Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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H. Heyer (Ed.)

Probability Measures on Groups IX

Proceedings, Oberwolfach 1988

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Editor

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PREFACE

The 1988 Conference "Probability Measures on Groups" was designed, in deviation from the previous meetings on the subject, as a forum on which the interplay between structural probability theory and seemingly quite distant topics of pure and applied mathematics were to be discussed and documented. The various interrelations were made explicit in the expository talks presented on special invitation. The organizers are very grateful to their colleagues for having prepared these special talks and for having submitted the underlying manuscripts in elaborated form to the Proceedings. Here is the list of the survey speakers and the titles of their talks

N.H. BINGHAM, The Royal Holloway and Bedford New College, Egham, England

Tauberian methods in probability theory

T. HIDA, Nagoya University, Nagoya, Japan

Infinite dimensional rotation group and unitary group

M.M. RAO, University of California at Riverside, Riverside, USA

Bimeasures and harmonizable processes

A. TERRAS, University of California at San Diego, La Jolla, USA

The central limit theorem for symmetric spaces of GL(3)

G.S. WATSON, Princeton University, Princeton, USA

Statistics of rotations.

In the following we shall briefly comment on the research contributions contained in this volume.

There has been a wide range of topics with some emphasis on convolution semigroups of measures, their potential theory and harmonic analysis. While at the preceding conferences on probability measures on groups the basic structure was mainly a group or a semigroup, the recent development emphasizes more general group-like structures.

These generalized translation or hypergroup structures, probabilistic in nature, appear in duals of non-Abelian groups, orbit and double-coset spaces, but also in the spaces $\mathbb{Z}_+$, $\mathbb{R}_+$, the unit interval or the unit disk where they differ significantly from the usual semigroup structures.

The research articles of this volume can be classified in four sections.

Probabilities and potentials on hypergroups. In this new branch of structural probability theory the concentration is on the study of random walks and more general space-homogeneous Markov processes. W.R. BLOOM and H. HEYER characterize the potentials of transient convolution semigroups in terms of Deny type fundamental families. L. GALLARDO supplements his previous paper on the Poisson representation of infinitely divisible probability measures. A. KUMAR and A.I. SINGH present a dichotomy theorem for random walks. In R. LASSER's paper stationary processes over polynomial hypergroups are studied. M. VOIT contributes to the problem of Schoenberg's duality for positive and negative definite functions. An extension of the theory of L^p-improving measures to hypergroups is given by R.C. VREM. Hm. ZEUNER continues in his article his previous work on one-dimensional hypergroups.

Probability measures on semigroups and groups. In this section the authors present new results pertaining to well-established open problems. M.S. BINGHAM proves the existence of a conditional probability distribution on an Abelian group. S.G. DANI and M. McCRUDDEN deal with a special aspect of the embedding problem for infinitely divisible probability measures on Lie groups. The paper of W. HAZOD and S. NOBEL concerns the convergence-of-types theorem which has implications to the theory of stable and semistable measures, a topic which is covered in E. SIEBERT's article. A. MUKHERJEA's results on the convergence of convolution products of probability measures shed new light on an important problem. To study roots of Haar measure on a compact group is the aim of a paper by G. TURNWALD.

Structure of special distributions and processes. This section documents in a particularly impressive way a variety of problems where general structural insight helps analyzing classical problems. J.L. DUNAU and H. SENATEUR discuss some generalizations of the Cauchy distribution. Y. HE applies an extended Delphic theory to point processes. In G. HÖGNÄS' paper the semigroup of analytic mappings is studied from some probabilistic point of view. J. KISYŃSKI presents in his contribution a detailed treatment of Feller generators and their processes. G. LETAC deals with the classification problem for exponential families. Divisibility problems immanent to the Hungarian contributions to structural probability theory are the topic of A. ZEMPLENI's article.

Harmonic analysis in probability theory on semigroups, groups and algebras. A. DERIGHETTI's paper is on the induction of convolution operators. Ph. FEINSILVER and R. SCHOTT treat stochastic processes with values in a Lie group by replacing their trajectories by smooth vector-valued curves. N. OBATA studies the Lévy Laplacian and its potential theory. Mixtures of characters of semigroups constitute the contents of P. RESSEL's article. In M. SCHÜRMANN's contribution infinitely divisible states and their impact to quantum stochastics are studied. K. YLINEN presents further results in the theory of random fields over a locally compact group.

From this short survey we are justified to conclude that the theory of probability on algebraic-topological structures not only develops by deepening and by extending the fundamental problems towards important generalizations and applications, but also by broadening its scope into other fields of mathematics, physics and statistics. Here we mention classical analysis (Tauberian theory), number theory (Helgason transfrom), stochastic analysis (Hida calculus), quantum stochastics (infinitely divisible states and instruments), data analysis (stationary processes and fields) and invariant decision theory (statistics on homogeneous spaces).

It goes without special mention that the 52 participants of the Conference who travelled to Oberwolfach from 13 different countries (including Germany) enjoyed the outstanding hospitality of the Forschungsinstitut and appreciated its inspiring atmosphere. The well-known recognition of Oberwolfach as a research center of international rank made it possible to attract specialists from all over the world to cooperate and give shape to the meeting.

The organizers of the Conference and the editor of these Proceedings extend their heartfelt thanks to all participants and contributors.

Tübingen, October 1988

Herbert Heyer

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The authors Y. He and A. Zempléni were kind enough to submit their manuscripts although they were prevented from participating at the conference.

A Fourier-analytic proof that conditional probability distributions exist on a group

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Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a measurable space $(G, \mathcal{B})$. If $\mathcal{A}$ is a sub-σ-field of $\mathcal{F}$, it is well known that the conditional probability distribution of X given $\mathcal{A}$ does not necessarily exist unless suitable extra conditions are assumed. The purpose of this paper is to present a new proof based on Fourier analysis of the following existence result.

Theorem Let G be a locally compact second countable abelian group with Borel σ-field $\mathcal{B}$ and let X be a G-valued, $\mathcal{B}$-measurable random variable defined on a probability space $(\Omega, \mathcal{F}, P)$. Then for any sub-σ-field $\mathcal{A}$ of $\mathcal{F}$ there exists on $(G, \mathcal{B})$ a conditional probability distribution of X given $\mathcal{A}$.

From the above theorem we can deduce the following well known more general result.

Corollary Let X be a random variable defined on the probability space $(\Omega, \mathcal{F}, P)$ and taking values in the standard Borel space $(E, \mathcal{E})$. Then for any sub-σ-field $\mathcal{A}$ of $\mathcal{F}$ there exists on $(E, \mathcal{E})$ a conditional probability distribution of X given $\mathcal{A}$.

Conditional probability distributions on standard Borel spaces are discussed in Parthasarathy (1967). See also Breiman (1968).

Throughout this article we shall suppose that G is a locally compact second countable abelian group and that $\hat{G}$ is the dual group of G. Thus $\hat{G}$ consists of all the (continuous) characters of G, i.e. the continuous homomorphisms of G into the unit circle group. $\hat{G}$ is endowed with its natural group structure and the compact open topology, which together make $\hat{G}$ into a locally compact second countable abelian group. Complex finite linear combinations of elements of $\hat{G}$ are called trigonometric polynomials on G. References for information on locally compact abelian groups and their duals include Hewitt and Ross (1963, 1970) and Rudin (1962).

Proof of the Theorem. Choose a countable dense subset of C of $\hat{G}$ and assume without loss of generality that the identity e of $\hat{G}$ is in C. Choose also an increasing sequence (L_m) of compact subsets of G with union G and define

$M_{m+1} := L_{m+1} \setminus L_m \quad (m \geq 1), \; M_1 := L_1.$

For each $y \varepsilon C$ and $m = 1,2,\ldots$ choose a version $\phi(y,m)$ of the conditional expectation $E[y(X)1(X\varepsilon M_m)|\mathcal{A}]$, where $1(F)$ indicates the indicator function of the event F. Then there exists $A\varepsilon\mathcal{A}$ with $P(A) = 1$ such that

$$\sum_{m=1}^{\infty} \phi(e,m) = 1 \tag{1}$$

holds for all $\omega\varepsilon A$. For each $\omega \notin A$ redefine $\phi(e,m)$ to be $P(X\varepsilon M_m)$ for all m. Then (1) holds for all $\omega\varepsilon\Omega$ and $\phi(e,m)$ is a version of $E[1(X\varepsilon M_m)|\mathcal{A}]$ for all m.

Now choose a sequence (U_k) of subsets of $\hat{G}$ which form a basis for the topology of $\hat{G}$ at e. Then for all positive integers m, ℓ we have

$$\sup_{x\varepsilon M_m} |y_1(x) - y_2(x)| \leq \sup_{x\varepsilon L_m} |(y_1 - y_2)(x) - 1| < \frac{1}{\ell}$$

whenever $y_1, y_2 \varepsilon C$, $y_1 - y_2 \varepsilon U_k$ and k is sufficiently large. Then

$|\phi(y_1,m) - \phi(y_2,m)| < 1/\ell$ a.s.P

whenever $y_1, y_2 \varepsilon C$, $y_1 - y_2 \varepsilon U_k$ and k is sufficiently large. As C is countable, it follows that there exists $A_m \varepsilon \mathcal{A}$ with $P(A_m) = 1$ such that

$$\phi(y_1,m) - \phi(y_2,m) \to 0 \text{ for all } \omega\varepsilon A_m$$

as $y_1 - y_2 \to e$ with $y_1, y_2 \varepsilon C$. Now take $A' := \bigcap_{m=1}^{\infty} A_m \varepsilon \mathcal{A}$.

Then $P(A') = 1$ and, for each $\omega\varepsilon A'$, each of the functions $y \to \phi(y,m)$ is uniformly continuous on C. For $\omega \notin A'$ redefine $\phi(y,m)$ to be $P(X\varepsilon M_m)$.

Next,

$$|y(X)1(X\varepsilon M_m)| = 1(X\varepsilon M_m)$$

so

$$|\phi(y,m)| \leq \phi(e,m) \tag{2}$$

holds a.s.P for each $y\varepsilon C$ and each m. As C is countable there exists $A''\varepsilon\mathcal{A}$ with $P(A'') = 1$ such that (2) holds for all $\omega\varepsilon A''$, all $y\varepsilon C$ and all m. Define

$$\phi(y) := \begin{cases} \sum_{m=1}^{\infty} \phi(y,m) & \text{if } \omega\varepsilon A'' \\ 1 & \text{if } \omega \notin A'' \end{cases}$$

Then $\phi(y)$ is always finite and, by the dominated convergence theorem, ϕ is a uniformly continuous function of y on C for each $\omega\varepsilon\Omega$. Also $\phi(e) = 1$ for all $\omega\varepsilon\Omega$ and $\phi(y)$ is a version of $E[y(X)|\mathcal{A}]$ for all $y\varepsilon C$.

For each positive integer n, complex numbers $\alpha_1, \ldots, \alpha_n$ and characters $y_1,\ldots,y_n$ in C we have

$$\sum_{r=1}^{n}\sum_{s=1}^{n} \alpha_r\overline{\alpha_s}\,\phi(y_r - y_s) = E[\,|\sum_{r=1}^{n}\alpha_r y_r(X)|^2\,|\mathcal{A}] \geq 0 \text{ a.s.P.}$$

Therefore there exists $A''' \epsilon \mathcal{A}$ with $P(A''') = 1$ such that

$$\sum_{r=1}^{n}\sum_{s=1}^{n} \alpha_r\,\overline{\alpha_s}\,\phi(y_r - y_s) \geq 0 \tag{3}$$

holds for all positive integers n, all complex numbers $\alpha_1,\ldots\alpha_n$ (with rational real and imaginary parts) and all $y_1,\ldots y_n \epsilon C$ whenever $\omega\epsilon A'''$. Redefine $\phi(y) = 1$ for all $y\epsilon C$ whenever $\omega\not\epsilon A'''$.

For each $\omega\epsilon\Omega$ the function $y \to \phi(y)$ is uniformly continuous on C and so has a unique extension to a continuous function on $\hat{G}$, which we again denote by ϕ. Then, for each fixed $y\epsilon\hat{G}$, $\phi(y)$ is a version of $E[y(X)|\mathcal{A}]$ and for each fixed $\omega\epsilon\Omega$ we have

(i) $\phi(e) = 1$

(ii) ϕ is continuous

(iii) (3) holds for all choices of positive integer n, complex numbers $\alpha_1,\ldots\alpha_n$ and $y_1\ldots,y_n\epsilon\hat{G}$.

By Bochner's theorem (see for example Parthasarathy (1967), Heyer (1978), Rudin (1962) or Hewitt and Ross (1970)) there is, for each fixed $\omega\epsilon\Omega$, a probability measure

$$B \to \mu(B,\omega) \quad , \quad B\epsilon\mathcal{B}$$

on $(G,\mathcal{B})$ such that

$$\int y(x)\mu(dx,\omega) = \phi(y)(\omega)$$

for all $y\epsilon\hat{G}$ and all $\omega\epsilon\Omega$.

To complete the proof we show that $\mu:\mathcal{B}\times\Omega\to[o,1]$ is a conditional probability distribution of X given $\mathcal{A}$. In order to do so it remains to be proved that, for each fixed $B\epsilon\mathcal{B}$,

$$\omega\to\mu(B,\omega) \quad \text{is} \quad \mathcal{A}\text{-measurable} \tag{4}$$

and

$$\int_A \mu(B,\omega)\, dP(\omega) = P([X\epsilon B]\cap A) \text{ for all } A\epsilon\mathcal{A} \tag{5}$$

or, in other words, that

$$\omega\to\int f(x)\mu(dx,\omega) \text{ is } \mathcal{A}\text{-measurable} \tag{6}$$

and

$$\int_A (\int f(x)\mu(dx,\omega))dP(\omega) = \int_A f(X)\, dP \text{ for all } A\epsilon\mathcal{A} \tag{7}$$

whenever f is the indicator function of a set $B\epsilon\mathcal{B}$.

By our choices of μ and ϕ we know that (6) and (7) hold whenever $f\epsilon\hat{G}$. Therefore they also hold whenever f is a trigonometric polynomial on G.

Let B be a closed subset of G and let f be the indicator function of B. For each positive integer n define

$$g_n(x) = \frac{1}{nd(x,B)+1}, \quad x \varepsilon G$$

where d is a metric for the topology of G. Each g_n is a real bounded continuous function on G and by the proof of (31.4) in Hewitt and Ross (1970) there exists a trigonometric polynomial f_n on G such that

$$\sup_{x \varepsilon L_n} |g_n(x) - f_n(x)| < 1/n$$

and

$$\sup_{x \varepsilon G} |f_n(x)| \leq \sup_{x \varepsilon G} |g_n(x)| + \frac{1}{n} \leq 2.$$

Thus, f is the pointwise limit of a uniformly bounded sequence of trigonometric polynomials (f_n). The dominated convergence theorem enables us to deduce the validity of (6) and (7) for f from their validity for each f_n.

The class of sets B in $\mathcal{B}$ for which (4) and (5) hold is obviously a sub-σ-field of $\mathcal{B}$ and the last paragraph shows that it contains all the closed subsets of G. Therefore (4) and (5) hold for all $B \varepsilon \mathcal{B}$ and the proof is complete.

To deduce the corollary, first note that any standard Borel space is Borel isomorphic to a Borel subspace of $(M, \mathcal{B}_M)$ where M denotes the space of all infinite sequences of 0's and 1's (with the product topology) and $\mathcal{B}_M$ is the Borel σ-field of M (See Theorem 2.3, page 8 of Parthasarathy (1967)). In turn $(M, \mathcal{B}_M)$ can obviously be identified with a Borel subspace of $(T^\infty, \mathcal{B}_{T^\infty})$, the compact second countable abelian group obtained by taking the product of an infinite sequence of copies of the unit circle group T. Thus, it is enough to establish the corollary for the case when E is a Borel subset of T^∞ and $\mathcal{E} = E \cap \mathcal{B}_{T^\infty}$, where $\mathcal{B}_{T^\infty}$ is the Borel σ-field of T^∞.

Let μ be the conditional distribution of X given by the theorem when X is regarded as T^∞-valued. Since $P(X \varepsilon E) = 1$ there is a set $A \varepsilon \mathcal{A}$ with $P(A) = 1$ such that

$\mu(E,\omega) = 1$ for all $\omega \varepsilon A$.

Choose an arbitrary probability measure ν on $(E, \mathcal{E})$ and for each $\omega \notin A$ redefine $\mu(.,\omega) = \nu(.)$. Then μ restricted to $(E, \mathcal{E})$ is a conditional probability distribution for X.

Alternatively the corollary can be deduced as in Breiman (1968) from the special case of the theorem in which G is the real line.

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TAUBERIAN THEOREMS IN PROBABILITY THEORY

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§1. WIENER THEORY

We begin with the classical Wiener Tauberian theory for the real line, in various formulations. The basic result is

Wiener's Approximation Theorem for $L^1(\mathbb{R})$. For $f \in L^1$, the following are equivalent:

(i) The Fourier transform $\hat{f}(t) := \int_{\mathbb{R}} e^{ixt} f(x)dx$ has no zeros $t \in \mathbb{R}$,

(ii) Linear combinations of translates of f are dense in L^1,

(iii) For $g \in L^\infty(\mathbb{R})$, the integral equation

$$(f*g)(x) := \int_{\mathbb{R}} f(x-y)g(y)dy \equiv 0$$

has only the trivial solution $g = 0$ a.e.

For proof, see Wiener (1932), (1933), Reiter (1968), I. As a corollary, one obtains

Wiener's First Tauberian Theorem. If $f \in L^1$ and $\hat{f}$ has no real zeros, $g \in L^\infty$, then

$$(f*g)(x) \to c \int f \qquad (x \to \infty)$$

implies

$$(h*g)(x) \to c \int h \qquad (x \to \infty)$$

for all $h \in L^1$.

A useful variant is obtained by demanding more of f and less of g. Write M for the subclass of L^1 consisting of continuous functions f with

$$\Sigma_{n=-\infty}^{\infty} \sup_{[n,n+1)} |f(.)| < \infty, \tag{1}$$

M^* for the class of signed measures $\mu = \mu_+ - \mu_-$ whose 'modulus' $|\mu| := \mu_+ + \mu_-$ satisfies

$$\sup_x |d\mu|([x,x+1)) < \infty \tag{2}$$

(thus if $\mu = \int g$ with $g \in L^{\infty}$, then $\mu \in M^*$). One has (Wiener (1932), Reiter (1968))

<u>Wiener's Second Tauberian Theorem.</u> If $f \in M$ with $\hat{f}$ non-vanishing, $\mu \in M^*$, then

$$(f*\mu)(x) := \int f(x-y)\,d\mu(y) \to c \int f \quad (x \to \infty)$$

implies

$$(h*\mu)(x) \to c \int h \quad (x \to \infty)$$

for all $h \in M$.

The Wiener theory above extends from $L^1(\mathbb{R})$ to $L^1(G)$ with G a locally compact abelian group; see e.g. Rudin (1963), Reiter (1968). For generalisations to locally compact groups, we refer to Reiter (1974), Leptin & Poguntke (1979), and the survey Leptin (1984), §5. The appropriate context for generalisations of the second form of the Tauberian theorem is that of <u>Segal algebras</u>; see Reiter (1968), (1971), Liu et al. (1974), Feichtinger (1981). For the duality between M and M*, see Edwards (1958), Goldberg (1967). The condition $\mu \in M^*$ is of <u>amalgam norm</u> type; for background see Fournier & Stewart (1985) and the references cited there.

§2. BENEŠ' THEOREM

We now examine Wiener's second Tauberian theorem in the case of positive measures, important for applications in probability theory and elsewhere. The result below is due to Beneš (1961); we sketch a streamlined approach to its proof.

<u>Beneš' Tauberian Theorem.</u> If μ is a positive measure satisfying

$$\sup_x \int_x^{x+1} d\mu < \infty, \qquad (2')$$

and f satisfies (1) with $\hat{f}$ non-vanishing, then

$$\int f(x-y)\,d\mu(y) \to c \int f$$

implies

$$\int g(x-y)\,d\mu(y) \to c \int g$$

for all g satisfying (1) and continuous almost everywhere.

<u>Proof.</u>

Step 1. Replace f by its convolution with the Gaussian kernel $e^{-\frac{1}{2}x^2}$. This preserves our hypothesis and non-vanishing of transforms, and induces continuity. We may thus suppose without loss that f is con-

tinuous.

Step 2. For g continuous, we can use Wiener's second theorem above. For g an indicator of an interval [a,b], approximate g above and below by continuous piecewise linear functions g_i, $g_1 \le g \le g_2$:

$$g_1 := 0 \text{ outside } [a,b], \quad 1 \text{ in } [a+\varepsilon, b-\varepsilon],$$

$$g_2 := 0 \text{ outside } [a-\varepsilon, b+\varepsilon], \quad 1 \text{ in } [a,b].$$

Then as μ is positive,

$$\int g_1(x-y)\,d\mu(y) \le \int g(x-y)\,d\mu(y) \le \int g_2(x-y)\,d\mu(y).$$

By continuity of g_i, $\int g_i(x-y)\,d\mu(y) \to c \int g_i$, and $\int g_i$ may be made arbitrarily close to $\int g$ by choice of ε. The result follows for indicators g.

Step 3. The result extends to simple functions (linear combinations of indicators) by linearity.

Step 4. If g has compact support and is continuous almost everywhere, g is Riemann integrable, and so is approximable above and below by simple functions g_i. Extend from g_i to g as above.

Step 5. If g is continuous a.e. and satisfies (1), we may for each $\varepsilon > 0$ use (2') to choose n so large that contributions from outside $[-n,n]$ are less than ε. We approximate to g by $g_n := g\, I_{[-n,n]}$ and use the step above.

Step 6. Finally, a counter-example due to Smith (see Beneš (1961)) shows that the result may fail if the discontinuity-set of g has positive measure. //

Note that, writing λ for Lebesgue measure, it suffices to have positivity of $\mu + b\lambda$ for some b rather than of μ (as $\int f(x-y)\,d\lambda(y) = \int f$). A variant on this yields the following classical one-sided form of Wiener's theorem, due to Pitt (see e.g. Widder (1941), V.13).

<u>Corollary</u>. If f is non-negative, continuous, satisfies (1) and has $\hat{f}$ non-vanishing, p(.) is bounded below, and $\int f(x-y)p(y)\,dy$ is bounded, then

$$\int f(x-y)\; p(y)\; dy \to c \int f$$

implies

$$\int g(x-y)\; p(y)\; dy \to c \int g$$

for all g satisfying (1) and continuous a.e.

<u>Proof</u>. Since f is continuous and non-negative with $\hat{f}$ non-vanishing, f is positive on some interval: $f(.) \ge \varepsilon > 0$ on $[a-\delta, a]$ say. If $p(.) \ge -b$, replacing p(.) by p(.)+b we may suppose $p(.) \ge 0$. Then for some M and all x,

$$M \ge \int f(x+a-y)p(y)\,dy = \int f(y)p(x+a-y)\,dy \ge \int_{a-\delta}^{a} \ge \varepsilon \int_0^{\delta} p(x+t)\,dt,$$