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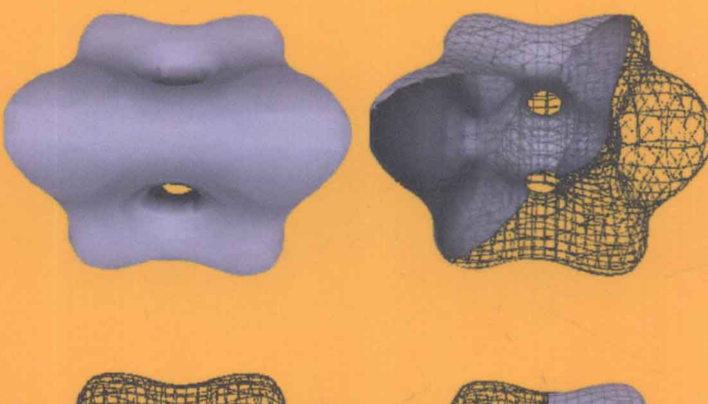
Symmetries of Compact Riemann Surfaces



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To Álvaro and my grandchildren
To my family
To Belén, Álvaro and Irene
To the memory of my parents

Preface

The content of this monograph is situated in the intersection of important branches of mathematics like the theory of one complex variable, algebraic geometry, low dimensional topology and, from the point of view of the techniques used, combinatorial group theory. The main tool comes from the Uniformization Theorem for Riemann surfaces, which relates the topology of Riemann surfaces and holomorphic or antiholomorphic actions on them to the algebra of classical cocompact Fuchsian groups or, more generally, non-euclidean crystallographic groups. Foundations of this relationship were established by A. M. Macbeath in the early sixties and developed later by, among others, D. Singerman.

Another important result in Riemann surface theory is the connection between Riemann surfaces and their symmetries with complex algebraic curves and their real forms. Namely, there is a well known functorial bijective correspondence between compact Riemann surfaces and smooth, irreducible complex projective curves. The fact that a Riemann surface has a symmetry means, under this equivalence, that the corresponding complex algebraic curve has a real form, that is, it is the complexification of a real algebraic curve. Moreover, symmetries which are non-conjugate in the full group of automorphisms of the Riemann surface, correspond to real forms which are birationally non-isomorphic over the reals. Furthermore, the set of points fixed by a symmetry is homeomorphic to a projective smooth model of the real form.

The monograph consists of an extensive Introduction, a compilation of basic results in the Preliminaries, four principal Chapters and a short Appendix on asymmetric Riemann surfaces. After the Preliminaries, in Chap. 2, we focus our attention on the quantitative results concerning upper bounds for the number of conjugacy classes of symmetries. We divide our study into three cases, according to the nature of the set of points fixed by the symmetries. Namely we distinguish whether this set is empty or not and, accordingly, consider just symmetries with fixed points, just symmetries without fixed points and finally hybrid configurations allowing both types of symmetries simultaneously.

Chapter 3 can be seen as a variation on the classical Harnack theorem, that states that the set of points fixed by a symmetry of a Riemann surface of genus g has at most $g+1$ connected components, all of them being closed Jordan curves, called ovals in Hilbert's terminology introduced in the nineteenth century. We first deal with the problem of finding the total number of ovals of a specified number of non-conjugate symmetries. We next consider the same problem for all the symmetries (conjugate or not) of a Riemann surface. We finally deal with the total number of ovals of a pair of symmetries in terms of the order of its product and the genus of the surface.

The monograph is actually devoted to the symmetries of Riemann surfaces of genus at least two since they are the ones uniformized by the hyperbolic plane. The theory of symmetries of the remaining surfaces, that is, the Riemann sphere and the tori, is well-known for a long time but, for the sake of completeness and the reader's convenience, we devote the main part of Chap. 4 to this subject. We also outline the classification of the symmetry types of hyperelliptic Riemann surfaces as being the double covers of the Riemann sphere.

Finally, Chap. 5 is dedicated to the symmetries of Riemann surfaces with large groups of automorphisms. Such surfaces are important since on the one hand they are determined by a 2-generator presentation of their groups of automorphisms, and on the other hand they can be defined over the algebraic numbers due to the celebrated theorem of Belyi from the late seventies. Furthermore, by a recent result of B. Köck and D. Singerman, these algebraic numbers can be chosen to be reals if the surface is symmetric. The foundations for the study of symmetries of such surfaces were established by Singerman, who found necessary and sufficient algebraic conditions in terms of the mentioned above generating pair for such a surface to be symmetric. In the first section, apart from Singerman's proof, we give formulae to compute the number of ovals of these symmetries, to which we refer as Singerman symmetries. Using these formulae we deal, in the next two sections, with the significant families of Macbeath-Singerman and Accola-Maclachlan and Kulkarni surfaces. Finally we describe the symmetries of the last two families by means of algebraic formulae.

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Introduction

By a *symmetry* σ of a compact Riemann surface S we mean an antianalytic involution $\sigma : S \to S$. A Riemann surface which admits a symmetry is called *symmetric*. Under the well known functorial bijective correspondence between compact Riemann surfaces and smooth, irreducible, complex projective curves, symmetric surfaces correspond to curves definable over the field $\mathbb{R}$ of real numbers. If $\sigma : S \to S$ is a symmetry then the pair (S, σ) is usually called a *real algebraic curve*, see the foundational monograph [4] by Alling and Greenleaf to justify this definition. Some topological features of the real curve (S, σ) can be obtained from its associated symmetry σ. For instance, the set of real points of the curve is homeomorphic to the fixed point set $\mathrm{Fix}(\sigma)$ of the symmetry. In addition, symmetries which are non-conjugate within the full group $\mathrm{Aut}(S)$ of automorphisms of S correspond to real curves which are non-isomorphic over the real numbers but are isomorphic over the complex numbers.

With a language closer to the one we will use here, let us show an example of two non-birationally $\mathbb{R}$-isomorphic real algebraic curves whose complexifications are birationally $\mathbb{C}$-isomorphic. Let us consider for $t = 0, 1$, the degree 3 homogeneous polynomial

$$F_t(x, y, z) = y^2 z - x(x^2 + (-1)^t z^2).$$

An easy computation shows that at any point in the complex projective plane $\mathbb{P}^2(\mathbb{C})$, the partial derivatives of F_t are not simultaneously zero. So, for $t = 0, 1$, each set

$$S_t = \{[x : y : z] \in \mathbb{P}^2(\mathbb{C}) : F_t(x, y, z) = 0\}$$

admits a structure of compact Riemann surface. In fact, S_0 and S_1 are birationally $\mathbb{C}$-isomorphic as complex algebraic curves via the isomorphism

$$\varphi : S_0 \to S_1 \ ; \ [x : y : z] \mapsto [\xi x : \xi^4 y : \xi^3 z],$$

where $\xi = e^{i\pi/4}$. However, their sets of $\mathbb{R}$-rational points, that is, the real curves $S_0(\mathbb{R})$ and $S_1(\mathbb{R})$, are not birationally $\mathbb{R}$-isomorphic. Indeed, both are smooth but $S_0(\mathbb{R})$ is connected while $S_1(\mathbb{R})$ has two connected components. The paper [32] by Cirre and Gamboa presents many other examples of non-isomorphic real algebraic curves with isomorphic complexifications.

These phenomena lead naturally to the following problems we deal with in this monograph:

(1) Is a complex smooth algebraic curve $\mathcal{C}$ definable over the reals?
(2) Assume that this question has an affirmative answer. How many non-birationally $\mathbb{R}$- isomorphic real algebraic curves admit $\mathcal{C}$ as its complexification? The projective smooth models of such real curves are usually called the *real forms* of $\mathcal{C}$.
(3) What can be said about the topology of the real forms of $\mathcal{C}$?

The expository work by Gromadzki [51] can be understood as the first attempt to survey the known answers to these questions. Because of the methods to be used, it seems convenient to translate these questions into a more suitable language. To that end we use the terminology introduced at the beginning. In particular, the first of the above problems reads off: is a compact Riemann surface symmetric?

Let σ and τ be symmetries of the compact Riemann surface S. The pairs (S, σ) and (S, τ) are real forms of S; they are said to be *isomorphic* if there exists an automorphism φ of S such that $\sigma = \varphi \circ \tau \circ \varphi^{-1}$. In this way the second problem to be treated is the counting of the number of conjugacy classes of symmetries with respect to the group $\mathrm{Aut}(S)$ of analytic and antianalytic automorphisms of the Riemann surface S.

Finally, the topological type of a symmetry σ of S is determined, together with the genus of S, by the number of connected components, or *ovals* (in the nineteenth century Hilbert's terminology) of the fixed point set $\mathrm{Fix}(\sigma) = \{p \in S : \sigma(p) = p\}$ and the connectedness character of its complement $S \setminus \mathrm{Fix}(\sigma)$ in S. More precisely, the triple (g, k, ε) is said to be the *topological type* of a symmetry σ of a genus g surface S if the set $\mathrm{Fix}(\sigma)$ has k connected components, and $\varepsilon = 1$ or $\varepsilon = 0$ according to whether $S \setminus \mathrm{Fix}(\sigma)$ is connected or not. We say that σ is *non-separating* if $\varepsilon = 1$ and *separating* otherwise.

A classical result due to Harnack [59] and Weichold [127] states that the necessary and sufficient conditions for a triple to be *admissible*, that is, to be the topological type of some symmetry σ, are the following:

$$
\begin{array}{ll}
1 \leq k \leq g+1 & \text{if } \varepsilon = 0 \text{ with } g+1 \equiv k \pmod{2}; \\
0 \leq k \leq g & \text{if } \varepsilon = 1.
\end{array}
$$

The pair (k, ε) is usually codified by the symbol $+k$ if $\varepsilon = 0$ and $-k$ if $\varepsilon = 1$. It is called the *species* of the symmetry σ and denoted by $\mathrm{sp}(\sigma)$.

It has to be mentioned that the orbit space $X_\sigma = S/\langle\sigma\rangle$ of the compact Riemann surface S under the symmetry σ is usually called a compact *Klein surface.* The fixed point set $\mathrm{Fix}(\sigma)$ is homeomorphic to the topological boundary of X_σ. Both are in fact homeomorphic to the set of real points of the projective smooth irreducible real algebraic curve associated to the symmetry σ. If this set is empty then we say that the corresponding real curve (S, σ) is *purely imaginary*. These pairs correspond to complex algebraic curves which can be defined over the reals but have no $\mathbb{R}$-rational points. It is well known that the set $S \setminus \mathrm{Fix}(\sigma)$ is either connected or it has two connected components. In the first case, i.e., if σ is non-separating, then the orbit space $X_\sigma = S/\langle\sigma\rangle$ is non-orientable, while in the separating case X_σ is orientable.

An expository account of the functorial correspondence between real algebraic curves and Klein surfaces can be found in [45], see also the condensed versions [103, 104] by Natanzon.

As we shall see throughout this monograph, a fundamental component to approach the problems mentioned above is the knowledge of the full automorphism group $\mathrm{Aut}(S)$ of the analytic and antianalytic automorphisms of S and its subgroup $\mathrm{Aut}^+(S)$ consisting of the analytic ones. Moreover, to determine the topology of a given symmetry σ of S, the centralizer $\mathrm{C}(\mathrm{Aut}(S), \sigma)$ of σ in $\mathrm{Aut}(S)$ plays a fundamental role. Although automorphism groups do not constitute the core of this work, we will need them very frequently. It is worth mentioning that the factor group $\mathrm{C}(\mathrm{Aut}(S), \sigma)/\langle\sigma\rangle$ is isomorphic to the group of automorphisms of the Klein surface $S/\langle\sigma\rangle$. There is a vast literature concerning groups of automorphisms of such surfaces. Among them we should mention [22], [58], [79]–[90], [93], the pioneering papers [114] and [40] and the exceptionally complete work [105].

We now describe briefly the content of this monograph. We also quote the contributions of different authors to the development of the employed techniques and related topics.

Although in Chap. 4 we study the symmetries of the sphere and the tori, we will mainly be concerned with compact Riemann surfaces of genus bigger than one. By the Uniformization Theorem, such a surface S can be presented as the orbit space of the hyperbolic plane $\mathcal{H}$ under the action of a surface Fuchsian group Γ. Moreover, using covering theory, it can be proved that each automorphism group of $S = \mathcal{H}/\Gamma$ is a factor group Λ/Γ, where Λ is a non-euclidean crystallographic (NEC in short) group containing Γ as a normal subgroup. The key point now is that the algebraic structure of both Fuchsian and NEC groups is well known and this is why we devote Sect. 1.1 to the presentation of some basic facts about these groups.

The above shows that in order to move ahead with the combinatorial approach of the study of symmetries of Riemann surfaces it is essential to understand the relation between the presentations of two NEC groups Γ and Λ, where the first is a normal subgroup of the second one. This task is mainly due to E. Bujalance, who developed in a series of papers [10–12] at the beginning of the eighties, an efficient method to solve this problem based on surgery of fundamental regions. It is also worth mentioning the article by J. A. Bujalance [27] concerning this problem. These results appear, without proofs, in Sect. 1.2.

One of the main elements in the combinatorial approach to the study of symmetries of compact Riemann surfaces is the analysis of the centralizers of hyperbolic reflections in NEC groups. Singerman found in his Ph. D. Thesis [115], see also [119], the isomorphism type of centralizers of reflections in NEC groups. Going a bit more into the details of Singerman's proof, explicit generators of these groups can be obtained, see the papers [48, 51] by G. Gromadzki. We present them in Sect. 1.3.

Section 1.4 concerns uniformization of compact Riemann and Klein surfaces by means of Fuchsian and NEC groups, respectively, and its consequences. We pay special attention to the explanation of the notions of maximal Fuchsian or NEC groups and maximal signatures, and the relation between them. Although we have not included proper proofs of the results we will be using throughout the monograph, for which the reader is referred to [22, Chap. 5], we present carefully the main concepts.

To finish this preliminary chapter, we explain the basics about symmetries in Sect. 1.5. We recall the notions of *topological type* and *species* of a symmetry and the classical Harnack-Weichold necessary and sufficient conditions for a given triple to be the topological type of some symmetry. We also approach the problem of deciding whether a Riemann surface is symmetric. This depends, in general, on its analytic type. However, there is an exception, pointed out by Singerman, who showed in [118] that if the group $\mathrm{Aut}^+(S)$ of analytic automorphisms of S is large enough then the symmetrical character of S depends only on the group $\mathrm{Aut}^+(S)$. Moreover, Singerman obtained a necessary and sufficient condition for the surface S to be symmetric and here we provide a slightly different proof of his criterion.

Surfaces S with large analytic automorphism group $\mathrm{Aut}^+(S)$ are rather special and, perhaps, the most interesting ones. In particular they are *Belyi surfaces* since $\mathrm{Aut}^+(S)$ can be uniformized by a triangle Fuchsian group. This implies, by Belyi's Theorem, see [7], that S can be defined by polynomial equations whose coefficients are algebraic numbers. Furthermore, by the recent results of Köck and Singerman [66] and Köck and Lau [67] on symmetric Riemann surfaces with large group of automorphisms, these algebraic numbers can be chosen to be real.

Chapter 2 is devoted to quantitative aspects of the theory; we deal with the problem of finding the number of conjugacy classes of symmetries of Riemann surfaces. The study of symmetries that fix points comes back to the seminal work of Natanzon [95] who proved, using deep topological methods, that a Riemann surface of genus g has at most $2(\sqrt{g}+1)$ non-conjugate symmetries that fix points. Moreover, he showed that this upper bound is attained for each value g of the form $g=(2^{n-1}-1)^2$. Later on, Bujalance, Gromadzki and Singerman proved in [24] that these are the only values of g for which Natanzon's bound is sharp. Moreover, if the bound is attained then all the symmetries are non-separating. In the same article the authors found an upper bound for the number of conjugacy classes of separating symmetries of a surface of genus g.

At a first sight this bound seems to be a strictly increasing function of the genus, but later on it was discovered that this is so only up to some extent. Indeed, Gromadzki and Izquierdo proved in [53] that a Riemann surface of even genus has at most four non-conjugate symmetries that fix points. This result was extended to surfaces of odd genus by Bujalance, Gromadzki and Izquierdo in [23]. In that paper, and for each odd genus, the authors found sharp upper bounds for the number of such symmetries. We reprove these results in Sect. 2.2 of this chapter.

The search of an upper bound for the number of conjugacy classes of fixed point free symmetries is much more involved. In Sect. 2.3 we provide an upper bound valid for those surfaces which have no symmetry with fixed points. The bound, which depends only on the 2-adic part of $g-1$, was obtained originally in [18] and it was shown to be attained for infinitely many values of g.

Finally, in Sect. 2.4 we obtain an upper bound for the number of conjugacy classes of symmetries of a genus g surface allowing both fixed point free symmetries and symmetries with ovals. Once more it turns out that this bound depends only on the 2-adic part of $g-1$.

Chapter 3 deals with several enumerations of ovals of the symmetries of a Riemann surface. Section 3.1 is crucial for the rest of the monograph; its main result allows us to find the number of ovals of a symmetry of a Riemann surface S from the algebraic structure of the full automorphism group $\mathrm{Aut}(S)$ and from the topological type of the action of $\mathrm{Aut}(S)$ on S. It was originally established in [49]. As we mentioned, a Riemann surface of even genus has at most four non-conjugate symmetries and, as an application of the result just quoted, Gromadzki and Izquierdo found in [54] the maximal total number of ovals of such extremal configuration of symmetries.

The problem of finding the maximal number of ovals of a fixed number k of non-conjugate symmetries of a Riemann surface of genus g has been investigated by many authors throughout the years. However, it has been solved in its full generality just recently [56]. The first results, concerning low values of k, were obtained by Natanzon in [96, 100, 105], where he showed that an upper bound for such number is $2g+2^{k-1}$ for $k=2,3,4$ and characterized the pairs (g,k) for which this bound is attained.

Later on, Singerman in [121] showed that for each non-negative integer k there exist infinitely many values of g for which there exists a Riemann surface of genus g admitting k non-conjugate symmetries having $2g-2+2^{k-3}(9-k)$ ovals in total. In his work, Singerman also conjectured that this is in fact the best possible upper bound. This was shown by Gromadzki in [50] to be false for $k>9$ by showing that, for $k\geq 9$, the maximal possible number of ovals is $2g-2+2^{r-3}(9-k)$, where r is the smallest positive integer for which $k\leq 2^{r-1}$. Moreover, this bound is attained, for arbitrary $k\geq 9$, for infinitely many values of g. Later on Natanzon proved in [107] that Singerman's conjecture is true under the additional assumption that the symmetries are separating. The presentation of these results is the main goal of Section 3.2.

It is worth mentioning that Singerman's conjecture was found to be true for $k=9$ in [50] and it was conjectured to be also true for k in range $5\leq k\leq 8$. This has recently been answered in the affirmative by Gromadzki and Kozłowska-Walania in [56].

Section 3.3 concerns the total number of ovals of all symmetries of a Riemann surface. Recall that a simple closed curve on a Riemann surface S is said to be an *oval of S* if it is an oval of some symmetry of S. Let $\|S\|$ be the number of ovals of S and let $\nu(g)$ be the maximum of $\|S\|$ where S runs over all Riemann surfaces of genus g. Using topological methods, Natanzon proved in [105] that $\nu(g)\leq 42(g-1)$, and Gromadzki improved this bound in [49] by using combinatorial methods. We present the complete proofs of these results in this section.

Finally, Sect. 3.4 is devoted to the study of pairs of symmetries of Riemann surfaces. A lot of work has been done in this topic, and we include in this section some of the most relevant results. Once more the first and fundamental steps in this kind of questions are due to Natanzon, who classified topologically in [102] pairs of commuting symmetries.

Natanzon in [105] and later on Bujalance, Costa and Singerman in [21], found an upper bound for the total number of ovals of two symmetries in terms of the genus of the surface and the order of their product. A finer bound, which involves the number of points fixed by the product of these symmetries, has been obtained by Gromadzki and Kozłowska-Walania in [55].

On the other hand, it was proved in [21] that two symmetries σ_1 and σ_2 of a genus g Riemann surface S having k_1 and k_2 ovals, where $k_1 + k_2 \geq g + 3$, always commute. In a recently published paper by Kozłowska-Walania [69], this bound has been proved to be optimal to guarantee the commutativity of each pair of symmetries of S, with one exception in each genus $g > 2$.

Another interesting result concerning pairs of symmetries was obtained by Bujalance and Costa, who calculated in [19] upper bounds for the degree of hyperellipticity of the product of two commuting symmetries. These upper bounds vary according to the separating character of the symmetries and they depend just on the numbers of their ovals. A nice improvement has been published by Kozłowska-Walania in [68], where the upper bounds for the degree of hyperellipticity are substituted by its precise values.

Izquierdo and Singerman showed in [63] that the existence of a symmetry whose number of ovals is extremal, that is, either 0 or $g + 1$ where g is the genus of the surface, imposes restrictions on the number of ovals of any other symmetry of the same surface. They also found extra restrictions if the separating character of the symmetries is considered. Later on, Costa and Izquierdo [34] showed that for every admissible triple (g, k, ε) there exists a genus g surface admitting symmetries σ and τ with topological types (g, k, ε) and $(g, 1, 1)$, respectively. This result has a deep consequence: the locus of symmetric Riemann surfaces of fixed genus $g \geq 2$ is a connected subspace of the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g. Of course this result is not new, but what is new is its proof. Klein conjectured it and Seppälä provided a modern and complete proof in [111] by using strong deformation of curves.

The study of the number of connected components of distinguished subspaces of $\mathcal{M}_g$ is a recurrent theme in algebraic geometry. In fact the connectedness of the most important subspaces is rather exceptional, as it was shown, for example, by Buser, Seppälä and Silhol in [28]. In this article the authors study the subset of the moduli space of stable curves of genus bigger than one consisting of curves admitting a given finite group as a group of analytic automorphisms. They prove that this subset is always compact, is not connected in general, and it is connected for the group of order 2. In the same vein, it is worth mentioning that in the already quoted paper [34], Costa and Izquierdo proved the disconnectedness of the subspace of p-gonal Riemann surfaces of genus g for fixed values of p and g. This extends an earlier theorem by Gross and Harris [59] only valid for $p = 3$.

The class of p-gonal surfaces has attracted the interest of many authors. In what concerns symmetries, we quote here the result of Costa and Izquierdo in [35] where they study the symmetries of cyclic p-gonal Riemann surfaces by means of Fuchsian and NEC groups. To finish, it is worth mentioning the paper by Bujalance, Costa and Gromadzki [20], where the behaviour of symmetries with maximal number of ovals under non-ramified coverings is studied.

Chapter 4 is devoted to the presentation of classical selected examples. To begin with, we study the Riemann sphere Σ in Sect. 4.1. It is elementary to show that the maps $\sigma_1 : z \to \bar{z}$ and $\sigma_2 : z \to -1/\bar{z}$ are symmetries of Σ and that they are the only ones, up to analytic conjugation. Section 4.2 is devoted to classify the symmetries of the tori, for which we follow closely the approach by Alling [3]. Each torus is presented as the orbit space $\mathbb{C}/\mathfrak{L}$ for a suitably arranged lattice $\mathfrak{L}$. The symmetrical character of the torus and the topological type of its symmetries are expressed in terms of the lattice $\mathfrak{L}$. As it is classical, the analysis requires the cases of square or hexagonal lattices to be treated separately.

In Section 4.3 we explain how the complete classification of the symmetries of hyperelliptic Riemann surfaces was obtained by the authors of this monograph in their previous work [14]. This work is too extensive even to be completely summarized here, but we explain an example in detail, showing how both the combinatorial approach and the use of algebraic equations, combined with a topological method, are fruitful in this case.

In Chap. 5 we deal with symmetries of surfaces S whose group of analytic automorphisms $\mathrm{Aut}^+(S)$ is large enough. Following [52], we call these symmetries *Singerman symmetries*. As mentioned above, the symmetrical character of such surfaces depends only on $\mathrm{Aut}^+(S)$. In Sect. 5.1 we give formulae for the number of ovals of the symmetries of such surfaces in terms of the orders of the isotropy groups of some automorphisms acting on $\mathrm{Aut}^+(S)$, and the orders of some distinguished elements in $\mathrm{Aut}^+(S)$. These results constitute a fundamental component in the development of the next sections of this chapter.

The understanding of the symmetries of the so called Macbeath-Singerman surfaces is the goal of Sect. 5.2. These are genus g surfaces admitting the projective special linear group PSL(2, q), where q is a prime power, as its group of analytic automorphisms of the maximal order $84(g-1)$. Klein [65] was the first to discover the existence of such surfaces, as he showed that the group of analytic automorphisms of the genus 3 surface

$$S = \{[x : y : z] \in \mathbb{P}^2(\mathbb{C}) : x^3y + y^3z + z^3x = 0\},$$

known as the Klein quartic, is the projective special group PSL(2, 7) of order 168. Macbeath [72] proved much later the existence of a unique Riemann surface of genus 7 on which the group PSL(2, 8) of order 504 acts as its full group of analytic automorphisms.

Following ideas of Singerman from [118], we show that all Macbeath-Singerman surfaces are symmetric. We also determine the number of symmetries they admit, which we call Macbeath-Singerman symmetries, and the topological type of each of them. Remarkably, all of them are non-separating. These results were proved for the first time by Broughton, Bujalance, Costa, Gamboa and Gromadzki in [8]. The proof we present here is quite different and relies heavily on the results of the previous section of this chapter.

In the 1960's, Accola [1] and Maclachlan [77] proved, independently, that for every integer $g \geq 2$ there is a compact Riemann surface X_g of genus g whose automorphism group has order $8g + 8$. It is called the *Accola-Maclachlan surface* and it is defined by the polynomial equation $y^2 = x^{2g+2} - 1$. The result is interesting as $8g + 8$ is the largest order of an automorphism group that can be attained for every genus g. Much later, Kulkarni [70] considered the question of uniqueness of the surfaces attaining this bound. It turns out that the Accola-Maclachlan surface X_g is the unique one if $g \equiv 0, 1, 2 \pmod 4$ and g sufficiently large. However, for large enough $g \equiv 3$ (mod 4), Kulkarni also proved that, in addition to X_g, there exists exactly one other surface, called *Kulkarni surface*, of genus g whose automorphism group also has order $8g + 8$.

In Sect. 5.3 we show that these surfaces are symmetric and, moreover, we determine the number of conjugacy classes of symmetries they admit and the topological type of each of them. As in the example of Sect. 5.2, the proof proposed here relies on the results in Sect. 5.1 and it is quite different from the original one which appeared in [9].

It must be pointed out that the examples selected to this chapter are in some sense exceptional because it has been possible to decide successfully the separating character of each symmetry. But, of course, they are not the only ones. In their paper [2], Akbas and Singerman not only calculated the number of ovals of the symmetries of the modular surfaces $X_0(N) = \mathcal{H}/\Gamma_0(N)$, but also showed that they are separating for $N = 2, 3, 5, 7, 13$ and non-separating for all other primes N. The situation is slightly worse for the symmetries of the modular surfaces $X(N) = \mathcal{H}/\Gamma(N)$. All of them are non-separating in case $N \equiv 3 \,(\mathrm{mod}\ 4)$ is prime but, as far as we know, there is no general answer for primes $N \equiv 1 \,(\mathrm{mod}\ 4)$.

Another interesting example, that we do not explain in the monograph, is due to Tyszkowska [126], who obtained sharp upper bounds for the number of ovals of the symmetries of the Belyi surfaces admitting PSL(2, p) as its group of automorphisms.

Section 5.4 is devoted to finding polynomial equations of the sets of points fixed by the symmetries of families of Riemann surfaces studied in the precedent ones. The key point is the Galois theory of finite coverings, as explained to the authors by P. Turbek. In fact Turbek is responsible for the original finding of equations of the symmetries of the Accola-Maclachlan surfaces occurring in [9], but in this monograph we have chosen a more geometrical approach. However, the presentation of the part of this section concerning defining equations of the sets of points fixed by the symmetries of the Kulkarni surfaces follows closely Turbek's article [124].

It is convenient to explain a little bit the method employed. We begin with a plane model of our Riemann surface S, possibly with singularities, defined as the zero set in $\mathbb{C}^2$ of a polynomial $P \in \mathbb{C}[X, Y]$. A symmetry σ of S can be seen as an involution of the quotient field E_P of the coordinate ring $\mathbb{C}[X, Y]/(P)$ of S. We look for a different polynomial $Q \in \mathbb{R}[X, Y]$ which also defines S. Then the quotient fields

E_P and E_Q are isomorphic via a birational isomorphism, say $\varphi : E_P \to E_Q$. With respect to these new coordinates, the symmetry $\sigma' = \varphi^{-1} \circ \sigma \circ \varphi$ acts as complex conjugation:

$$\sigma'(i) = -i\,; \quad \sigma'(X) = X\,; \quad \sigma'(Y) = Y,$$

where $i = \sqrt{-1}$. Observe that the fixed points of σ' are the real solutions of the equation $Q(X,Y) = 0$.

We finish this work about symmetries with a few words about *asymmetric surfaces*, that is, surfaces admitting no symmetry. Such surfaces have recently played an important role in the study of deformations and moduli of complex surfaces, as in the paper [29] by Catanese, where the author finds a counterexample to a conjecture of Friedman and Morgan relating diffeomorphisms and deformations of such complex surfaces.

Let $\mathcal{M}_g$ be the moduli space of complex isomorphism classes of complex algebraic curves of genus $g \geq 2$. Since $\mathcal{M}_g$ is a quasiprojective variety defined in some projective space $\mathbb{P}^n(\mathbb{C})$ by means of polynomials with real (in fact rational) coefficients, complex conjugation induces an anticonformal involution $\sigma_g^* : \mathcal{M}_g \to \mathcal{M}_g$. Let $\mathcal{M}_g^{\mathbb{R}}$ be the *complex* moduli space of real algebraic curves of genus g, which consists of complex isomorphism classes of complex algebraic curves that are defined by real polynomials. It is clear that the set $\mathrm{Fix}(\sigma_g^*)$ of points fixed by σ_g^* contains $\mathcal{M}_g^{\mathbb{R}}$ but, as observed by Clifford Earle in [40], the inclusion $\mathcal{M}_g^{\mathbb{R}} \subset \mathrm{Fix}(\sigma_g^*)$ is proper. The asymmetric curves are precisely those whose isomorphism classes occur in the difference $\mathrm{Fix}(\sigma_g^*) \setminus \mathcal{M}_g^{\mathbb{R}}$. Seppälä showed in [110] that every asymmetric curve is in fact a covering of a real algebraic curve.

It is classical that for any integer $g > 2$ there exists a compact Riemann surface of genus g whose group of analytic automorphisms is trivial. Indeed, Greenberg proved in [47] that outside a proper analytic subset of the Teichmüller space, all compact Riemann surfaces of genus $g \geq 3$ have the identity as its only analytic automorphism. However, it is not easy to construct examples of such surfaces. It is worth mentioning the paper by Mednykh [91] who constructed, for each pair of integers (p, r), where $p > 3$ is prime and $r \geq 2p$, a fundamental region of a Fuchsian group which uniformizes a compact Riemann surface of genus $g = (p-1)(r-1)/2$ with trivial automorphism group.

Later on, Everitt in [42] found new examples for all $g > 2$, using Schreier coset graphs for subgroups of triangle groups. Combining covering theory with Galois theory of algebraic function fields in one variable, Turbek [123, 125] provided defining equations of compact Riemann surfaces with trivial group of analytic automorphisms.

In the same vein, Earle in [40] was the first to find examples of pseudo-real Riemann surfaces, that is, surfaces without symmetries but with orientation reversing automorphisms. Later on, Bujalance and Turbek constructed in [26] algebraic equations of the elements of an infinite family of pseudo-real Riemann surfaces. The construction we present in Chap. 6 is a particular case of the one in [26].

More recently, Bujalance, Conder and Costa in [17] have shown that there exist pseudo-real Riemann surfaces of genus g for each $g \geq 2$ and, furthermore, that the maximum number of automorphisms of such a surface is $12(g-1)$. This bound turns out to be sharp for infinitely many values of g.

Another instance of pseudo-real surfaces occurs in [13], where Riemann surfaces of even genus g with an orientation reversing automorphism of order $2g$ are studied. These surfaces constitute a family of real dimension three and "most" (but not all) of them are asymmetric. In fact, a defining algebraic equation depending on three real parameters can be given for each such surface and it turns out that those which are symmetric depend just on two parameters.

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