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Introduction

The present book deals with the axiomatic potential theory of local character and its corresponding continuous standard processes. This subject has already been treated by several authors.

P.A. Meyer [26] constructed a Hunt process on a Brelot space such that the class of all excessive functions and the class of all positive hyperharmonic functions should coincide. This result was generalized and completed by N.Boboc, C.Constantinescu and A.Cornea [8], W.Hansen [19], I.Cuculescu [16], H.Bauer [2], C.Constantinescu and A.Cornea [13]. The converse problem, the construction of an axiomatic potential theoretic structure associated to a given continuous Markov process, was first approached by Ph. Courrège and P.Priouret [14]. Then J.C. Taylor [40] and J.Bliedtner and W.Hansen [5] proved that each continuous standard process, whose potential kernel is strong Feller, yields a harmonic space, in the meaning of C.Constantinescu and A.Cornea [13].

The classical examples of the axiomatic potential theory of local type and of continuous Markov processes are associated to second order elliptic or hypoelliptic differential operators. On a locally compact space a fairly good substitute for the differential operators are the local operators. The notion of a local operator was introduced by E.B. Dynkin [17] p.145. He associated a local operator to each continuous standard process. The relation between this notion and the kernels from the axiomatic potential theory was pointed out by G.Mokobodzki and D.Sibony [31] Th. 21. Further boundary value problems associated to local operators were considered by G.Lumer [23], [24] and J.P. Roth [34].

The main axiomatic potential theoretic object studied here is a local operator, L, on a locally compact space, X, with a countable base. Similar to elliptic differential operators our local operator is assumed to obey a maximum principle and to have a base of open sets that are regular for the Poisson-Dirichlet problem. We note that G.Lumer (in [23]) was the first to consider a similar framework, but the spirit of the present approach differs much from his.

In Chapter I we construct a continuous standard process with state space X such that its characteristic operator extends L and its transition function is unique. Then we characterize the excessive functions of the process by means of the Dirichlet problem for L.

The sheaf of all excessive functions and the hitting distributions (or harmonic measures) may be viewed as the invariants under the random time change transformations. In Chapter II we suggest an axiomatic approach for these objects, taking as axioms some of the properties proved in the first chapter. First we construct the potential kernel associated to a potential (function) by computing it in terms of the given harmonic measures. In Section 6. of Chapter II we construct an open covering $\{U_i | i \in I\}$ and for each i a continuous strict potential, p_i, on U_i such that $p_i - p_j$ is harmonic on $U_i \cap U_j$ for $i \neq j$. This is somewhat analogous with the problem of constructing the random time change for two processes with identical hitting distributions. By analogy an inequality similar to the one proved by R.M. Blumenthal and R.K. Getoor [6] turns out to be important. This inequality is proved in Section 3. From it we deduce some remarkable properties of the potential kernel associated to a continuous strict potential. Finally a local operator possessing the properties considered in Chapter I is associated to the family $\{U_i , p_i\}$.

In Chapter III we study the topological properties of the transition function of the process constructed in Chapter I. Using the results of Chapter II we show that the transition function maps the cone of all lower semicontinuous functions into itself and the range of the resolvent has a "local density" property. If X is compact and if there is a function $h>0$ such that $Lh=0$, then the transition function maps the space of all continuous functions into itself.

The study of a product space is a classical theme in potential theory. While the early papers (K. Gowrisankaran [18], R. Cairoli [12]) study functions on the product space which are related to the structures of the terms of the product, in Chapter IV of the present work we follow the idea of the probabilistic work of R. Cairoli [11], constructing a structure on the product space and studying this structure. Namely we construct local operators on product spaces. This subject is a particular aspect of the general problem of constructing the notion of product in potential theory (a problem suggested by N.Boboc).

In Chapter IV we first consider two local operators L^1, L^2 on locally compact spaces X_1 , X_2 which possess bases of regular sets. Then we construct the sum L^1+L^2 on $X_1 \times X_2$ and prove that the product of two regular sets is regular (for L^1+L^2). Then we prove a similar result for the sum of a series of local operators on the product of a sequence of compact spaces. Further we consider a local operator, L, and construct the operator L-d/dt. (A similar construction within a different framework was made by J.P. Roth [34]). Then we are interested in those local operators which yield Bauer spaces and the operator L-d/dt allows us to characterize those operators with the property that, by addition they also yield Bauer spaces on product spaces. (The problem was also treated by E.Popa [33] and U. Schirmeier [36] in the frame of harmonic spaces in the sense of C.Constantinescu and A.Cornea. The key technical result is Lemma 5.5). Finally it is shown that the sum of a series of local operators preserves these properties under suitable conditions. This result extends a (more precise) result of C.Berg to compact spaces. (He constructed a Brelot space on the infinite dimensional torus [3]).

In Section 1. of Chapter V we consider the case when the state space, X, is a locally compact abelian group. We show that for a given translation invariant structure of the type considered in Chapter II, there exists a unique translation invariant local operator associated to it. In Section 2.we show that local operators can be used on a harmonic space in the sense of Constantinescu and Cornea (althogh there is no base of regular sets) and all the results from the previous sections rest valid in a natural analogous form.

Chapter VI is devoted to Feller resolvents. In Section 1. we present an improvement of a wellknown result on convex cones of lower semicontinuous functions. In Section 2. we give a very general construction of Hunt processes. Section 3. contains an excessiveness criterion. Section 4.presents a characterisation of those Feller resolvents which yield continuous Hunt processes (Corollary 4.12).

By analogy with the study made in Section 5.of Chapter IV, the final note gives a characterisation for the semigroups of compact contractions in Hilbert spaces.

Most of the material in this book was previously presented at the Potential Theory Seminar in Bucharest.

I would like to express my thanks to professors N.Boboc, Gh. Bucur, A.Cornea and I.Cuculescu from whom I learned potential theory and Markov Processes.

The expert typing was done by Camelia Minculescu, to whom I want to express my gratitude.

C O N T E N T S

NOTATION

For a locally compact space with a countable base, T, we shall denote by $C(T)$ the space of all real continuous functions and by $C_o(T)$, $C_c(T)$, $C_b(T)$ the subspaces of functions vannishing to infinity, of functions of compact support, of bounded functions. The space of all real Borel functions on T will be denoted by $\mathcal{B}(T)$ and the subspace of bounded Borel functions by $\mathcal{B}_b(T)$.

A kernel on T will be a positive linear operator V from $\mathcal{B}_b(T)$ into $\mathcal{B}(T)$ such that for each $x \in T$ the map $f \longrightarrow Vf(x)$ defines a Radon measure. The measure associated to x is denoted by V^x, i.e. $V^x(f)=$ $=Vf(x)$.

All terminology and notation on Markov processes will be that of [6]. Particularly if $(\Omega, M, M_t, X_t, \theta_t, P^x)$ is a standard process with state space $(E,\mathcal{E})$, f is a nearly Borel positive function and A is a nearly Borel set we use the notation $T_A=\inf\{t>0 / X_t \in A\}$, $P_A^\lambda f(x)=$ $=E^x[\exp(-\lambda T_A).f(X_{T_A}), T_A<\infty]$ and $P_A=P_A^0$.

We say that a standard process is continuous if $t \longrightarrow X_t$ is a.s. continuous on $[0,\zeta)$.

For the terminology and notation from the theory of harmonic spaces which is not specifically explained here we refer to [13].

I. LOCAL OPERATORS

1. General Properties

1.1. A sheaf of vector spaces of real continuous functions on a locally compact space, X, is a family $\{\mathcal{A}(U)/U$ open set$\}$ such that:

1^o For each open set U, $\mathcal{A}(U)$ is a vector space of real continuous functions on U;

2^o If $U_1 \subset U_2$ are open sets and $f \in \mathcal{A}(U_2)$ then $f_{|U_1} \in \mathcal{A}(U_1)$;

3^o If $\{U_i / i \in I\}$ is a family of open sets, $U=\bigcup_{i\in I} U_i$ and $f \in C(U)$ satisfies $f_{|U_i} \in \mathcal{A}(U_i)$, then $f \in \mathcal{A}(U)$.

1.2. A local operator, L, on a locally compact space, X, is a pair $(\{D(U,L)/U$ open set$\}$, $\{(L,U)/U$ open set$\})$, where $\{D(U,L)/U$ open set$\}$ is a sheaf of vector spaces of real continuous functions on X and $\{(L,U)/U$ open set$\}$ is a family of linear operators such that:

1^o $(L,U) : D(U,L) \longrightarrow C(U)$ is a linear operator.

2^o If U, V are open sets, $U \subset V$ and $f \in D(V,L)$ then $(L,U)(f_{|U}) = ((L,V)f)_{|U}$ (i.e. L is a sheaf morphism from $\{D(U,L)/U$ open set$\}$ into the sheaf of all continuous functions).

We shall use the notation $(L,U)f = Lf$ for any open set U and any $f \in D(U,L)$ (just like in the case of differential operators in R^n).

1.3. <u>In this section</u> we shall consider a locally compact space with a countable base, X, and a local operator, L, on X. For each $\lambda > 0$ we denote by L_λ the operator defined as follows: $D(U,L_\lambda) = D(U,L)$ and $L_\lambda f = Lf - \lambda f$ for any any open set, U, and any $f \in D(U,L)$.

1.4. Suppose that U is a relatively compact open set such that $\partial U \neq \Phi$.

U will be called Dirichlet regular (or D-regular) if:

1^o $(\forall)$ $f \in C(\partial U)$, $(\exists) u \in C(U)$ unique such that $u_{|\partial U} = f$, $u_{|U} \in D(U,L)$, and $Lu=0$ on U,

2^o if $f \geqslant 0$, then the associated function, u, satisfies $u \geqslant 0$.

If U is D-regular and $f \in C(\overline{U})$ we shall denote by $H^U f = u$, the function associated to $f_{|\partial U}$, via 1^o in the above definition. H^U may be regarded as a linear operator, $H^U : C(\overline{U}) \longrightarrow C(\overline{U})$, which extends to a kernel on $\overline{U}$. If U is D-regular with respect to L_λ, then we shall use the notation H^U_λ for the analogous object.

U will be called Poisson regular (or P-regular) if:

1^o $(\forall)$ $f \in C(\overline{U})$, $(\exists)$ $u \in C_o(\overline{U}) \cap D(U,L)$ unique such, that $Lu=-f$,

2^o if $f \geqslant 0$, then the associated function, u, satisfies $u \geqslant 0$,

3^o the space $D_o(U)$ is dense in $C_o(U)$, where

(1) $$D_o(U) = f \in \{D(U,L) \cap C_o(U) / Lf \in C_o(U)\}.$$

If U is both P-regular and D-regular we shall call it P-and D-regular.

If U is P-regular and $f \in C(\overline{U})$ we shall denote by $G^U f = u$, the function associated to f via 1^o in this definition. This way we get a positive linear operator G^U: $C(\overline{U}) \longrightarrow C(\overline{U})$, which extends to a kernel on $\overline{U}$. Condition 3^o shows that for any $x \in U$ the measure $G^{U,x}$ is nonnul, and hence $G^U 1 > 0$ on U. If U is P-regular with respect to L_λ, then G^U_λ will denote the analogous object (of course $G^U_o = G^U$). If U is P-regular with respect to L_λ for any $\lambda > 0$ and $f \in C(U)$, then

$$L(G^U_\alpha f - G^U_\beta f) - \alpha(G^U_\alpha f - G^U_\beta f) = (\alpha - \beta) G^U_\beta f \quad ,$$

which leads to $G^U_\alpha - G^U_\beta = (\beta - \alpha) G^U_\alpha G^U_\beta$, $\alpha, \beta > 0$, i.e. the rezolvent equation.

1.5. The operator L will be called locally closed (24(II) p.207) if:

$(\forall)$ U open set, $(\forall)$ $\{f_n / n \in N\} \subset D(U,L)$, $f_n \longrightarrow f$, $Lf_n \longrightarrow g$

uniformly on each compact set

$$\Longrightarrow f \in D(U,L) \text{ and } Lf=g.$$

We remark that L is locally closed provided there exists a base of open sets which are P- and D-regular. In order to see this we consider an open set, U, and a P- and D-regular set, V, such that $\overline{V} \subset U$; then we deduce

$$\varphi = H^V_\varphi + G^V(-L\varphi) \quad \text{on } V, \qquad (\forall)\ \varphi \in D(U,L). \tag{2}$$

Writting this formula for the sequence $\{f_n\}$ and letting $n \longrightarrow \infty$ we get $f = H^V f + G^V(-g)$, which shows $f \in D(V,L)$ and $Lf=g$ on V.

1.6. The operator L will be called locally dissipative if it obeys the following maximum principle:

$(\forall)$ U open set, $(\forall)$ $f \in D(U,L)$, $(\forall)$ $x \in U$,

$f(x) \geqslant f$ on U, $f(x) \geqslant 0 \Longrightarrow Lf(x) \leqslant 0$.

<u>From now on we suppose that L is locally dissipative</u>. Then L_λ, $\lambda > 0$ are locally dissipative. First we are going to state a very useful form of the minimum principle. Versions of it were proved in several places (see [17 (I)] p.145 and [24](II) p.210).

1.7. <u>Proposition</u>

Suppose that U is an open set and $\varphi \in D(U,L)$ satisfies $|\varphi| \leqslant 1$ and $L\varphi < 0$. If $f \in D(U,L)$, $Lf \geqslant 0$ and $\limsup_{x \to \infty(u)} f(x) \leqslant a$, $a \in R_+$, where $\infty(U)$ is the Alexandrov point associated to the locally compact space U, then $f \leqslant a$ on U.

<u>Proof</u>

Let us suppose that $f(x) \geqslant a+\alpha$, $\alpha > 0$ for some $x \in U$. Then $f-(\alpha/2)\varphi = g$ verifies

$$g(x) \geqslant a+\alpha/2 \quad \text{and} \quad \limsup_{y \to \infty(U)} g(y) \leqslant a+\alpha/2.$$

There exists a maximum point, $y \in U$, such that $g(y) \geqslant g$ on U. Then $g(y) \geqslant g(x) \geqslant 0$, and hence $Lg(y) \leqslant 0$. On the other hand

$$Lg(y) = Lf(y) - (\alpha/2)L\varphi(y) > 0,$$

which is a contradiction. Our supposition failed, and hence $f(x) \leqslant a$ for any $x \in U$.

Now we are going to introduce the "local closure" of L. First we need a version of a result from [34] p.55.

1.8. Proposition

Let V be an open set, $\{f_n/n\in N\}\subset D(V,L)$ a sequence such that $f_n \longrightarrow f$, $Lf_n \longrightarrow \psi$ uniformly on each compact set, and f has nonnegative local maximum in $x_o\in V$. Assume that for any neighbourhood, W, of x_o there exist an open set, U, such that $x_o\in U$, $\overline{U}\subset W$ and $g\in C_o(U)\cap D(U,L)$ such that $g(x_o)>0$, $Lg\in C_b(U)$. Then $\psi(x_o)\leqslant 0$.

Proof

Let us suppose that $\psi(x_o)>0$. We choose an open set, U, such that $\overline{U}\subset V$, $x_o\in U$, $\psi>\alpha$ on U, $\alpha\in R$, $\alpha>0$, $f(x_o)\geqslant f$ on U and $g\in C_o(U)$ $\cap D(U,L)$ such that $g(x_o)=\beta>0$, $|Lg|\leqslant\alpha/2$. Further we choose $n\in N$ such that $|f_n-f|<\beta/2$ on $\overline{U}$ and $|Lf_n-\psi|<\alpha/2$ on $\overline{U}$. Then we have

$$f_n(x_o)+g(x_o)>f(x_o)+\beta/2\ ,$$

$$f_n(y)+g(y)=f_n(y)<f(y)+\beta/2\leqslant f(x_o)+\beta/2\ ,\quad (\forall)\ y\in\partial U$$

and

$$Lf_n+Lg=Lf_n-\psi+\psi+Lg>-\alpha/2+\alpha-\alpha/2=0\ .$$

This contradicts 1.7, and hence $\psi(x_o)\leqslant 0$.

1.9. Corollary

Let us assume that the following condition holds:
$(\forall)$ $x\in X$, $(\forall)$ V a neighbourhood of x, $(\exists)$ U open set, $x\in U$, $\overline{U}\subset V$, $(\exists)$ $g\in C_o(U)\cap D(U,L)$ such that $g(x)>0$ and $Lg\in C_b(U)$.

Then for each open set, V, and each sequence, $\{f_n\}\subset D(V,L)$, such that $f_n\longrightarrow 0$ and $Lf_n\longrightarrow\psi$ uniformly on the compact subsets of V it holds $\psi\equiv 0$.

If the requirement from the above corollary is fulfilled, we may define $\tilde{L}$, the local closure of L, as follows:

If U is an open set, a function $f\in C(U)$ belongs to $D(U,\tilde{L})$ if and only if there exist a function $\psi\in C(\overline{U})$, an open covering of U, $\{U_i/i\in I\}$, and for any $i\in I$ there exists a sequence $\{\varphi_n^i/n\in N\}\subset D(U_i,L)$ such that $\varphi_n^i\longrightarrow f$ and $L\varphi_n^i\longrightarrow\psi$ $(n\longrightarrow\infty)$ uniformly on the compact subsets of U_i. Furthermore we put $\tilde{L}f=\psi$.

We note that from 1.8 one deduces $\tilde{L}$ is also locally dissipative.

The next proposition is a criterion of P-regularity and also shows that the kernel G^U is supported by U.

1.10. Proposition

Suppose that U is an open set such that $\partial U\neq\emptyset$ and for any

$f \in C(\overline{U})$ there exists a function $u \in C_o(U) \cap D(U,L)$ which fulfils $Lu=-f$ and the space $C_o(U) \cap D(U,L)$ is dense in $C_o(U)$. Then U is P-regular and $G^U(\partial U)=0$.

Proof

Let $u \in C_o(U) \cap D(U,L)$ be such that $Lu=-f$, $f \in C(U)$, $f \geqslant 0$. Proposition 1.7 implies $u \leqslant 0$. Thus 2^o and the unicity assertion of 1^o within the definition of P-regularity are fulfilled. Further the operator G^U exists and may be extended to a kernel on U. Next we are going to prove $G(\partial U)=0$. Let $\{\psi_n\} \subset C(U)$ be a sequence such that $0 \leqslant \psi_{n+1} \leqslant \psi_n \leqslant 1$, $\overline{\{\psi_n < 1\}} \subset U$, and $\bigcup_n \overbrace{\{\psi_n=0\}}^{o} = U$. From 1.7 we get

$$\| G\psi_n \| \leqslant \sup \{ G\psi_n(x)/\psi_n(x) > 0\} \leqslant \sup \{ G1(x)/\psi_n(x) > 0\} \ .$$

But $G1 \in C_o(U)$ and so $G(\partial U) = \lim_{n \to \infty} G(\psi_n) = 0$.

Now let $u \in C_o(U) \cap D(U,L)$, $Lu \in C(\overline{U})$, $Lu \leqslant 0$. For $x \in U$ we have $u(x)=G(-Lu)(x)= \lim_{n \to \infty} G((-Lu)(1-\psi_n))(x)$. The limit being increasing it is uniforme. This leads to condition 3^o from the definition of P-regularity.

1.11. The next theorem due to G.A. Hunt will be used several times in our paper. For a proof we refer to [30] p.223-224 or [27] X T 10. An extend study of this subject can be found in [22].

Theorem

Let $V : C_b(X) \longrightarrow C_b(X)$ be a positive linear operator satisfying the complete maximum principle, i.e.:

if $f,g \in C_{b+}(X)$, $Vf(x) \leqslant Vg(x)+1$, $(\forall)\ x \in \{f>0\}$,

then $Vf \leqslant Vg+1$.

Then there exists a unique family $\{V_\lambda / \lambda \geqslant 0\}$ of positive linear operators on $C_b(X)$ such that

1^o $V_\alpha - V_\beta = (\beta-\alpha) V_\alpha V_\beta$, $\alpha,\beta \geq 0$,

2^o $\lambda V_\lambda 1 \leq 1$, $\lambda > 0$,

3^o $V_o = V$.

1.12. Now we are going to depict several relations between the various kinds of regularity.

1^o If an open set, U, is P-regular, then it is P-regular with respect to L_λ for any $\lambda>0$ and the resolvent $\{G_\lambda^U / \lambda>0\}$ is sub-Markov: $\lambda G_\lambda^U 1 \leq 1$.

This is a consequence of the above theorem applied to the operator G^U. The complete maximum principle for G^U results from 1.7.

2^o If U is P-regular with respect to L_α , for some $\alpha>0$, then U is P-regular with respect to L_λ for any $\lambda>0$ and the resolvent $\{G_\lambda^U/\lambda>0\}$ is sub-Markov.

This results from Theorem III.3.1 of F.Hirsch [22].

3^o If U is P-regular with respect to L_λ for any $\lambda>0$ then U is P-regular (with respect to L) provided there exists $f\in C_b(U)\cap D(U,L)$ such that $Lf\in C_b(U)$ and $Lf\leq -1$.

In order to prove this we choose $\alpha>0$ such that $\alpha||f||<1/2$ and put $\varphi=2G_\alpha^U(-L_\alpha f)$. Then

$$L\varphi=2(Lf-\alpha(f+G_\alpha^U L_\alpha f)) \quad .$$

Using 1.7 we get $|f+G_\alpha^U L_\alpha f|\leq ||f||$, and hence $L\varphi\leq -1$, which leads to $G_\lambda^U 1\leq\varphi$ for any $\lambda>0$. Now using the kernel $G^U=\lim_{\lambda\to 0} G_\lambda^U$ it is easy to deduce that U is P-regular.

4^o Let U be D-regular and suppose there exists $\lambda>0$ such that U is also P-regular with respect to L_λ. Then U is D-regular with respect to L_λ and

(3) $$H_\lambda^U=H^U-\lambda G_\lambda^U H^U.$$

5^o Let U be an open set. Assume that L is locally closed and there exists a P-regular set V such that $\overline{U}\subset V$. If U is P-regular, then it is also D-regular and for any $f\in C_b(V)$,

(4) $$G^V f-G^U f=H^U G^V f \qquad \text{on } U.$$

Condition 2^o and the unicity assertion from 1^o within the definition of D-regularity are consequences of 1.7. In order to prove the existence assertion we firstly consider the case when $f\in C(\partial U)$ is of the form $f=G^V g_{|\partial U}$ for suitable $f\in C_b(V)$. Then $u=G_g^V-G_g^U$ fulfils $u_{|\partial U}=f$ and $Lu=0$ on U. For a general $f\in C(\partial U)$ one makes an approximation.

6^o Let U be an open set and V a P-regular set such that $\overline{U}\subset V$. If U is D-regular then it is also P-regular. The proof of this assertion is similar with the preceding one.

2. The Markov Process Associated to a Local Operator

Let X be a locally compact space with a countable base. In this section we study a local operator on X, L, which is locally dissipative and suppose that the family of all P- and D-regular sets forms a topological base. Then from 1.12, 1° we deduce similar properties with respect to the operators L_{λ} , $\lambda>0$.

2.1. Proposition

a) Let V be a P-regular set. There exists a continuous Hunt process $(\Omega, M, M_t, X_t, \theta_t, P^x)$ with state space V such that for each $\varphi \in C_o(V)$ and $t>0$ the function $\psi(x)=E^x[\varphi(X_t)]$ satisfies $\psi \in C_o(V)$ and

(1) $$G_{\lambda}^{V} f(x)=E^x[\int_0^{\infty} \exp(-\lambda t) f(X_t)\,dt], \quad (\forall)\ x \in V, \quad \lambda \geq 0,$$

$$f \in C_b(V).$$

b) If U is a D-regular set, $\overline{U} \subset V$, then each point $x_o \in \partial U$ is regular, i.e. $E^{x_o}[T_{V \setminus U}>0]=0$, and the following equalities hold:

(2) $$H_{\lambda}^{U} f(x)=P_{V\setminus U}^{\lambda} f(x), \quad (\forall)\ x \in \overline{U}, \quad \lambda \geq 0, \quad f \in C(V),$$

(3) $$G_{\lambda}^{U} f(x)=E^x[\int_0^{T_{V\setminus U}} \exp(-\lambda t) f(X_t)\,dt], \quad (\forall)\ x \in U,\ \lambda \geq 0,\ f \in C_b(U).$$

In order to prove this proposition we need the next three lemmas:

2.2. Lemma. Let $(\Omega, M, M_t, X_t, \theta_t, P^x)$ be a standard process with state space E. Assume that there exists a sequence $\{B_n\}$ of nearly Borel sets such that $\bigcup_n B_n=E$ and $R(B_n)(x)=E^x[\int_o^{\infty} \chi_{B_n}(X_t)\,dt]$, $(x \in E)$ is a bounded function for each $n \in N$. Suppose that A is a nearly Borel set and H is a kernel on E such that

1° $H(E\setminus A)=0$,

2° $Hf \leq f$ for any excessive function f,

3° there exists a family $\mathcal{A}$ of excessive functions such that:
(a) any two measures on E, μ and υ, coincide provided $\mu(f)=\upsilon(f)$ for any $f \in \mathcal{A}$; (b) Hf is excessive for any $f \in \mathcal{A}$ and Hf=f on A.

Then $P_A=H$ and all points in A are regular.

Proof. Let $f \in \mathcal{A}$, and g be an excessive function such that $f \leq g$ on A; them from 1° and 2° we get $Hf \leq Hg \leq g$ and on account of 3°(b) deduce

$$Hf=\inf\{g/\text{excessive},\ f \leq g \text{ on } A\}.$$

On the other hand Hunt's balayage theorem ([6] p.141) gives us $P_A f \leq Hf$ and $P_A f(x)=Hf(x)$ except possibly for those points, x, in A which are not regular, i.e. except for a semipolar set ([6] p.80). But $P_A f$ is also excessive ([6] p.73), hence $Hf=P_A f$. Now condition 3°(a) implies

$H=P_A$.

If $x\in A$ then conditions 3^o(a), (b) show that $H^x=\varepsilon_x$. Thus $R(B_n)(x)=E^x[R(B_n)(X_{T_A})]$ or $E^x[\int_o^\infty \chi_{B_n}(X_t)dt]=E^x[\int_{T_A}^\infty \chi_{B_n}(X_t)dt]$ for any $n\in N$. We deduce $E^x[\int_o^{T_A}\chi_{B_n}(X_t)dt]=0$ for any $n\in N$, and hence $E^x[T_A>0]=0$.

2.3. Lemma. Let $g\in C_{b+}(V)$ and put

$$h(x)=\begin{cases} g(x) & \text{if } x\in V\setminus U \\ H^U g(x) & \text{if } x\in U \end{cases}$$

Then k is excessive for the resolvent $\{G_\lambda^V \mid \lambda\geq 0\}$, i.e. $\lambda G_\lambda^V h\to h$, as $\lambda\to\infty$, and $\lambda G_\lambda^V h\leq h$, for $\lambda>0$.

Proof. Since V is P-regular we know that $\overline{G_\lambda^V(C_o(V))=C_o(V)}$ for for each $\lambda\geq 0$. Therefore $\lambda G_\lambda^V f\to f$, as $\lambda\to\infty$, for each $f\in C_o(V)$. Since $h\in C_o(V)$ we have only to prove the inequality $\lambda G_\lambda^V h\leq h$. From 1.7 we get $h\leq g$ and $\lambda G_\lambda^V h\leq\lambda G_\lambda^V g\leq g=h$ on $V\setminus U$. On the other hand $L_\lambda(\lambda G_\lambda^V h-h)=0$ on U. Again 1.7 gives us $\lambda G_\lambda^V h\leq h$ on U.

The next Lemma was proved by Ph.Courrège and P.Priouret in Annexe 1 of [14].

2.4. Lemma. Let $(\Omega,M, M_t,X_t,\theta_t,P^x)$ be a standard process with state space E. If there exists a base of open sets, $\mathcal{U}$, such that

$$P_{CU}(E\setminus\overline{U})(x)=0, \quad \text{for each } U\in\mathcal{U} \text{ and each } x\in U,$$

then the process is continuous.

Proof of Proposition2.1. a) The resolvent $\{G_\lambda^V \mid \lambda>0\}$ satisfies the conditions from the Hille-Yosida theorem, on the Banach space $C_o(V)$. Thus we get a (C_o)-class semigroup of positive sub-Markov operators on $C_o(V)$. Further we apply the theorem from [6] p.46 and get a standard process $(\Omega,M,M_t,X_t,\theta_t,P^x)$ with state space V which fulfils relation (1). The continuity of the process results from Lemma 2.4 by using relation (2), which will be proved below.

b) In order to prove (2) we are going to apply Lemma 2.2 with respect to the kernel H^U (extended to V by taking $H^{U,x}=\varepsilon x$ for $x\in V\setminus U$), the set $A=V\setminus U$ and the family $\mathcal{A}=G^V(C_{b+}(V))$.

Conditions 1^{o} and 3^{o} (a) from Lemma 2.2 are obviously fulfilled. From 1.7 we get $H^{U}G^{V}g \leqslant G^{V}g$ for each $g \in \mathcal{C}_{b+}(V)$. Then the monotone class theorem shows that this inequality is still valid for each $f \in \mathcal{B}_{b+}(V)$. Further we get $H^{U}f \leqslant f$ for each excessive function, by approximating f with potentials. This cheks condition 2^{o} from 2.2. Condition 3^{o}(b) 2.2 results from Lemma 2.3. Thus relation (2) follows from Lemma 2.2.

Now let $f \in \mathcal{C}_{b}(V)$. The strong Markov property gives us

$$H^{U}G^{V}f(x)=E^{x}[E^{X_{T_{V\setminus U}}}[\int_{0}^{\infty} f(X_{t})\,dt]]=E^{x}[\int_{T_{V\setminus U}}^{\infty} f(X_{t})\,dt].$$

This relation together with 1(4) leads to (3).

2.5. <u>Theorem</u>. There exists a continuous standard process $(\Omega, \mathcal{M}, \mathcal{M}_{t}, X_{t}, \theta_{t}, P^{x})$ with state space X such that for any P-regular set, U,

$$(4)\qquad G_{\lambda}^{U}f(x)=E^{x}[\int_{0}^{T_{E\setminus U}} \exp(-\lambda t) f(X_{t})\,dt], \qquad (\forall)\ x \in U,$$

$$(\forall)\ f \in \mathcal{C}_{b}(U), \quad (\forall)\ \lambda \geq 0.$$

If another continuous standard process fulfils (4), then it has the same transition function.

This theorem is a consequence of the next theorem proved by Ph.Courrège and P.Priouret [15] 2.4.2. (See also P.A. Meyer [42] and M.Nagasawa [43]).