xi_p$.

There is another easy consequence of (1.1.7) worth remarking. If we apply (1.1.7) to the zero section o with $f \equiv 0$ we get

$$\tilde{D}o \equiv 0, \tag{1.1.11}$$

i.e., $\tilde{D}o_p(u) = o_p$ for all $p \in M$ and $u \in T_pM$.

We are now ready to introduce the third incarnation of horizontal bundles.

Definition 1.1.6: A *non-linear connection* is a map $\tilde{D}: \mathcal{X}(TM) \to \mathcal{X}(T^*M \otimes TM)$ satisfying (1.1.10) and (1.1.11). $\tilde{D}\xi$ is called the *covariant differential* of the vector field $\xi \in \mathcal{X}(TM)$, and $\tilde{D}\xi_p(u)$ (which we shall denote by $\tilde{\nabla}_u\xi$) is the *covariant derivative* of ξ in the direction of $u \in T_pM$.