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Asymptotic Combinatorics with Applications to Mathematical Physics

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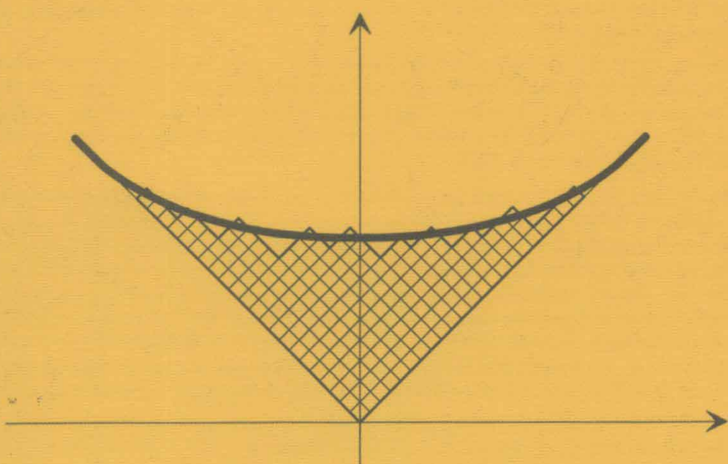

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Asymptotic Combinatorics with Applications to Mathematical Physics

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Preface

A European Mathematical Summer School entitled "ASYMPTOTIC COMBINATORICS WITH APPLICATIONS TO MATHEMATICAL PHYSICS" was held at St. Petersburg, Russia, 9–22 July 2001. This meeting was at the same time a NATO Advanced Studies Institute.

It was cosponsored by NATO Science Committee, European Mathematical Society, and Russian Fund for Basic Research.

This volume contains mathematical lectures from the school. Another part of the materials presented at the School, more related to mathematical physics, is already published[1]. Information about the School, its participants, program, etc. can be found at the end of this volume.

The present volume contains lecture courses, as well as several separate lectures, which have mainly mathematical rather than physical orientation. They are aimed mostly at non-specialists and beginners who constituted the majority of the participants of the School. I would like to emphasize that splitting the lectures into "physical" and "mathematical" ones is relative. Moreover, the idea of the School was to unite mathematicians and physicists working essentially on the same problems but following different traditions and notations accepted by their communities.

The last few years were marked by an impressive unification of a number of areas in mathematical physics and mathematics. The Summer School presented some of these major – and until recently mutually unrelated – topics: matrix problems (the study of which was initiated by physicists about 25 years ago), asymptotic representation theory of classical groups (which arose in mathematics approximately at the same time), the theory of random matrices (also initiated by physicists but intensively studied by mathematicians), and, finally, the theory of integrable nonlinear problems in mathematical physics with a wide range of related problems. As a result of the new

[1] *Proceedings of NATO ASI Asymptotic Combinatorics with Application to Mathematical Physics,* V. Malyshev and A. Vershik, Eds., Kluwer Academic Publishers, 2002, 328pp.

interrelations dicovered, all these young theories, which constitute the essential part of modern mathematics and mathematical physics, have become part of one large mathematical area. These interconnections are mainly combinatorial. An illustration of this phenomenon is the perception of the fact that the asymptotic theory of Young tableaux and the theory of spectra of random matrices is essentially the same theory, since the asymptotic microstructure of a random Young diagram with respect to the Plancherel measure coincides with the microstructure of the spectrum of a random matrix in the Gaussian ensemble. The corresponding asymptotic distributions are new for the probability theory. They were originally found by Riemann–Hilbert problem techniques (the Riemann–Hilbert problem arises in calculation of the diagonal asymptotics of orthogonal polynomials). In the present volume this direction is represented by the lectures by P. Deift and A. Borodin. Later these distributions were obtained by another method, that calculates the correlation functions directly using direct relations to integrable problems and hierarchies (A. Borodin, A. Okounkov, G. Olshansky).

The lectures by A. Vershik, G. Olshansky, R. Hora, and partially by A. Borodin and P. Biane are devoted to the asymptotic representation theory. This theory studies the asymptotic behavior of characters of classical groups as the rank of the group grows to infinity. It was started in the beginning of 1970s by works of Vershik–Kerov and Logan–Shepp and one of the first results was the proof of the asymptotic behaviour of the characters of symmetric group and Young diagrams. At that time the similarity and relations to quantum chromodynamics and matrix problems were anticipated but not yet clearly understood. Now these relations are well understood; they have become precise statements rather than vague analogies. These relations are also considered in the lectures by E. Bresin and V. Kasakov which have appeared in the other volume of the School proceedings.

From this point of view, four lectures by A. Okounkov take a particular position. They contain a sketch of the complete proof (obtained jointly with R. Pandharipande) of the Witten–Kontsevich formula relating the generating function of important combinatorial numbers of algebraic geometrical origin and the τ-function of the KdV equation hierarchy. In these lectures, special attention is paid to the role of the theory of symmetric functions and asymptotic representation theory, as well as to relations to random matrices.

The lectures by R. Speicher, M. Nazarov, and by M. Bożejko and R. Szwarc are devoted to more special topics which, however, fit in the same context. Although there are at present hundreds of journal papers on all these subjects, however the time for accomplished presentations is yet to come. The published lectures of the School should stimulate this process. The reader should keep in mind that the references cited in the lectures are not exhaustive. Of course, the relations between asymptotic combinatorics and mathematical physics extend farther than the topics touched upon during the School. For example, closely related combinatorial problems play a key role in conformal field theory which is now developing fast. I hope that the lectures presented in this volume will be useful for beginners as well as for specialists who want to familiarize themselves with this fascinating area of modern mathematics and mathematical physics and start working in it.

Two prominent mathematicians, Anatoly Izergin (1948–1999) and Sergey Kerov (1946–2000), died a year before the conference which had been planned with their active participation. Their contribution to areas of mathematical physics and mathematics related to the topics of the conference was enormous. One of the sessions of the conference was devoted to their memory.

All the work on preparation of this manuscript was carried out by Yu. Yakubovich, to whom I am very grateful. I would like to express my gratitude to the following organizations which helped greatly in the organization of the School: the European Mathematical Society, the NATO Science Committee, the Euler International Mathematical Institute, the St. Petersburg Department of the Mathematical Institute of the Russian Academy of Sciences and the St. Petersburg Mathematical Society.

Anatoly M. Vershik

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Asymptotic representation theory and Riemann–Hilbert problem

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Summary. We show how the Riemann–Hilbert problem can be used to compute correlation kernels for determinantal point processes arising in different models of asymptotic combinatorics and representation theory. The Whittaker kernel and the discrete Bessel kernel are computed as examples.

Introduction

A (discrete or continuous) random point process is called *determinantal* if its correlation functions have the form

$$\rho_n(x_1, \ldots, x_n) = \det[K(x_i, x_j)]_{i,j=1}^n,$$

where $K(x, y)$ is a function in two variables called the *correlation kernel.* A major source of such point processes is Random Matrix Theory. All the "unitary" or "$\beta = 2$" ensembles of random matrices lead to determinantal point processes which describe the eigenvalues of these matrices.

Determinantal point processes also arise naturally in problems of asymptotic combinatorics and asymptotic representation theory, see [6]–[9], [5], [15], [21]. Usually, it is not very hard to see that the process that we are interested in is determinantal. A harder problem is to compute the correlation kernel of this process explicitly. The goal of this paper is to give an informal introduction to a new method of obtaining explicit formulas for correlation kernels. It should be emphasized that in representation theoretic models which we consider the kernels cannot be expressed through orthogonal polynomials, as it often happens in random matrix models. That is why we had to invent something different.

The heart of the method is the *Riemann–Hilbert problem* (RHP, for short). This is a classical problem which consists of factorizing a matrix–valued function on a contour in the complex plane into a product of a function which is holomorphic inside the contour and a function which is holomorphic outside the contour. It turns out that the problem of computing the correlation kernels can be reduced to solving a RHP of a rather special form. The input of the RHP (the function to be factorized) is always rather simple and can be read off the representation theoretic quantities such as dimensions of irreducible representations of the corresponding groups. We also employ a discrete analog of RHP described in [2].

The special form of our concrete RHPs allows us to reduce them to certain linear ordinary differential equations (this is the key step), which have classical special functions as their solutions. This immediately leads to explicit formulas for the needed correlation kernels.

The approach also happens to be very effective for the derivation of (nonlinear ordinary differential) Painlevé equations describing the "gap probabilities" in both random matrix and representation theoretic models, see [4], [3]. However, this issue will not be addressed in this paper.

The paper is organized as follows. In Section 1 we explain what a determinantal point process is and give a couple of examples. In Section 2 we argue that in many models correlation kernels give rise to what is called "integrable integral operators". In Section 3 we relate integrable operators to RHP. In Section 4 we derive the Whittaker kernel arising in a problem of harmonic analysis on the infinite symmetric group. In Section 5 we derive the discrete Bessel kernel associated with the poissonized Plancherel measures on symmetric groups.

This paper is an expanded text of lectures the author gave at the NATO Advanced Study Institute "Asymptotic combinatorics with applications to mathematical physics" in July 2001 in St. Petersburg. It is a great pleasure to thank the organizers for the invitation and for the warm hospitality. The author would also like to thank Grigori Olshanski and Percy Deift for helpful discussions.

This research was partially conducted during the period the author served as a Clay Mathematics Institute Long-Term Prize Fellow. This work was also partially supported by the NSF grant DMS-9729992.

1 Determinantal point processes

Definition 1. *Let $\mathfrak{X}$ be a discrete space. A probability measure on $2^{\mathfrak{X}}$ is called a determinantal point process if there exists a function $K : \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$ such that*

$$\operatorname{Prob}\left\{A \in 2^{\mathfrak{X}} \mid A \supset \{x_1, \dots, x_n\}\right\} = \det[K(x_i, x_j)]_{i,j=1}^{n}$$

for any finite subset $\{x_1, \dots .x_n\}$ of $\mathfrak{X}$. The function K is called the correlation kernel. The functions

$$\rho_n : \{n\text{-point subsets of } \mathfrak{X}\} \to [0,1]$$
$$\rho_n : \{x_1, \dots, x_n\} \mapsto \operatorname{Prob}\{A \mid A \supset \{x_1, \dots, x_n\}\}$$

are called the correlation functions.

Example 1. Consider a kernel $L : \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$ such that

- $\det[L(x_i, x_j)]_{i,j=1}^k \geq 0$ for all k-point subsets $\{y_1, \dots, y_k\}$ of $\mathfrak{X}$.
- L defines a trace class operator in $\ell^2(\mathfrak{X})$, for example, $\sum_{x,y \in \mathfrak{X}} |L(x,y)| < \infty$ or L is finite rank. In particular, this condition is empty if $|\mathfrak{X}| < \infty$.

Set

$$\mathrm{Prob}\{\{y_1, \dots, y_k\}\} = \frac{1}{\det(1+L)} \cdot \det[L(y_i, y_j)]_{i,j=1}^k .$$

This defines a probability measure on $2^{\mathfrak{X}}$ concentrated on finite subsets. Moreover, this defines a determinantal point process. The correlation kernel $K(x,y)$ is equal to the matrix of the operator $K = L(1+L)^{-1}$ acting on $\ell^2(\mathfrak{X})$. See [10], [5], Appendix for details.

Definition 2. *Let $\mathfrak{X}$ be a finite or infinite interval inside $\mathbb{R}$ (e.g., $\mathbb{R}$ itself). A probability measure on locally finite subsets of $\mathfrak{X}$ is called a determinantal point process if there exists a function $K : \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$ such that*

$$\lim_{\Delta x_1, \dots, \Delta x_n \to 0} \frac{\mathrm{Prob}\{A \in 2^{\mathfrak{X}}_{loc.fin.} \mid A \textit{ intersects } [x_i, x_i + \Delta x_i] \textit{ for all } i = 1, \dots, n\}}{\Delta x_1 \cdots \Delta x_n} = \det[K(x_i, x_j)]_{i,j=1}^n$$

for any finite subset $\{x_1, \dots, x_n\}$ of $\mathfrak{X}$. The function K is called the correlation kernel and the left–hand side of the equality above is called the nth correlation function.

Example 2. Let $w(x)$ be a positive function on $\mathfrak{X}$ such that all the moments $\int_{\mathfrak{X}} x^n w(x) dx$ are finite. Pick a number $N \in \mathbb{N}$ and define a probability measure on N-point subsets of $\mathfrak{X}$ by the formula

$$P_N(dx_1, \dots, dx_N) = c_N \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 \prod_{1 \leq k \leq N} w(x_k) dx_k .$$

Here $c_N > 0$ is a normalizing constant. This is a determinantal point process. The correlation kernel is equal to the Nth Christoffel–Darboux kernel $K_N(x,y)$ associated with $w(x)$, multiplied by $\sqrt{w(x)w(y)}$. That is, let

$$p_0 = 1, \; p_1(x), \; p_2(x), \dots$$

be monic (= leading coefficient 1) orthogonal polynomials on $\mathfrak{X}$ with the weight function $w(x)$:

$$p_m(x) = x^m + \text{ lower order terms },$$

$$\int_{\mathfrak{X}} p_m(x) p_n(x) w(x) dx = h_m \delta_{mn}, \quad m, n = 0, 1, 2, \dots .$$

Then the correlation kernel is equal to

$$K_N(x,y) = \sum_{k=0}^{N} \frac{p_k(x)p_k(y)}{h_k} \sqrt{w(x)w(y)}$$
$$= \frac{1}{h_{N-1}} \frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{x-y} \sqrt{w(x)w(y)}.$$

The construction of this example also makes sense in the discrete setting. See [12], [18], [19], [15] for details.

Remark 1. The correlation kernel of a determinantal point process is not defined uniquely! In particular, transformations of the form $K(x,y) \to \frac{f(x)}{f(y)}K(x,y)$ do not change the correlation functions.

2 Correlation kernels as integrable operators

Observe that the kernel $K_N(x,y)$ of Example 2 has the form

$$K_N(x,y) = \frac{\phi(x)\psi(y) - \psi(x)\phi(y)}{x-y}$$

for appropriate ϕ and ψ. Most kernels appearing in "$\beta = 2$ ensembles" of Random Matrix Theory have this form, because they are either kernels of Christoffel–Darboux type as in Example 2 above, or scaling limits of such kernels. However, it is an experimental fact that integral operators with such kernels appear in many different areas of mathematics, see [11].

Definition 3. *An integral operator with kernel of the form*

$$\frac{f_1(x)g_1(y) + f_2(x)g_2(y)}{x-y} \tag{1}$$

is called integrable. Here we assume that $f_1(x)g_1(x) + f_2(x)g_2(x) = 0$ so that there is no singularity on the diagonal. Diagonal values of the kernel are then defined by continuity.

The class of integrable operators was singled out in the work of Its, Izergin, Korepin, and Slavnov on quantum inverse scattering method in 1990 [14].

We will also call an operator acting in the ℓ^2-space on a discrete space integrable if its matrix has the form (1). It is not obvious how to define the diagonal entries of a discrete integrable operator in general. However, in all concrete situations we are aware of, this question has a natural answer.

Example 3 (poissonized Plancherel measure, cf. [5]). Consider the probability measure on the set of all Young diagrams given by the formula

$$\mathrm{Prob}\{\lambda\} = e^{-\theta}\theta^{|\lambda|}\left(\frac{\dim\lambda}{|\lambda|!}\right)^2. \tag{2}$$

Here $\theta > 0$ is a parameter, $\dim\lambda$ is the number of standard Young tableaux of shape λ or the dimension of the irreducible representation of the symmetric group $S_{|\lambda|}$ corresponding to λ. Denote by $(p_1,\dots,p_d|q_1,\dots,q_d)$ the Frobenius coordinates of λ (see [17], §1 for the definition of Frobenius coordinates). Here d is the number of diagonal boxes in λ. Set $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2} = \{\pm\frac{1}{2}, \pm\frac{3}{2}, \dots\}$.

Let us associate to any Young diagram $\lambda = (p\,|\,q)$ a point configuration $\mathrm{Fr}(\lambda) \subset \mathbb{Z}'$ as follows:

$$\mathrm{Fr}(\lambda) = \left\{p_1 + \tfrac{1}{2}, \dots, p_d + \tfrac{1}{2}, -q_1 - \tfrac{1}{2}, \dots, -q_d - \tfrac{1}{2}\right\}.$$

It turns out that together with (2) this defines a determinantal point process on $\mathbb{Z}'$. Indeed, the well-known hook formula for $\dim\lambda$ easily implies

$$\begin{aligned}\mathrm{Prob}\{\lambda\} &= e^{-\theta}\left(\det\left[\frac{\theta^{\frac{p_i+q_j}{2}}}{(p_i-\frac{1}{2})!(q_j-\frac{1}{2})!(p_i+q_j)}\right]_{i,j=1}^{d}\right)^2 \\ &= e^{-\theta}\det[L(y_i,y_j)]_{i,j=1}^{2d}\end{aligned}$$

where $\{y_1,\dots,y_{2d}\} = \mathrm{Fr}(\lambda)$, and $L(x,y)$ is a $\mathbb{Z}' \times \mathbb{Z}'$ matrix defined by

$$L(x,y) = \begin{cases} 0, & \text{if } xy > 0, \\ \dfrac{\theta^{\frac{|x|+|y|}{2}}}{(|x|-\frac{1}{2})!(|y|-\frac{1}{2})!}\dfrac{1}{x-y}, & \text{if } xy < 0. \end{cases}$$

In the block form corresponding to the splitting $\mathbb{Z}' = \mathbb{Z}'_+ \sqcup \mathbb{Z}'_-$ it looks as follows

$$L(x,y) = \begin{bmatrix} 0 & \dfrac{\theta^{\frac{x-y}{2}}}{(x-\frac{1}{2})!(-y-\frac{1}{2})!}\dfrac{1}{x-y} \\ \dfrac{\theta^{\frac{-x+y}{2}}}{(-x-\frac{1}{2})!(y-\frac{1}{2})!}\dfrac{1}{x-y} & 0 \end{bmatrix}.$$

The kernel $L(x,y)$ belongs to the class of integrable kernels. Indeed, if we set

$$f_1(x) = g_2(y) = \begin{cases} \dfrac{\theta^{\frac{x}{2}}}{(x-\frac{1}{2})!}, & x > 0, \\ 0, & x < 0, \end{cases} \qquad f_2(x) = g_1(y) = \begin{cases} 0, & x > 0, \\ \dfrac{\theta^{-\frac{x}{2}}}{(-x-\frac{1}{2})!}, & x < 0, \end{cases}$$

then it is immediately verified that $L(x,y) = (f_1(x)g_1(y)+f_2(x)g_2(y))/(x-y)$. Comparing the formulas with Example 1, we also conclude that $e^{\theta} = \det(1+L)$.[1]

[1] Since $\sum_{x,y\in\mathbb{Z}'}|L(x,y)| < \infty$, the operator L is trace class, and $\det(1+L)$ is well–defined.

What we see in this example is that L is an integrable kernel. We also know, see Example 1, that the correlation kernel K is given by $K = L(1+L)^{-1}$. Is this kernel also integrable? The answer is positive; the general claim in the continuous case was proved in [14], the discrete case was worked out in [2].

Furthermore, it turns out that in many situations there is an algorithm of computing the correlation kernel K if L is an integrable kernel which is "simple enough". The algorithm is based on a classical problem of complex analysis called the *Riemann–Hilbert problem* (RHP, for short).

Let us point out that our algorithm is not applicable to deriving correlation kernels in the "$\beta = 2$" model of Random Matrix Theory. Indeed, the Christoffel–Darboux kernels have norm 1, since they are just projection operators. Thus, it is impossible to define the kernel $L = K(1-K)^{-1}$, because $(1-K)$ is not invertible. In this sense, RMT deals with "degenerate" determinantal point processes.

On the other hand, the orthogonal polynomial method of computing the correlation kernels, which has been so successful in RMT, cannot be applied directly to the representation theoretic models like Example 2.2 above (see, however, [15]). The algorithm explained below may be viewed as a substitute for this method.

3 Riemann–Hilbert problem

Let Σ be an oriented contour in $\mathbb{C}$. We agree that $(+)$-side is on the left of the contour, and $(-)$-side is on the right of the contour. Let v be a 2×2-matrix valued function on Σ.

Definition 4. *We say that a matrix function* $m : \mathbb{C}\setminus\Sigma \to \mathrm{Mat}(2,\mathbb{C})$ *solves the RHP* (Σ, v) *if*

(1) m *is analytic in* $\mathbb{C}\setminus\Sigma$*;*

(2) $m_+ = m_- v$ *on* Σ*, where* $m_\pm(x) = \lim\limits_{\zeta\to x \text{ from } (\pm)\text{-side}} m(\zeta)$.

We say that m *solves the normalized RHP* (Σ, v) *if, in addition, we have*

(3) $m(\zeta) \to I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *as* $\zeta \to \infty$.

Next we explain what is a *discrete* Riemann–Hilbert problem (DRHP, for short).

Let X be a locally finite subset of $\mathbb{C}$, and let w be a 2×2-matrix valued function on X.

Definition 5. *We say that a matrix function* $m : \mathbb{C}\setminus X \to \mathrm{Mat}(2, C)$ *solves the DRHP* (X, w) *if*

(1) m *is analytic in* $\mathbb{C}\setminus X$*;*

(2) m *has simple poles at the points of* X*, and*

$$\mathrm{Res}_{\zeta=x}\, m(\zeta) = \lim_{\zeta\to x}\big(m(\zeta)w(x)\big) \quad \text{for any } x \in X.$$

We say that m solves the normalized DRHP (X, w) if

(3) $m(\zeta) \to I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *as* $\zeta \to \infty$.

If the set X is infinite, the last relation should hold when the distance from ζ to X is bounded away from zero.

Our next step is to explain how to reduce, for an integrable operator L, the computation of the operator $K = L(1+L)^{-1}$ to a (discrete or continuous) RHP.

3.1 Continuous picture [14]

Let L be an integrable operator on $L^2(\Sigma, |d\zeta|)$, $\Sigma \subset \mathbb{C}$, with the kernel ($x, y \in \Sigma$)

$$L(x,y) = \frac{f_1(x)g_1(y) + f_2(x)g_2(y)}{x-y}, \qquad f_1(x)g_1(x) + f_2(x)g_2(x) \equiv 0.$$

Assume that $(1+L)$ is invertible.

Theorem 1. *There exists a unique solution of the normalized RHP (Σ, v) with*

$$v = I + 2\pi i \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \begin{bmatrix} g_1 & g_2 \end{bmatrix} = \begin{bmatrix} 1 + 2\pi i f_1 g_1 & 2\pi i f_1 g_2 \\ 2\pi i f_2 g_1 & 1 + 2\pi i f_2 g_2 \end{bmatrix}.$$

For $x \in \Sigma$ set

$$\begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix} = \lim_{\zeta \to x} m(\zeta) \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix},$$

$$\begin{bmatrix} G_1(x) \\ G_2(x) \end{bmatrix} = \lim_{\zeta \to x} m^{-t}(\zeta) \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix}.$$

Then the kernel of the operator $K = L(1+L)^{-1}$ has the form ($x, y \in \Sigma$)

$$K(x,y) = \frac{F_1(x)G_1(y) + F_2(x)G_2(y)}{x-y} \quad \text{and} \quad F_1(x)G_1(x) + F_2(x)G_2(x) \equiv 0.$$

Example 4. Let Σ be a simple closed curve in $\mathbb{C}$ oriented clockwise (so that the (+)-side is outside Σ), and let L be an integrable operator such that the functions f_1, f_2, g_1, g_2 can be extended to analytic functions inside Σ. Then the solution of the normalized RHP (Σ, v) has the form

$$m = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{outside } \Sigma, \\ I - 2\pi i \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \begin{bmatrix} g_1 & g_2 \end{bmatrix} & \text{inside } \Sigma. \end{cases}$$

Then we immediately obtain $F_i = f_i$, $G_i = g_i$, $i = 1, 2$; and $K = L(1+L)^{-1} = L$. On the other hand, this is obvious because $\int_\Sigma L(x,y)L(y,z)dy = 0$ by Cauchy's theorem which means that $L^2 = 0$.