Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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Local Theory of Nonlinear Analytic Ordinary Differential Equations

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PREFACE

These notes present a course of lectures given by the author at the Division of Applied Mathematics, Brown University during the second semester of the academic year 1975-1976. They are based on a course, on the theory of the stability of the motion, which the author gave at the Department of Mathematics and Mechanics at the University of Leningrad during the last several years, and on some recent publications by the author.

The author is very grateful to Professor Jack K. Hale, and members of the Division for their warm hospitality and useful discussions. The author thanks Messrs. R. Malek-Madani, K. Lyons and N. Alikakos for their help in preparing the manuscript, and Miss Sandra Spinacci for her meticulous typing. The author is also grateful to R. Malek-Madani for his careful proofreading of the material.

June, 1976

Yuri N. Bibikov
Providence, R. I.

Basic Notation

a) Generalities

$\mathbb{R}^n$ is the n-dimensional real space;

$\mathbb{C}^n$ is the n-dimensional complex space;

$\mathbb{Z}$ is the set of integers;

$\mathbb{Z}_+$ is the set of non-negative integers;

$\mathbb{N}$ is the set of natural numbers;

$\mathfrak{M}^{(n\times m)}$ is the set of $n \times m$ matrices;

$\dot{x} = \frac{dx}{dt}$

NF stands for "normal form";

NFIS stands for "normal form on invariant surface";

QNF stands for "quasi-normal form";

"$\times$" denotes the end of a proof.

b) Vectors

If $x \in R^n$ or $x \in \mathbb{C}^n$ then $x = (x_1,\ldots,x_n)$;

Also, $x = (x',x'')$, where $x' = (x_1,\ldots,x_m)$,

$x'' = (x_{m+1},\ldots,x_n)$ $(1 \leq m \leq n)$;

$||x|| = \max_k |x_k|$;

e_k denotes the k^{th} unit vector $(0,\ldots,1,\ldots,0)$;

$q = (q_1,\ldots,q_n)$, where $q_k \in \mathbb{Z}_+$ $(k = 1,\ldots,n)$,

$|q| = q_1 + \ldots + q_n$;

in Chapter III $q = (q_1,\ldots,q_n,\overline{q}_1,\ldots,\overline{q}_n)$, where

$q_k,\overline{q}_k \in \mathbb{Z}_+$, $|q| = q_1 + \ldots + \overline{q}_n$;

$p = (p_1,\ldots,p_n)$, where $p_k \in \mathbb{Z}$ $(k = 1,\ldots,n)$,

$|p| = |p_1| + \ldots + |p_n|$;

$x \leq y$ $(x, y \in R^n$ or $\mathbb{C}^n)$ means that $x_k \leq y_k$ $(k = 1,\dots,n)$;

$\bar{x}$ is the complex conjugate vector to x (this is not applied to q);

vectors are always considered as columns, but for convenience are written as rows.

c) Matrices

$A \in \mathfrak{M}(n\times n)$;

$\kappa = (\kappa_1,\dots,\kappa_n)$ is the vector whose components are eigenvalues of A;

$$J = \begin{pmatrix} \kappa_1, & & 0 \\ \sigma_2, \kappa_2, & & \\ 0 & \ddots & \sigma_n, \kappa_n \end{pmatrix}$$ is the Jordan canonical form of A;

$(q,\kappa) = q_1\kappa_1 + \dots + q_k\kappa_k$.

d) Power Series

By a (vector) power series, we mean (unless stated on the contrary) an expression:

$$X(x) = (X_1(x),\dots,X_m(x)),$$

$$X_k(x) = \sum_{|q|=2}^{\infty} X_k^{(q)} x^q, \quad (k = 1,\dots,m),$$

where

$$x^q = x_1^{q_1} \dots x_n^{q_n}, \quad X_k^{(q)} \in \mathbb{C};$$

$\tilde{X}(x',x'')$ denotes a power series with linear terms in x';

$\hat{X}(x)$ denotes a majorant power series with respect to $X(x)$;

$X(x) \prec Y(x)$ means that $Y(x)$ majorizes $X(x)$;

$$\lceil X \rceil_k^N = \sum_{|q|=2}^{N} X_k^{(q)} x^q .$$

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§0. Introduction

The principle problem of the local theory of ordinary differential equations is the investigation of the neighborhood of a given solution. In general, in applications the right-hand sides of differential equations are expanded in convergent power series. The objective of this volume is to present an approach to a solution of this problem based on the use of these series.

We consider the case where the system at hand is autonomous and the given solution corresponds to an equilibrium point. Setting the origin at this point, we obtain a system

$$\dot{x} = Ax + X(x) \tag{0.1}$$

where x is an n-vector, A is a constant matrix, $n \times n$, the co-ordinates X_k of X are convergent power series in powers of $x_1,\ldots,x_n$, without constant and linear terms. From the point of view of applications the phase space should be real, $X(x)$ should be a real analytic function. In our presentation, we will deal with both complex and real cases.

The most attention is paid to the stability problem and to the closely connected problem of existence of periodic and quasi-periodic solutions. In particular, the course presents all results from the fundamental thesis of Liapunov [1], concerning autonomous systems. More recent results [2-7] on periodic and quasi-periodic solutions are presented too.

The general approach is based on ideas and methods developed by Poincaré [8], Liapunov [1], Dulac [9], Birkhoff [10], Siegel [2], Kolmogorov [11], Malkin [12], Brjuno [13].

In practice, differential equations usually depend analytically on a parameter ε which can be considered as changing in a small neighborhood of the origin in $\mathbb{R}^m$. This case can also be included into the considered case by adding to the system the new equation $\dot{\varepsilon} = 0$ and obtaining a n+1-dimensional system (0.1). In §7 this method is used in one particular case. In the same fashion the neighborhood of a periodic solution of a periodic time dependent system can be investigated (see Appendix or §11 in [13]).

For an understanding of the text, it is sufficient to know the foundations of the theory of functions of several complex variables in depth similar to the exposition in Chapter II of monograph [14].

CHAPTER I

ANALYTIC FAMILIES OF SOLUTIONS

§1. Auxiliary Lemma

Consider the function $X^{(\ell)}: \mathbb{C}^n \to \mathbb{C}^m$, given by

$$X_k^{(\ell)} = \sum_{|q|=\ell} X_k^{(q)} x^q, \qquad (k = 1,\ldots,m) \tag{1.1}$$

(see list of notations) which we shall call a vector homogeneous polynomial (v.h.p) of order ℓ. Every v.h.p. can be defined by the set of its coefficients $X_k^{(q)}$ written in a certain order. We order the coefficients as follows: (k_*,q_*) precedes (k^*,q^*) if and only if the first non-zero difference $k^* - k_*$, $q_1^* - q_{1*}, \ldots, q_n^* - q_{n*}$ is positive. This order will be called canonical. Under canonical order the space $\mathfrak{G}(\ell,m,n)$ of v.h.p. of order ℓ can be identified with the N-dimensional complex vector space $\mathbb{C}^N$, $N = N(\ell,m,n)$.

Consider a linear operator $L(\ell,m,n)$: $\mathfrak{G}(\ell,m,n) \to \mathfrak{G}(\ell,m,n)$, defined by

$$LX^{(\ell)} = \frac{\partial X^{(\ell)}}{\partial x} Ax - BX^{(\ell)} \tag{1.2}$$

where $A \in \mathfrak{M}^{(n\times n)}$, $B \in \mathfrak{M}^{(m\times m)}$.

Lemma 1.1. The eigenvalues Λ_j $(j = 1,\ldots,N)$ of the operator $L(\ell,m,n)$ are given by the formula:

$$\Lambda_j = (q,\kappa) - \lambda_k \qquad (k = 1,\ldots,m;\ |q| = \ell), \tag{1.3}$$

where (q,κ) is the scalar product given by $q_1\kappa_1 + \cdots + q_n\kappa_n$, $\kappa_1,\ldots,\kappa_n$ being eigenvalues of matrix A; $\lambda_1,\ldots,\lambda_m$ being eigenvalues of matrix B.

Proof. Let $h(x)$ be a non-trivial v.h.p., $S \in \mathfrak{M}^{(n\times n)}$, $T \in \mathfrak{M}^{(m\times m)}$ are non-singular matrices and set

$$h(x) = Tg(y), \quad x = Sy.$$

We have:

$$Lh = T\frac{\partial g}{\partial y}(y)S^{-1}Ax - BTg(y). \tag{1.4}$$

Consider the equation for the definition of the eigenvalues of operator L:

$$Lh = \Lambda h,$$

which by virtue of (1.4) coincides with

$$\frac{\partial g}{\partial y}(y)S^{-1}ASy - T^{-1}BTg(y) = \Lambda g(y).$$

Hence, eigenvalues of L are the same as those of L^*:

$$L^*X^{(\ell)} = \frac{\partial X^{(\ell)}}{\partial x} S^{-1}ASx - T^{-1}BTX^{(\ell)}.$$

Let S,T be such that $J_1 = S^{-1}AS$, $J_2 = T^{-1}BT$ are lower triangular Jordan canonical matrices with non-diagonal elements σ_k $(k = 2,\dots,n)$, τ_k $(k = 2,\dots,m)$ respectively.

Now, we are looking for the matrix $Z \in \mathfrak{M}^{(N\times N)}$ of the operator L^*. Setting $L^*h = f$, we observe

$$f_k(x) = \sum_{i=1}^{n} \frac{\partial h_k}{\partial x_i} (\kappa_i x_i + \sigma_i x_{i-1}) - \lambda_k h_k - \tau_k h_{k-1}, \quad (k = 1,\dots,m),$$

hence coefficients of f_k are of the form

$$f_k^{(q)} = [(q,\kappa) - \lambda_k] h_k^{(q)} + \sum_{i=2}^{n} (1+q_i)\sigma_i h_k^{(q-e_{i-1}+e_i)} - \tau_k h_{k-1}^{(q)},$$

where $e_i = (0,\dots,1,\dots0)$ is the i^{th} unit vector. Since we use the canonical order, this implies that Z is a lower triangular matrix with numbers $\Lambda_j = (q,\kappa) - \lambda_k$ $(k = 1,\dots,m;$ $|q| = \ell)$ at the diagonal. ×

<u>Corollary</u>. Consider the partial differential equation:

$$\frac{\partial V}{\partial x} Ax = U(x) \tag{1.5}$$

where $x \in \mathbb{C}^n$, $A \in \mathfrak{M}^{(n\times n)}$, $U(x)$ is a scalar quadratic form. An operator in the left-hand side of (1.5) is $L(2,n,n)$ with $B = 0$. By Lemma 1.1 its eigenvalues are $\kappa_k + \kappa_j$ $(k, j = 1,\dots,n)$.

Therefore, if

$$\kappa_k + \kappa_j \neq 0, \quad (k,j = 1,\ldots,n) \tag{1.6}$$

then the equation (1.5) has unique solution $V(x)$ which is a quadratic form.

§2. Normal Form

Consider a vector power series $X = \sum_{\ell=2}^{\infty} X^{(\ell)}$, $X^{(\ell)}$ being v.h.p (1.1). If there exists a neighborhood of the origin where all coordinate series are (absolutely) convergent then we say that series $X(x)$ is convergent. If there is no assertion of the convergence of a series then we say that $X(x)$ is a formal series (calculus of formal power-series is described in [14], Ch. I, §1).

Consider two formal systems of ordinary differential equations

$$\dot{x} = Ax + X(x) \tag{2.1}$$

and

$$\dot{y} = Ay + Y(y), \tag{2.2}$$

where X, Y are formal power-series.

Definition 2.1. We say that systems (2.1) and (2.2) are formally equivalent if there exists a change of variables:

$$x = y + h(y) \qquad (2.3)$$

where $h(y)$ is a formal power series, which reduces (2.1) to (2.2).

Definition 2.2. If X, Y, h in Definition 2.1 are convergent series, then we say that systems (2.1) and (2.2) are analytically equivalent.

Let $\kappa = (\kappa_1, \ldots, \kappa_n)$ be the vector whose co-coordinates are eigenvalues of matrix A.

Theorem 2.1. If

$$(q, \kappa) - \kappa_k \neq 0 \quad (k = 1, \ldots, n; \ |q| \geq 2), \qquad (2.4)$$

then system (2.1) is formally equivalent to any system (2.2) and $h(y)$ in (2.3) is uniquely determined.

Proof. Differentiating (2.3) with respect to t and taking into account (2.1) and (2.2) we obtain

$$\frac{\partial h}{\partial y} Ay - Ah = X(y+h) - \frac{\partial h}{\partial y} Y - Y. \qquad (2.5)$$

Represent h as $\sum_{\ell=2}^{\infty} h^{(\ell)}$, where $h^{(\ell)}$ is a v.h.p. of order ℓ. Then we have from (2.5):

$$\frac{\partial h^{(\ell)}}{\partial y} Ay - Ah^{(\ell)} = f^{(\ell)}(h^{(i)}, Y^{(j)}, X^{(k)}) - Y^{(\ell)} \qquad (\ell = 2,3,\ldots) \tag{2.6}$$

where $i < \ell$, $j < \ell$, $k \leq \ell$. Hence for $\ell = 2$ the right-hand part of (2.6) is a known v.h.p. and in general, if $h^{(2)},\ldots,h^{(\ell-1)}$ are known, then the right-hand part of (2.6) is known too. By Lemma 1.1 the eigenvalues of the operator of the left-hand part are $(q,\kappa) - \kappa_k$, $(k = 1,\ldots,h;\ |q| = \ell)$ and they are not zero by (2.4). Hence all $h^{(\ell)}$ are uniquely determined. ×

Corollary. If (2.4) holds then the system (2.1) is formally equivalent to its linear approximation

$$\dot{y} = Ay. \tag{2.7}$$

Note that condition (2.4) is not necessary for (2.1) and (2.7) to be formally equivalent.

Suppose now that (2.4) does not hold. Then some equations of (2.6) can be unsolvable. This means that system (2.1) is not formally equivalent to any system (2.2). However, since equations (2.6) can still be solvable, there exist systems (2.2) which are formally equivalent to (2.1).

We seek the simplest form of such a system. It is convenient to assume that $A = J$ is a Jordan canonical matrix. This can be achieved by means of linear non-singular change of variables. So we consider a system

$$\dot{x} = Jx + X(x). \tag{2.8}$$

Equation (2.6) reduces to

$$\frac{dh^{(\ell)}}{dy} Jy - Jh^{(\ell)} = f^{(\ell)}(h^{(i)}, Y^{(j)}, X^{(k)}) - Y^{(\ell)} \quad (i < \ell,\ j < \ell,\ k \leq \ell). \tag{2.9}$$

Using the canonical order of the coefficients of $h^{(\ell)}$ we obtain the matrix of operator in the left-hand side of (2.9) in lower triangular form with numbers $(q,\kappa) - \kappa_k$ $(k = 1,\ldots,n;\ |q| = \kappa)$ at the diagonal (see the proof of Lemma 1.1). Hence the coefficient $h_k^{(q)}$ of v.h.p. $h^{(\ell)}$ is determined by the equation

$$[(q,\kappa) - \kappa_k]h_k^{(q)} = g_k^{(q)} - Y_k^{(q)}, \tag{2.10}$$

where $g_k^{(q)}$ can be considered as a known number because it depends on coefficients of $h^{(i)}$, where $i < \ell$, and on the preceding coefficients in a canonical order of $h^{(\ell)}$. If