DIGITAL SIGNAL PROCESSING

AND

TIME SERIES ANALYSIS

by

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and
Manuel T. Silvia

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PILOT EDITION

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PREFACE

During the past decade digital signal processing has emerged as a major discipline in electrical engineering. This development has been the result of the marked progress in computer technology. As computers become more accessible, users find new applications and generate new demands for even more sophisticated technology. These events have had a profound impact on the growth of digital signal processing into a distinct subject in the university curriculum.

This book has grown out of our teaching and research experience in the field of digital signal processing. We have written the book in the form of a standard mathematics textbook in the following sense. The book is self-contained and all concepts are defined in mathematical terms as they are introduced. The chapters are divided into articles, and approximately ten problems are given at the end of each article. These problems deal with the subject matter of the article, and the successful working of these problems indicates a mastery of the material in the article. The answers to all the problems are given at the end of the book so the student can check his results himself. Altogether there are over 500 problems, and this collection makes up what we consider to be the most important part of the book.

This book is written primarily for use in electrical engineering departments. It is designed as a text for a senior or first-year graduate level course. It is assumed that the reader has a background in advanced calculus. Because the first chapter is a review of complex variable theory, many instructors will choose to omit Chapter I. When we teach the course, we usually assign selected problems from this chapter during the first week. In this way we can spot any students that might need extra help, and we can direct those students to selected portions of the chapter. Much of Chapter II is numerical analysis, so the material is already quite familiar to students at the senior and graduate level. As a result, the speed with which material is covered is always at the discretion of the instructor. The inclusion of such introductory material is a special feature of this book. It eases the student into digital signal processing without too much of a jolt by appealing to his knowledge of standard mathematical areas as complex variable theory and numerical analysis.

Another special feature of the book is its introductory inclusion of time series analysis. The rapid engineering expansion of computers and digital processing has made possible much more complete analyses of time series data. Signal processing algorithms and their implementation go hand in hand with the study of time series data generated by physical, economic, or biological processes.

This book can be used as a supplementary text in a course in time series analysis as given in mathematics, statistics, and economics departments. The mathematical approach of the book and the inclusion of the introductory material and the problems make the book accessible to students without prior knowledge of electrical engineering. In this way, the book provides an opportunity for a cross-fertilization of ideas between the digital signal processing methods of electrical engineering and time-series methods.

We have found the book especially helpful in teaching industrial courses. All companies today realize the importance of keeping their people abreast of the latest scientific developments. The special features of this book represent a valuable asset in the training of people who have been away from college studies for several years.

Thus, the present book is written for people interested in obtaining a working knowledge of the combined fields of digital signal processing and time series analysis. The emphasis is on the development of an understanding of the concepts involved and an appreciation of the problems faced in applications. The material is presented with sufficient depth to provide the reader with the background necessary to understand the details of the methods. These methods are now being applied to such diverse areas as acoustics, biomedical engineering, data communication and telephony, economics and business analysis, geophysics, picture and image enhancement, nuclear science, oceanography, sonar, and speech communication and recognition.

In summary, this book presents in a mathematical way the main body of knowledge of digital signal processing. It is aimed at the engineering student to ease his entrance into this field, to the time-series student so he can incorporate these methods into his area of research, and to the practicing engineer or scientist to facilitate his job of staying abreast new developments in his field.

We want to express our sincere thanks to Professor George B. Thomas of the Massachusetts Institute of Technology for many valuable insights in the writing of this book. We want to thank Mrs. Gardiner W. White for her excellent work in the typing of the manuscript.

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CHAPTER I

COMPLEX VARIABLES AND PHASORS

1.1 The real and complex number systems

The majority of the techniques and considerations in digital signal processing are based on a knowledge of complex numbers. In other areas of study, e.g. the Laplace transform, the frequency response of linear systems, the solution of differential equations, the use of complex numbers are necessary or at least convenient. In electrical engineering, complex numbers are useful for determining the steady-state behavior of electrical circuits and they appear in the complex exponential form of Fourier series. As we shall see, the study of digital signals and systems is simplified by the introduction of complex numbers. A satisfactory discussion of the main concepts of digital signal processing must be based on an accurately defined number concept. In this context, let us review the real and complex number systems.

In ancient times, the concept of length was an important use of numbers. For example, in terms of some reference measure (i.e. the foot of an individual), the height of a tree was 50 feet, the distance between two villages was 3500 feet, the length of a creek was 35,000 feet, and so on. It was natural and physically satisfying for length to be a positive number. Hence, people measured length in terms of positive *rational numbers,* i.e. numbers of the form m/n where m and n are integers and $m \neq 0$. For most situations, this system was sufficient. However, if one desired to know the length of the hypotenuse of a right triangle with equal sides of unit length, the answer could not be found in the system of rational numbers, for $\sqrt{2}$ is not a rational number. Thus, the rational number system was inadequate for a general measure of length and this led to the so-called *irrational numbers,* i.e. numbers not capable of being expressed exactly as the ratio of two integers. Further, by combining the rational numbers with the irrational numbers we form the *real number* system.

The purpose of the above discussion has been to show that the rational number system has certain gaps, in spite of the fact that between any two rationals there is another. The real number system fills these gaps. This is the principal reason for the fundamental role which it plays in analysis.

The concept of a real number is so familiar to all of us that we sometimes take for granted that our entire number system can be viewed as an extension of real numbers. For example, the length of a football field, the temperature in Alaska during the winter months, or the roots of the equation $x^2-1 = 0$ are all real numbers. The length of a football field (100 yards) is a positive real number. On the other hand, the temperature in Alaska during the winter months might reach -20°C, which is a negative real number. We say that the equation $x^2-1 = 0$ has real roots $x = -1$ and $x = +1$ and the sequence $\{1,0,3.14159,\sqrt{2},2/3\}$ represents a finite-length sequence of real numbers. Further, the fundamental units of mass, length, and time and dimensional quantities such as energy and speed are expressed as real numbers. However, let us consider the roots of the equation $x^2+1 = 0$. The solution of this equation is $x = \sqrt{-1}$ and $x = -\sqrt{-1}$, but since the square root of a negative number is not defined in the real number system, we need a new number system in order to furnish a solution for this situation. Just as the irrational numbers filled the gaps contained in the rational number system, the complex numbers fill the gaps contained in the real number system. The key to the complex number is the imaginary unit

$$i \equiv \sqrt{-1}$$

Thus, the solution of the equation $x^2 + 1 = 0$ is now represented by $x = \pm i$, where i is a 'new' unit not contained in the set of real numbers. Let us consider the solution of the system $x^2 - x + 1 = 0$. By the quadratic formula we obtain

$$x = \frac{1}{2} \pm \frac{\sqrt{-3}}{2}$$

Thus, the two roots of this system are $x = \frac{1}{2} + i\,\frac{\sqrt{3}}{2}$ and $x = \frac{1}{2} - i\,\frac{\sqrt{3}}{2}$.

We see that these roots are expressed as a combination of real numbers and the imaginary unit i. The plus sign (+) in the root

$$x = \frac{1}{2} + i\,\frac{\sqrt{3}}{2},$$

however, does not mean addition, for addition makes sense only when we are adding real numbers to real numbers. Similarly, the minus sign suggesting subtraction (-) in the root

$$x = \frac{1}{2} - i\,\frac{\sqrt{3}}{2}$$

does not denote subtraction, since subtraction is valid only when we subtract real numbers from real numbers. In general, if x is a real number, y is a real number, and i is the previously defined imaginary unit, then what does the representation $x + iy$ mean? At this point we need a definition.

We now define a *complex number* as an ordered pair of real numbers denoted by (x,y). Ordered means that (x,y) and (y,x) are regarded as distinct if $x \neq y$.

The complex numbers are subject to certain laws which we now give.

<u>Equality</u>: $(x_1,y_1) = (x_2,y_2)$ if and only if $x_1 = x_2$ and $y_1 = y_2$

<u>Addition</u>: $(x_1,y_1) + (x_2,y_2) = (x_1+x_2,y_1+y_2)$ (1)

<u>Multiplication</u>: $(x_1,y_1)(x_2,y_2) = (x_1x_2-y_1y_2,x_1y_2+y_1x_2)$;

$C(x_1,y_1) = (Cx_1,Cy_1)$ for any real number C.

We shall usually use the more customary notation

$$z = x + iy \tag{2}$$

to denote the complex number (x,y). The real number x is called the *real part* of z and is denoted as

$$x = \mathrm{Re}(z) \tag{3}$$

The real number y is called the *imaginary part* of z and is denoted as

$$y = \mathrm{Im}(z) \tag{4}$$

The above laws in the customary notation are

<u>Equality</u>: $x_1+iy_1 = x_2+iy_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$

<u>Addition</u>: $(x_1+iy_1) + (x_2+iy_2) = (x_1+x_2) + i(y_1+y_2)$

<u>Multiplication</u>: $(x_1+iy_1)(x_2+iy_2) = (x_1x_2-y_1y_2) + i(x_1y_2+y_1x_2)$;

$C(x_1+iy_1) = Cx_1+iCy_1$ for any real number C.

We see that two complex numbers x_1+iy_1 and x_2+iy_2 are equal if and only if the real and imaginary parts of the first are equal respectively to the real and imaginary parts of the second. Thus, in considering equality between two complex numbers, we must include both components of the complex number, namely, the real part and the imaginary part. Moreover, the vanishing of a complex number implies not one but two conditions, namely, that both the real part and the imaginary part of the given number are zero.

The complex number

$$z^* = x - iy \tag{5}$$

is called the *complex conjugate* of z, or simply the *conjugate* of z. The following rules hold:

(a) $(z^*)^* = z$

(b) $z + z^* = 2\mathrm{Re}(z)$

(c) $z - z^* = 2i\mathrm{Im}(z)$

(d) $(z_1+z_2)^* = z_1^*+z_2^*$ (6)

(e) $z_1z_2^* = z_2^*z_1$

(f) $(z_1z_2)^* = z_1^*z_2^*$

(g) zz^* is real and positive (except when $z = 0$)

The proofs of (a) - (f) follow directly by use of the definitions (2) - (5). To prove (g), we form

$$\begin{aligned} zz^* &= (x+iy)(x-iy) \\ &= x^2+y^2 \end{aligned} \tag{7}$$

and note that x^2+y^2 is a positive real number.

If z is a complex number, its *absolute* value $|z|$ is defined as the non-negative square root of zz^*; that is

$$|z| = (zz^*)^{1/2} \qquad (8)$$

The following rules hold

(a) $|z| > 0$ if $z \neq 0$ and $|z| = 0$ if $z = 0$.

(b) $|z^*| = |z|$

(c) $|z_1 z_2| = |z_1||z_2|$ (9)

(d) $|\mathrm{Re}(z)| \leq |z|$

(e) $|z_1+z_2| \leq |z_1|+|z_2|$ (triangle inequality)

The proofs of (a) and (b) follow from definition (8). To prove (c), we form

$$|z_1z_2|^2 = (z_1z_2)(z_1z_2)^* = (z_1z_2)(z_1^*z_2^*)$$
$$= (z_1z_1^*)(z_2z_2^*) = |z_1|^2|z_2|^2 \qquad (10)$$

and the desired result follows by taking the positive square root of each side. To prove (d), we note that $x^2 \leq x^2+y^2$. Hence

$$|x| = \sqrt{x^2} \leq \sqrt{x^2+y^2}$$

Since $|x| = |\mathrm{Re}(z)|$ and $|z| = \sqrt{x^2+y^2}$, the desired result follows.

To prove the triangle inequality (e), we note that $z_1^*z_2$ is the conjugate of $z_1z_2^*$. Using 6(b) it follows that

$$z_1z_2^* + z_1^*z_2 = 2\,\mathrm{Re}(z_1z_2^*) \qquad (11)$$

We now form

$$|z_1+z_2|^2 = (z_1z_2)(z_1+z_2)^*$$
$$= (z_1+z_2)(z_1^*+z_2^*)$$
$$= z_1z_1^* + z_1z_2^* + z_2z_1^* + z_2z_2^* \qquad (12)$$

Rewriting equation (12) with the aid of equation (11) we obtain

$$|z_1+z_2|^2 = |z_1|^2 + 2\mathrm{Re}(z_1z_2^*) + |z_2|^2 \qquad (13)$$

We observe that for any complex number z,

$$-|z| \leq \mathrm{Re}(z) \leq |z| \qquad (14)$$
$$-|z| \leq \mathrm{Im}(z) \leq |z|$$

Hence,

$$\mathrm{Re}(z_1 z_2^*) \le |z_1 z_2^*| = |z_1||z_2^*|$$

or

$$\mathrm{Re}(z_1 z_2^*) \le |z_1||z_2| \tag{15}$$

Substitution of equation (15) in (13) yields

$$|z_1+z_2|^2 \le |z_1|^2 + 2|z_1||z_2| + |z_2|^2$$

or

$$|z_1+z_2|^2 \le (|z_1|+|z_2|)^2 \tag{16}$$

The triangle inequality (e) follows by taking the square root of both sides of equation (16).

The definitions and algebraic manipulations discussed above play an essential role in digital signal processing. Frequently, we shall not only be concerned with the complex number z, its conjugate z*, and its absolute value $|z|$, but also with its reciprocal $1/z = z^{-1}$. In particular, if $z \neq 0$ then

$$\frac{1}{z} = \frac{z^*}{zz^*} = \frac{z^*}{|z|^2} \tag{17}$$

or

$$\frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} \tag{18}$$

The quotient z_1/z_2 can be simplified by multiplying both numerator and denominator by the conjugate of the denominator:

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)} \tag{19}$$

or

$$\frac{z_1}{z_2} = \frac{(x_1x_2+y_1y_2)}{x_2^2+y_2^2} + i\frac{(x_2y_1-x_1y_2)}{x_2^2+y_2^2} \tag{20}$$

The real numbers, which include both the rational and irrational numbers, are *ordered*. Thus, inequality relations are valid; for example: if $a<b$ and $k>0$, then we know that $ka<kb$. For example, $2(-2/3)<2(1)$ or $-4/3<2$. However, the use of inequality relations between complex numbers does not make sense. To see this, let us consider one of the simplest complex numbers, namely, the imaginary unit i. If the complex numbers could be ordered, then we would have a statement such as $-i<o<i$. Multiplying this inequality by i we obtain

$$-i^2<o<i^2$$

or

$$1<o<-1$$

which is a contradiction. Thus, the complex numbers cannot be ordered, i.e., the use of inequality relations between complex numbers is not meaningful. We note that in some of our manipulations involving $|z|$, Re(z), and Im(z) we used inequality relations. This usage is valid because $|z|$, Re(z), and Im(z) are *real numbers*.

PROBLEMS FOR ARTICLE 1.1

In Problems 1-8 reduce the numbers to the form $z = x + iy$

1. $(4 + 3i) - (2 + i)$

2. $(5 - 2i)(-3 + i)$

3. $i(3 - 5i)$

4. $(2 - i)/(1 + i)$

5. $(1 - i)^3$

6. $\frac{1}{10}(3 - i)(2 + i)(3 + i)$

7. $\dfrac{1 + 3i}{4-i} + \dfrac{2 - i}{5i}$

8. $\dfrac{2i}{(i-3)(i+2)}$

9. Solve for the real numbers x and y:

$$(2 - 3i)^2 - 3(x + iy) = x - iy$$

10. Show that each of $z = 1 + i$ and $z = 1 - i$ satisfies $z^2 - 2z + 2 = 0$.

11. Write $i = (0,1)$. Find i^2, $-i$, i^3, i^4 in this notation.

12. If a and b are complex numbers, under what conditions does the following inequality hold:

$$\left|\frac{a - b}{1-a^*b}\right| < 1$$

In Exercises 13-16 find Re(z), Im(z), z^*, $|z|$, and z^{-1}

13. $z = 3 - i3$

14. $z = -i4$

15. $z = 5$

16. $z = 5 + 2i$

1.2 The complex plane

Much of our work in digital signal analysis will be centered on discussions involving the complex plane. It is important that we define the concept of the complex plane and its relationship to the complex number z.

There is a one-to-one correspondence between ordered pairs of real numbers (x,y) and complex numbers $x + iy$. For example, corresponding to the ordered pair (3,-5) or $x = 3$, $y = -5$, is the complex number $3 - 5i$, and conversely. It is natural to associate the complex number $x + iy$ with the point that has rectangular Cartesian coordinates (x,y) in the x-y plane. We say a complex number z is in *rectangular* or *Cartesian* form if it is written as

$$z = x + iy \tag{1}$$

where x and y are real numbers.

In plotting the real function y(x) in the x-y plane, we plot x as the abscissa on the x-axis and y as the ordinate on the y-axis. In an analogous fashion, we plot $\mathrm{Re}(z) = x$ as the abscissa on the x-axis and $\mathrm{Im}(z) = y$ as the ordinate on the y-axis when representing a complex number z in Cartesian form. However, in the case of complex numbers, we shall refer to the x-axis as the *real axis* and the y-axis as the *imaginary axis*. This two-dimensional space is referred to as the *complex plane* or the *z-plane* and is shown in Figure 1-1.

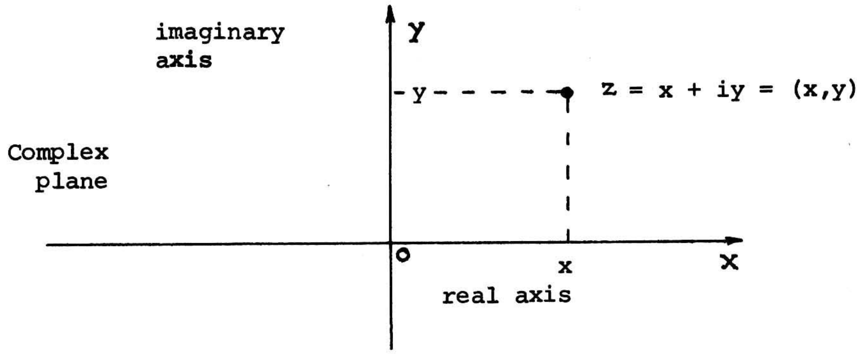


Figure 1-1

Cartesian or rectangular form of the complex number z in the complex plane.

Now consider the point $z = x + iy$ in the complex plane. This point also has polar coordinates (ρ,θ):

$$\rho = \sqrt{x^2 + y^2} = |z| \tag{2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

In evaluating the angle θ, care should be taken in determining the correct quadrant, for $\tan^{-1}(y/x) = \tan^{-1}(-y/-x)$ and $\tan^{-1}(y/-x) = \tan^{-1}(-y/x)$.

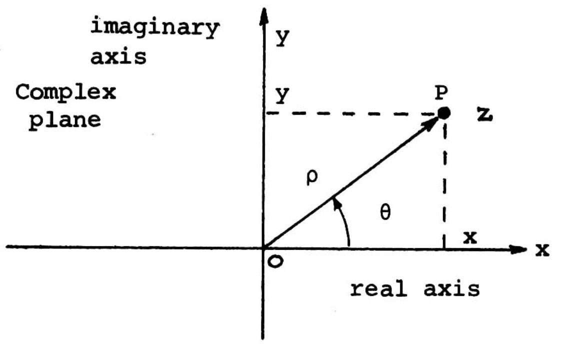
The type of representation given by equation (2) is known as the *polar* form of a complex number. The polar coordinate ρ is the absolute value of z; that is

$$\rho = |z|$$

The absolute value $|z|$ is called the *modulus* or *magnitude* of z. The polar coordinate θ is called the *argument* of z, which is written as

$$\theta = \arg(z)$$

The argument arg(z) is also called the *angle* or *phase* of z. In fact, we can think of the polar form of z as being represented by a vector or directed line segment which joins the origin (0,0) to the point (x,y). If z is represented as a vector, then the vector length is $\rho = |z|$ and the direction angle of this vector is $\theta = \arg(z)$. The polar form of the complex number z is illustrated in Figure 1-2, together with the relationship of the polar coordinates to the Cartesian coordinates.



$$\rho = |z| = \sqrt{x^2+y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$x = \rho \cos\theta$$
$$y = \rho \sin\theta$$

Figure 1-2

Polar form of the complex number z, represented by the vector OP with magnitude $|z|$ and angle θ.

From Figure 1-2, we see that

$$x = \rho \cos\theta = |z| \cos\theta$$
$$y = \rho \sin\theta = |z| \sin\theta \tag{3}$$

Thus, the complex number $z = x + iy$ can be written in the equivalent polar form

$$z = \rho \cos\theta + i\rho\sin\theta$$
$$= |z|(\cos\theta + i\sin\theta) \tag{4}$$

Since $\cos(\theta \pm 2n\pi) = \cos\theta$ and $\sin(\theta \pm 2n\pi) = \sin\theta$ for $n = 0,1,2,\ldots$, we notice that θ plus any integer multiple of 2π can be substituted in equation (4) without changing the value of z.

For the polar representation of complex numbers, we sometimes use the notation

$$z = \rho \angle \theta = |z| \angle \theta \tag{5}$$

The above is interpreted as "the complex number z with magnitude $|z|$ at an angle θ".

If we have two complex numbers z_1 and z_2 in polar form, their product can be written as

$$z_1 z_2 = \left[\rho_1(\cos\theta_1 + i\sin\theta_1)\right]\left[\rho_2(\cos\theta_2 + i\sin\theta_2)\right]$$
$$= \rho_1\rho_2\left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right] \tag{6}$$
$$= \rho_1\rho_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$

and their quotient (for $z_2 \neq 0$) can be written as

$$\frac{z_1}{z_2} = \frac{\rho_1(\cos\theta_1 + i\sin\theta_1)}{\rho_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{\rho_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{\rho_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)} \tag{7}$$

$$= \frac{\rho_1}{\rho_2}\,\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{\rho_1}{\rho_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$$

Further, with the notation defined in equation (5), these results can be written as

$$z_1 z_2 = (\rho_1\rho_2)\angle(\theta_1+\theta_2)$$
$$\frac{z_1}{z_2} = \left(\frac{\rho_1}{\rho_2}\right)\angle(\theta_1-\theta_2)\ ,\ \rho_2 \neq 0 \tag{8}$$