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Lutz Heindorf Leonid B. Shapiro

Nearly Projective Boolean Algebras

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With an Appendix
by Sakaé Fuchino

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To our parents

Preface

The history of this publication starts in April 1992, when the second author gave a talk at the Freie Universität Berlin.

As one result of this lecture part of the audience felt the desire to study some topological papers in more detail than before. A series of seminar talks was given by the first author and the present text grew from the accompanying notes.

The idea to turn those notes into a joint publication occured after the second author obtained some new results that would nicely complement the already gathered material. Work on this project was then begun separately in Moscow and Berlin with interchanges by mail.

The actual writing of the text was done by the first author who has to thank several people for their help. Sabine Koppelberg and Sakaé Fuchino read previous versions and gave helpful comments, which led to simplifications in some proofs. Many valuable remarks also came from Ingo Bandlow.

John Wilson improved the English and Ulrich Fuchs helped with TEX.

The final version was prepared jointly by both authors during March 1994 in Berlin. We want to thank the Deutsche Forschungsgemeinschaft for its financial support (Grant Number 436 RUS 17/192/93), which made the visit of the second author possible.

After the main text was ready, Sakaé Fuchino kindly wrote an appendix on set-theoretic methods in the field, which also includes some of his recent independence results.

Berlin, September 21, 1994 L. H. L.B.S.

Contents

Introduction

The first manifestation of interest in projective Boolean algebras seems to be the paper [22] by Halmos. He established the (now) familiar properties of projective objects in general and proved that all countable Boolean algebras are projective. This is in sharp contrast to what happens for other classes of algebras, where difficult questions arise already at the finite level.

The decisive tools for the study of projective Boolean algebras came from topology. As projective Boolean algebras can be embedded into free algebras, their dual spaces are dyadic. This class of spaces had been introduced by P. S. Alexandorff and studied by topologists for many years. Most results about them concerned cardinal functions in one form or the other. For a satisfactory structure theory the class of dyadic spaces turned out to be too wide, however. For the subclass of AE(0) spaces (its zero-dimensional members are exactly the dual spaces of projective Boolean algebras) such a theory started with the remarkable paper [23] by R. Haydon. He established that these spaces admit a special kind of inverse limit representation. By means of this description Haydon showed that the class AE(0) coincides with the class of so-called Dugundji spaces, which were earlier introduced by A. Pełczynski ([44]) via a functional analytic property. The name 'Dugundji space' has later become popular and is in the given context often used instead of AE(0).

Haydon's inverse limits $X = \underline{\lim}\{X_\alpha; p_\alpha^\beta; \alpha < \beta < \lambda\}$ have an ordinal as index set and are continuous, i.e. for limit ordinals $\gamma < \lambda$ the space X_γ is homeomorphic to the limit of the restricted system $\{X_\alpha; p_\alpha^\beta; \alpha < \beta < \gamma\}$. Such inverse systems are called 'transfinite spectra'. The most important feature of Haydon's spectra is, however, the specific nature of the bonding maps $p_\alpha^{\alpha+1}$.

Transfinite spectra were probably first used by L. S. Pontryagin under the name 'Lie series' in his analysis of the structure of compact groups. Before Haydon's paper transfinite spectra were used by S. Sirota to characterize the Cantor cube of weight $\aleph_1$. As a consequence of his characterization Sirota proved that the hyperspace (or exponential) of a dyadic space of weight at most $\aleph_1$ is dyadic again. This result, and the obvious question of what happens for bigger weights, had an essential influence on the further development of the theory.

In the mid seventies, investigating uncountable products of metrizable spaces, E. V. Ščepin introduced the class of κ-metrizable spaces. Using Haydon's characterization he proved the κ-metrizability of all Dugundji spaces. This led him to a spectral characterization of compact κ-metrizable spaces and to general questions about inverse limit representations.

Generalizing the concrete work of his predecessors Ščepin introduced the concept of a class of compact spaces and a class of continuous mappings being 'adequate'. Roughly speaking, spaces are classified according to whether they admit inverse limit representations in which the bonding maps are taken from a special class of mappings. This idea, in some sense, reduces the study of spaces to the study of mappings.

The class of Dugundji spaces is adequate to what Ščepin called 0-soft mappings. We shall be mainly concerned with the adequate pair that consists of κ-metrizable (otherwise known as open generated) spaces and open mappings. It was studied mainly by Ščepin, with important contributions coming from other Moscow topologists, L. V. Širokov and A. V. Ivanov to name just two. The latter established an important link between κ-metrizable and Dugundji spaces: a compact space is κ-metrizable iff its superextension is a Dugundji space. In fact, this theorem was an important step in the proof of the adequateness.

Another adequate pair, which will play a prominent role below, grew out of the theory of absolutes and co-absoluteness, which dates back to I. V. Ponomarev [45]. It turned out that the class of spaces co-absolute to Dugundji spaces is adequate for the class of mappings co-absolute to 0-soft ones. The starting point here was the second author's result that every dyadic space is co-absolute with (any compactification of) an at most countable sum of Cantor cubes of suitable weights.

In this work we shall use the language of Boolean algebras to present most of the results and concepts mentioned above. In other words, we confine ourselves to the zero-dimensional case. Due to that special case, many proofs become technically more transparent, which makes the ideas come out more clearly. Admittedly, some of the ideas that are important for higher-dimensional spaces get lost. Obviously, we do not touch the geometrical role of Dugundji spaces, more precisely their subclasses AE(n). For information about these aspects the interested reader may consult Dranišnikov's paper [10].

To assure the topologist reader that he is not wasting his time, it should be added that many interesting examples and counter-examples of the theory are zero-dimensional anyway.

We have tried to collect all the results that have been obtained over the years, mostly by Russian topologists. A word of caution is in place here. We often attribute Boolean algebraic results to topologists. This is to be understood in the wider sense. In most cases the corresponding topological result is more general being true for spaces of arbitrary dimension. In some cases, the corresponding topological result is only near in spirit to what we do.

The algebraic dual of an inverse limit decomposition of a space is the representation of an algebra as the union of a system of subalgebras. The basic idea of dealing with uncountable algebras by looking at suitable systems of well-embedded subalgebras has, independently of topological considerations, been developed by model theorists (cf. [38]) and set-theoretically oriented algebraists (cf. [11]).

One of the popular areas in this field is the study of almost free algebras in various classes. This is in spirit similar to what we do in the text, but will play no explicit role. The reader may consult [18] for further information and results concerning these problems for the class of Boolean algebras.

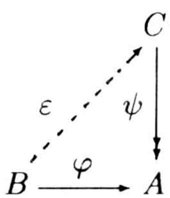
Having returned to the algebraic setting we have to mention the name of S. Koppelberg, who solved some of the problems from [22] and cultivated the technique of decomposing Boolean algebras into chains of subalgebras. Her survey [35] in the ***Handbook of Boolean Algebras*** made much of the material about projectivity available to people working in algebra and set-theory. The present work, which mainly deals with generalizations of projectivity, can be regarded as a continuation in the same direction.

We now briefly introduce and motivate the main concepts that will play a role in this work. We also try to summarize the contents in order to give the reader an idea of what he can expect. Some of the statements in this introduction may contain unexplained notions or be otherwise somewhat imprecise. Everything really needed will be vigorously repeated in the main text.

Overview

Projective and rc-filtered Boolean algebras

Projectivity for Boolean algebras is defined as in all other varieties by a diagram condition.



To be read:
A Boolean algebra B is projective iff for all homomorphisms $B \xrightarrow{\varphi} A \xleftarrow{\psi} C$, with ψ surjective, there exists a homomorphism $\varepsilon : B \to C$ such that $\psi \circ \varepsilon = \varphi$.

Putting $A = B$, $\varphi = id$ and letting C be free, we get that each projective Boolean algebra is a retract of a free Boolean algebra. It is easy to prove that this property characterizes projectivity.

It would be desirable to have a characterization of projectivity that refers only to the algebra itself. For other varieties of algebras such intrinsic characterizations have been found. For example, in [14] R. Freese and J.B. Nation characterize projective lattices by four conditions, one of which reads

(∗) *for each* $b \in B$ *there are two finite sets* $U(b) \subseteq \{c \in B : b \leq c\}$ *and* $L(b) \subseteq \{c \in B : c \leq b\}$ *such that, if* $a \leq b$*, then* $U(a) \cap L(b) \neq \emptyset$.

This condition makes sense for Boolean algebras, too, and for a while it was believed to characterize projective Boolean algebras.[1] It turns out that this is true for Boolean algebras of power at most $\aleph_1$ (cf. 2.2.7) and that all projective Boolean algebras have the property (cf. 2.2.6).

It is remarkable that the class of algebras satisfying (∗) arises in a totally different context. In [55] E.V. Ščepin introduced the class of so-called *openly generated*[2] compact spaces. These are spaces that can be represented as inverse limits of inverse systems $\{X_i, p^i_j, I\}$, where

(1) the partially ordered index set I is σ-complete, i.e. all countable chains $i_1 \leq i_2 \leq \ldots \leq i_n \leq \ldots$ have suprema in I,

(2) the system is continuous, i.e. if $j = \sup J$ exists for some $J \subseteq I$, then X_j is the inverse limit of the restricted system $\{X_j, p^i_j, J\}$,

(3) all X_i are compact and metrizable, and

(4) all bonding maps $p^i_j : X_i \to X_j$ are open.

To make the further explanations precise, we now give two definitions that will be fundamental for the whole work.

A *skeleton* of a Boolean algebra B is a collection $\mathcal{S}$ of subalgebras of B that is closed under unions of chains, i.e. $\bigcup \mathcal{K} \in \mathcal{S}$ whenever $\mathcal{K}$ is a subchain (under $\subseteq$) of $\mathcal{S}$, and absorbing in the sense that for each $X \subseteq B$ there exists some $S \in \mathcal{S}$ such that $X \subseteq S$ and $|S| \leq |X| + \aleph_0$.

A subalgebra A of a Boolean algebra B will be called *relatively complete* (symbolically $A \leq_{rc} B$) if for each $b \in B$ there exists a least element of A above b.

Noticing that relatively complete embeddings are dual to open mappings and that skeletons are (something like) duals of inverse systems, it should be no surprise that the Stone space of a Boolean algebra is openly generated iff the algebra itself has a skeleton consisting of relatively complete subalgebras. As an abreviation we use the expression 'rc-skeleton' and call the algebras that have such skeletons *rc-filtered*. It turns out that (cf. 2.2.3)

A Boolean algebra is rc-filtered iff it has the property (∗).

[1] The first author wants to thank M. Ploščica for drawing his attention to this problem.

[2] Also translated as *open generated*.

Comparison of the two classes

Having the property (∗), all projective Boolean algebras are rc-filtered. Moreover, the two classes coincide for Boolean algebras of cardinality at most $\aleph_1$.

There are several characterizations of both classes that demonstrate their similarity. In this introduction we just give two such pairs of characterizations. The diagram definition of projectivity can be modified in the following way (due to Širokov, cf. 2.4.3).

> *The Boolean algebra B is projective iff for all pairs of homomorphisms $B \xrightarrow{\varphi} A \xleftarrow{\psi} C$, with ψ surjective, there exists a mapping $\varepsilon : B \to C$ preserving* 0 *and* $\wedge$ *such that* $\psi \circ \varepsilon = \varphi$.

The diagram is the same as for projectivity, but ε need not be a homomorphism any more. The counterpart for rc-filtered Boolean algebras reads (cf. 3.2.7):

> *The Boolean algebra B is rc-filtered iff for all pairs of homomorphisms $B \xrightarrow{\varphi} A \xleftarrow{\psi} C$, with ψ surjective, there exists an order-preserving mapping $\varepsilon : B \to C$ that also preserves disjointness such that $\psi \circ \varepsilon = \varphi$.*

On the other hand, the property defining rc-filtered algebras, i.e. the existence of an rc-skeleton, also has a counterpart for projective algebras. It is due to Ščepin and says (cf. 1.3.2(4)):

> *The Boolean algebra B is projective iff it has a skeleton $\mathcal{S}$ such that for each subset $\mathcal{T} \subseteq \mathcal{S}$ the subalgebra generated by $\bigcup \mathcal{T}$ is relatively complete in B.*

We only mention one further connection between projective and rc-filtered Boolean algebras (due to A.V. Ivanov, cf. 3.2.6).

> *The Boolean algebra B is rc-filtered iff λB is projective,*

where λB is a Boolean algebra constructed from B in a way explained in section 3.2. Its topological dual is the so-called superextension of the Stone space of B.

Starting from cardinality $\aleph_2$ on, the two classes differ. Much of what follows will be devoted to the construction of rc-filtered Boolean algebras that are not projective. Most of them have additional properties which show that they are non-projective 'in a strong sense'. Let us list the most interesting of these. *There are rc-filtered Boolean algebras which are*

(1) *not embeddable into a free Boolean algebra* (cf. 3.3.11),

(2) *not projective but relatively complete subalgebras of free Boolean algebras* (cf. 3.4.7),

(3) *not co-complete to a projective Boolean algebra* (cf. 6.3.2).

Moreover, Fuchino proved (unpublished, cf. 6.4.2) that

there are $2^{\aleph_2}$ pairwise non-isomorphic rc-filtered Boolean algebras of power $\aleph_2$.

Let us mention that among them there are only $2^{\aleph_1}$ projective Boolean algebras (by a result of Koppelberg's [35] not reproduced here).

The class of rc-filtered Boolean algebras

In chapter 2 the class of rc-filtered Boolean algebras is studied in a rather systematic way. Let us just mention two results about the behaviour of rc-filtered Boolean algebras with respect to various operations (cf. 2.2.8 and 2.3.1).

Relatively complete subalgebras of rc-filtered Boolean algebras remain rc-filtered.
If B can be written as the union of a well-ordered continuous chain $(B_\alpha)_{\alpha<\lambda}$ of rc-filtered subalgebras such that $B_\alpha \leq_{rc} B_\beta$ for all $\alpha < \beta$, then B is itself rc-filtered.

We also study cardinal functions on rc-filtered Boolean algebras and their subalgebras. It turns out that with respect to the most popular functions these algebras behave like free ones. More precisely (cf. 2.7.10), *if B is a subalgebra of an rc-filtered Boolean algebra, then*

$$\begin{array}{c} \pi\chi = \mathrm{ind} = \pi = \mathrm{Irr} = \mathrm{t} = \mathrm{s} = \chi = \mathrm{hL} = \mathrm{hd} = \mathrm{Inc} = \text{h-cof} = |B| \\ \geq \\ d \\ \geq \\ \mathrm{Depth} = \mathrm{Length} = \mathrm{c} = \aleph_0. \end{array}$$

Co-completeness and weak projectivity

Two Boolean algebras will be called co-complete if they have isomorphic completions. Chapter 5 is devoted to the class of Boolean algebras that are co-complete with projective Boolean algebras. In want of a better name, we call them *weakly projective*. The main results are characterizations of weak projectivity. Two highlights from that chapter are theorem 5.3.11 saying that

every subalgebra of a projective Boolean algebra is weakly projective

and theorem 5.2.2, which determines weakly projective Boolean algebras up to co-completeness as the at most countable products of free Boolean algebras.

In the context of co-completeness the relevant type of embedding is called *regular*. We write $A \leq_{reg} B$ iff $sup^A M = sup^B M$ for each $M \subseteq A$ such that $sup^A M$ exists (i.e. the identical mapping preserves all infinite suprema existing in A).

It is rather easy to prove that each Boolean algebra that is co-complete to an rc-filtered one is 'regularly filtered', i.e. has a skeleton consisting of regular subalgebras. Whether the converse is also true, remains an open problem.

Adequate pairs

At the beginning of the investigations in connection with his 'spectral theorem' Ščepin defined the notion of a class $\mathcal{X}$ of compact spaces being adequate for a class Φ of continuous mappings. The following is a slightly modified Boolean algebraic version of this concept.

Let $\mathcal{B}$ be a class of Boolean algebras and $\mathcal{E}$ a class of embeddings. We write $A \leq_{\mathcal{E}} B$ to express that $A \leq B$ belongs to $\mathcal{E}$. We call the classes $\mathcal{B}$ and $\mathcal{E}$ ***adequate*** if the following conditions are satisfied.

($A1$) Every algebra in $\mathcal{B}$ has a skeleton $\mathcal{S}$ such that $S \leq_{\mathcal{E}} T$ for all $S \leq T$ belonging to $\mathcal{S}$.

($A2$) If $(B_\alpha)_{\alpha<\lambda}$ is a well-ordered continuous chain of Boolean algebras belonging to $\mathcal{B}$ such that $B_\alpha \leq_{\mathcal{E}} B_\beta$ for all $\alpha < \beta$, then $B = \bigcup_{\alpha<\lambda} B_\alpha$ belongs to $\mathcal{B}$ and $B_\alpha \leq_{\mathcal{E}} B$, for all $\alpha < \lambda$.

The class of rc-filtered Boolean algebras is adequate for the class of relatively complete embeddings and the class of regularly filtered Boolean algebras turns out to be adequate for the class of regular embeddings. In both cases condition ($A1$) is taken as definition and ($A2$) proved from it. The same is true for the third pair that we study in chapter 4. It is defined in terms of σ-embeddings, where $A \leq_\sigma B$ if, for each $b \in B$, the ideal $\{a \in A : a \leq b\}$ is countably generated. The results parallel those for the other two pairs.

Let us mention here that there are also classes of embeddings which are adequate for the class of projective and weakly projective Boolean algebras. Appropriately, they are called projective and weakly projective embeddings. The definitions are more complicated than in the above cases and can be found in sections 1.5 and 5.3, respectively.

Functors

In chapter 3 we consider three constructions that are, in fact, covariant functors of the category of Boolean algebras into itself: λ, exp, and SP^2. Their main purpose is to prove Ivanov's theorem and to construct the examples (1) and (2) mentioned on page 5 above. The algebras in question are $exp\,\mathrm{Fr}\,\omega_2$ and $SP^2(\mathrm{Fr}\,\omega_2)$, respectively, where $\mathrm{Fr}\,X$ denotes the free Boolean algebra on the set X of generators.

The examples show that the class of projective algebras is not closed under the functors exp and SP^2. This observation naturally leads to more general question of closedness under these functors. We concentrate on exp, where the principal results are 3.3.10, 3.3.6, and 5.4.5:

exp A is projective iff A is projective and $|A| \leq \aleph_1$.

exp A is rc-filtered iff A is rc-filtered.

exp A is weakly projective iff A is weakly projective.

It should be mentioned here that in the topological setting there is a theory of so-called normal functors and that (slight modifications of) the above and other results below hold for these in general. Our restriction[3] to exponentials has several reasons. First of all, *exp* is *the* typical normal functor. In that sense we dont loose much. Moreover, the proofs are technically more transparent for exponentials than in the general case. Finally, the very definition of a normal functor becomes rather clumsy and unnatural if translated into the Boolean algebraic language. The reader who wants to know more is referred to [55].

Set-theoretic appendix

The results in the main text are all obtained in ZFC by orthodox topological and algebraic methods. The appendix written by Sakaé Fuchino demonstrates another method to obtain results in ZFC. It uses elementary submodels of models of set theory and was first applied to topological questions independently by I. Bandlow and A. Dow.

Moreover, the appendix contains a number of recent independence results mostly due to Fuchino himself concerning rc-filtered Boolean algebras which answer some questions of Ščepin from [55].

Prerequisites and notation

We present all definitions and results in the language of Boolean algebras and the reader is supposed to have some experience with them. Our standard reference will be the *Handbook of Boolean algebras,* in particular its first volume [34]. Whenever possible we quote results from there. This is rather unjust to the original authors, but, hopefully, convenient for the reader.

Modulo the Handbook the text is more or less self-contained. Some 'Digressions' contain results that shed additional light on what is in the main text. Some of them are quoted without proof and qualified as 'Informations'. None of these results will be used in later proofs.

With very few exceptions we use standard notation, i.e. that of the Handbook. Let us dwell on some points that may differ from what the reader is used to.

Boolean operations

As a rule we consider Boolean algebras as complemented distributive lattices, i.e. with the fundamental operations of intersection = meet, union = join, and

[3]But notice that SP^2 is also normal.

complementation. We stick to the good old symbols $\wedge$, $\vee$, and $-$ (the latter is officially unary; but $a - b$ stands for $a \wedge -b$).

If $F = \{a_1, \ldots a_n\}$ is a finite set of elements of the Boolean algebra A, we alternatively write $\bigvee F$, $a_1 \vee \ldots \vee a_n$ or $\bigvee_{i=1}^{n} a_i$ to denote its join. The a_i are then called 'joinands'. If the set F is infinite, we still write $\bigvee F$ for its supremum (if it exists). The elements of F will still be 'joinands'. If several algebras are considered at the same time, it makes sense to indicate in which algebra the supremum is taken and we write $\bigvee^{A} F$. Similarly for finite and infinite meets and 'meetands'.

Sometimes (and then we emphasize this) it will be convenient to consider Boolean algebras as linear algebras (i.e. vector spaces with a multiplication) over the field $\mathbf{F}_2$ with two elements. That is why we use $\wedge, \vee$ and $-$ to denote the lattice-theoretic operations and $+$ and $\cdot$ for the ring-theoretic ones. The connection is well known:

$$a \cdot b = a \wedge b, \quad a + b = (a \vee b) - (a \wedge b), \quad a \vee b = a + b + a \cdot b, \quad -a = 1 + a.$$

Subalgebras and embeddings

$A \leq B$ means that A is a subalgebra of B. Formally, an *embedding* (sometimes also called *extension*) is a pair (A, B) such that $A \leq B$. We usually suppress the parentheses and write, e.g., 'let $A \leq B$ be an embedding...'. The more general concept of embedding hardly ever occurs in what follows and if it does, it will be called an injective homomorphism.

For $A \leq B$ and $b \in B$ we let $A\restriction b$ denote the ideal $\{a \in A : a \leq b\}$ of A. If $b \in A$ then $A\restriction b$ is a principal ideal, which can also be considered as a Boolean algebra, the so-called factor algebra of A corresponding to b. It will be clear from the context if we mean the factor algebra.

For a subset $X \subseteq A$ of a Boolean algebra, $\langle X \rangle_A$ denotes the subalgebra of A generated by X. Usually A will be clear from the context and we write $\langle X \rangle$ only. If C is a subalgebra of A and $X \subseteq A$, we sometimes write $C(X)$ instead of $\langle C \cup X \rangle$. If $X = \{x_1, \ldots, x_n\}$ is finite, this notation becomes $C(x_1, \ldots, x_n)$.

We shall often meet subalgebras of the form $\langle B \cup C \rangle_A$, where B and C are subalgebras of A. The elements of $\langle B \cup C \rangle$ have a particularly simple description, namely $\bigvee_{i=1}^{n} b_i \wedge c_i$, where $b_i \in B$ and $c_i \in C$. This trivial fact often makes life easier and will be tacitly used througout.

Free products

By $A \otimes B$ we denote the free product of A and B. (cf. subsection 11.1 of [34], where the notation $A \oplus B$ is used). We find it more illuminating to denote its canonical generators by $a \otimes b$ (instead of the $e_A(a) \wedge e_B(b)$ of the *Handbook*). So, each element of $A \otimes B$ can be written in the form $\bigvee_{i=1}^{n} a_i \otimes b_i$ for some $a_1, \ldots a_n \in A$ and $b_1, \ldots b_n \in B$. The characteristic property of free products is expressed by the following fact.

For each pair $\varphi : A \to C$ and $\psi : B \to C$ of homomorphisms there is a unique homomorphism $\varphi \otimes \psi : A \otimes B \longrightarrow C$ such that $(\varphi \otimes \psi)(a \otimes b) = \varphi(a) \wedge \psi(b)$.

The free product of an infinite family will be written as $\bigotimes_{i\in I} A_i$.

Sikorski's Extension Criterion

The following theorem (5.5 in [34]) will be used in several places and it seems appropriate to formulate it once in all detail.

Theorem 0.0.1 *Assume X generates the Boolean algebra A and φ maps X into a Boolean algebra B. For φ to extend to a homomorphism $A \to B$ it is necessary and sufficient that for all $x_1, \ldots x_n \in X$ and all $\varepsilon_1, \ldots, \varepsilon_n \in \{+1, -1\}$ if $\varepsilon_1 x_1 \wedge \ldots \wedge \varepsilon_n x_n = 0$ in A, then $\varepsilon_1\varphi(x_1) \wedge \ldots \wedge \varepsilon_n\varphi(x_n) = 0$ in B.*

Here $+1x$ means x and $-1x$ is $-x$. In practice the given condition often splits into three (collecting 'positive and negative' elements on different sides).

$$\begin{aligned} x_1 \wedge \ldots \wedge x_n = 0 &\implies \varphi(x_1) \wedge \ldots \wedge \varphi(x_n) = 0, \\ x_1 \vee \ldots \vee x_n = 1 &\implies \varphi(x_1) \vee \ldots \vee \varphi(x_n) = 1, \\ &\text{and} \\ x_1 \wedge \ldots \wedge x_m \leq x_{m+1} \vee \ldots \vee x_n &\implies \\ &\varphi(x_1) \wedge \ldots \wedge \varphi(x_m) \leq \varphi(x_{m+1}) \vee \ldots \vee \varphi(x_n). \end{aligned}$$

Set theory

Our notation is standard. As usual, we consider cardinal numbers as special ordinals. In particular, $\aleph_\alpha$ and ω_α denote the same object, considered under different aspects. It will be convenient to use the notation $|X|$ to denote the maximum of $\aleph_0$ and the cardinality of X, i.e. $|X|$ is always infinite.

Very little set theory is needed in the main text. In sections 2.10 and 6.4 we use stationary sets and some of their basic properties. Everything we need to know about these sets can be found in all modern standard texts. It is also contained in J. D. Monk's *Appendix on set theory* to volume 3 of the *Handbook* [34] (which the reader is likely to use anyway). The same is true of the (easiest version) of the Δ-Lemma, which occurs several times in the text. As with Sikorski's Theorem above, we feel obliged to once formulate it in full detail:

Theorem 0.0.2 *If κ is a regular uncountable cardinal and $(X_\alpha)_{\alpha<\kappa}$ is a family of finite sets, then there exists a subset $K \subseteq \kappa$ and a finite set Y such that $|K| = \kappa$ and $(X_\alpha)_{\alpha\in K}$ is a Δ-system with kernel Y, i.e. $X_\alpha \cap X_\beta = Y$ for all distinct $\alpha, \beta \in K$.*

On one occasion we also need the analogous statement for families of more than $2^{\aleph_0}$ countable sets. An unorthodox proof of the general form is contained in Fuchino's appendix (cf. A.1.13).