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The Geometry of Hamiltonian Systems

Tudor Ratiu
Editor

The Geometry of Hamiltonian Systems

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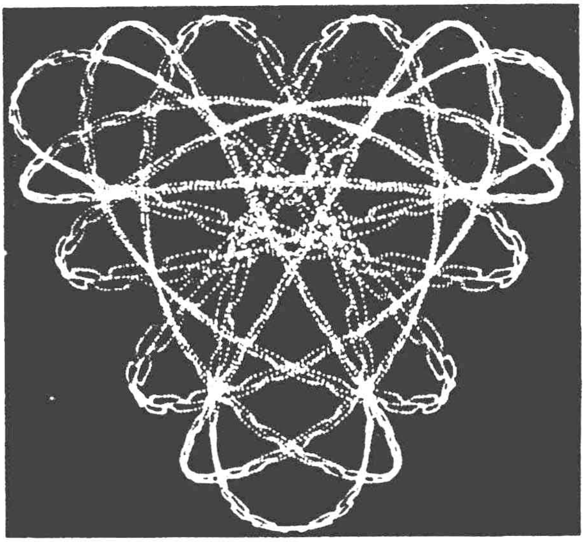

14,000 Iterations of a fourth-order symplectic integrator for the Hénon-Heiles Hamiltonian System. Photo by Clint Scovel and Paul Channell.

Preface

The papers in this volume are an outgrowth of the lectures and informal discussions that took place during the workshop on "The Geometry of Hamiltonian Systems" which was held at MSRI from June 5 to 16, 1989. It was, in some sense, the last major event of the year-long program on Symplectic Geometry and Mechanics. The emphasis of all the talks was on Hamiltonian dynamics and its relationship to several aspects of symplectic geometry and topology, mechanics, and dynamical systems in general. The organizers of the conference were R. Devaney (co-chairman), H. Flaschka (co-chairman), K. Meyer, and T. Ratiu.

The entire meeting was built around two mini-courses of five lectures each and a series of two expository lectures. The first of the mini-courses was given by A.T. Fomenko, who presented the work of his group at Moscow University on the classification of integrable systems. The second mini-course was given by J. Marsden of UC Berkeley, who spoke about several applications of symplectic and Poisson reduction to problems in stability, normal forms, and symmetric Hamiltonian bifurcation theory. Finally, the two expository talks were given by A. Fathi of the University of Florida who concentrated on the links between symplectic geometry, dynamical systems, and Teichmüller theory.

Due to the large number of participants, not everyone was able to speak at the conference. However, several special sessions were organized informally by some of the participants. Everyone was invited to participate, encouraged to speak, and submit a contribution to these proceedings. All papers that were received went through the refereeing process typical of a mathematics research journal and those that were accepted form the present volume. Unfortunately, there were several invited speakers who decided not to submit their contributions, so the present collection of papers does not accurately reflect all aspects of Hamiltonian dynamics that were discussed at the workshop.

As usual, getting a volume of proceedings of this magnitude off the ground entails a lot of work on the part of a number of people. First, thanks are due to the workshop co-chairmen, I. Kaplansky, the Director of MSRI, and V. Guillemin and A. Weinstein, the principal organizers of the entire special year, all of whom were of great help during the preparation of the workshop itself and beyond. The staff at MSRI has been particularly helpful, insuring that the workshop, one of the largest MSRI ever organized, did not collapse of its own weight. A. Baxter, the Manager of Business and Finance of MSRI, deserves the gratitude of all contributors for the excellent and difficult work she has done in shepherding the entire volume from its beginning right up to the last production stages. And last but not least, the participants at the conference must be thanked for offering everyone a two-week-long intense and highly interesting mathematical experience.

Tudor Ratiu

The Geometry of Hamiltonian Systems

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Heisenberg Algebras, Grassmannians and Isospectral Curves

MALCOLM R. ADAMS AND MAARTEN BERGVELT

Abstract. The connection between Heisenberg algebras and the geometry of branched covers of P^1 by Riemann surfaces is discussed using infinite Grassmannians.

§1. Introduction.

In recent years, several geometric and algebraic settings have been introduced for studying nonlinear partial differential evolution equations that display some properties common to Liouville completely integrable systems of ordinary differential equations. It is interesting to investigate the relations between these various approaches since this richness of structure allows one to use results that may be straightforward in one setting to obtain a deeper understanding of another setting.

For example, the Korteweg-de Vries equation (KdV) has been cast in the following three forms. The first is representation theoretic (see e.g., Date, Jimbo, Kashiwara, Miwa [1] or Kac, Raina [2]): the Plücker equations describing the group orbit through the highest weight vector of the basic representation $L(\Lambda_0)$ of the affine Kac-Moody algebra $A_1^{(1)}$ give the Hirota bilinear equations for the KdV hierarchy in the principal construction of $L(\Lambda_0)$. This construction uses the flows of the principal Heisenberg subalgebra of $A_1^{(1)}$ which correspond to the time variables of the KdV hierarchy. The next framework is more geometric (see Segal, Wilson [3]):the principal Heisenberg flows are realized as loop group flows on the infinite Grassmannian $Gr^{(2)}$. Finally, there is an algebro-geometric construction using linear flows of line bundles over Riemann surfaces (see Krichever [4]). (There is a fourth framework using Hamiltonian flows on the dual of a loop algebra (see e.g., Flaschka, Newell, Ratiu [5]), but we will not discuss this approach in this paper.)

Other hierarchies of equations can be found by choosing other Kac-Moody algebras and/or other constructions of the basic representation using other Heisenberg algebras. For example, the AKNS-Toda system may be studied using the homogeneous Heisenberg algebra of $A_1^{(1)}$. In fact, it is known in the theory of representations of affine Kac-Moody algebras that each conjugacy class of maximal Heisenberg subalgebras yields a construction of the basic module and a corresponding hierarchy of "integrable" equations in Hirota bilinear form. These hierarchies can be realized geometrically as the commutativity conditions for pairs of flows from the positive part of the Heisenberg algebra acting on an appropriate infinite Grassmannian. In this paper we want to describe the relation of the Heisenberg algebras in $A_n^{(1)}$ with the geometry of the Riemann surfaces. The details will appear in a later publication.

The authors would like to thank Robert Varley for many stimulating discussions.

§2. Heisenberg Algebras.

Let $\Lambda\mathrm{gl}(n,\mathbf{C})$ be the algebra of real analytic maps from the circle S^1 to $\mathrm{gl}(n,\mathbf{C})$. It is well known that the maximal Abelian subalgebras of $\Lambda\mathrm{gl}(n,\mathbf{C})$ (invariant under the Cartan involution) are, up to conjugacy, parametrized by the partitions of n (for a more precise statement and proof see ten Kroode [**6**]).

More explicitly, let $\underline{n} = (n_1 \geq n_2 \ldots \geq n_k > 0)$ be a partition of n, and let $\mathcal{H}^{\underline{n}}$ be the subalgebra of $\Lambda\mathrm{gl}(n,\mathbf{C})$ consisting of block matrices of the form

$$\begin{array}{c} \\ n_1 \\ n_2 \\ \vdots \\ n_k \end{array}\begin{array}{c} \begin{array}{cccc} n_1 & n_2 & \ldots & n_k \end{array} \\ \begin{pmatrix} B_1 & 0 & \ldots & 0 \\ 0 & B_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & B_k \end{pmatrix} \end{array},$$

where every block B_i of size $n_i \times n_i$ is a power series in

$$n_i \overset{n_i}{\begin{pmatrix} 0 & 0 & \ldots & 0 & z \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}}.$$

(Here $z = e^{i\theta}$ is the loop parameter on the circle). Then the $\mathcal{H}^{\underline{n}}$ are representatives for all the conjugacy classes of maximal abelian subalgebras if $\underline{n}$ runs over all partitions of n.

In the central extension of $\Lambda gl(n, \mathbf{C})$ the $\mathcal{H}^{\underline{n}}$ are Heisenberg algebras and for simplicity we will also refer to $\dot{\mathcal{H}}^{\underline{n}}$ (without central extension) as Heisenberg algebras.

§3. Grassmannians and vector bundles over P^1.

We need the following notation:

$$\Lambda \mathrm{Gl}(\mathrm{n}, \mathbf{C}) = \{g : S^1 \to \mathrm{Gl}(n, \mathbf{C}), \text{real analytic}\},$$

$$\mathrm{H}^{(n)} = L^2(S^1, \mathbf{C}^n),$$

$$\mathrm{H}_+^{(n)} = \left\{ \begin{array}{r} f \in \mathrm{H}^{(n)} \mid f \text{ extends to an holomorphic} \\ \text{map from the disk to } \mathbf{C}^n \end{array} \right\}.$$

Define then

$$Gr^{(n)} = \{W \subset \mathrm{H}^{(n)} \mid W = g\mathrm{H}_+^{(n)}, g \in \Lambda \mathrm{Gl}(\mathrm{n}, \mathbf{C})\}.$$

Note that

$$Gr^{(n)} \simeq \Lambda \mathrm{Gl}(\mathrm{n}, \mathbf{C})/\Lambda_+ \mathrm{Gl}(\mathrm{n}, \mathbf{C}),$$

where

$$\Lambda_+ \mathrm{Gl}(\mathrm{n}, \mathbf{C}) = \left\{ \begin{array}{r} g \in \Lambda \mathrm{Gl}(\mathrm{n}, \mathbf{C}) \mid g \text{ extends to a holomorphic map} \\ \text{from the disk to } \mathrm{Gl}(n, \mathbf{C}) \end{array} \right\}.$$

There is a geometric interpretation of $Gr^{(n)}$(see Wilson [7]). For $g \in \Lambda \mathrm{Gl}(\mathrm{n}, \mathbf{C})$ we will define a holomorphic rank n bundle over P^1 as follows. Let $\{U_0, U_\infty\}$ be a cover of P^1 such that $S^1 \subset U_0 \cap U_\infty$ and let $\{e_i^{(0)}\}_{i=1}^n$, $\{e_i^{(\infty)}\}_{i=1}^n$ be frames for the trivial bundles over U_0, U_∞. Then the bundle E_g is defined by gluing with g: we put in $U_0 \cap U_\infty$

$$e_i^{(\infty)} = g e_i^{(0)}.$$

This gives us a correspondence

$$g \leftrightarrow \{\mathcal{E}_g, \{e_i^{(0)}\}_{i=1}^n, \{e_i^{(\infty)}\}_{i=1}^n\},$$

where $\mathcal{E}_g$ is the sheaf of germs of holomorphic sections of E_g. Forgetting about the trivialisation $\{e_i^{(0)}\}_{i=1}^n$ corresponds to quotienting by $\Lambda_+ \mathrm{Gl}(\mathrm{n}, \mathbf{C})$, so we have the following correspondences:

$$W_g = g\mathrm{H}_+^{(n)} \leftrightarrow g \mod \Lambda_+ \mathrm{Gl}(\mathrm{n}, \mathbf{C}) \leftrightarrow \{\mathcal{E}_g, \{e_i^{(\infty)}\}_{i=1}^n\}.$$

§4. Line bundles over Curves.

We consider the following geometric data:

(1) $f : X \to P^1$, an n-fold cover of P^1 by some algebraic curve X.
(2) $L \to X$, a holomorphic line bundle over X, with sheaf of germs of holomorphic sections $\mathcal{L}$,
(3) ϕ, a trivialisation of $\mathcal{L}$ over $f^{-1}(U_\infty)$,
(4) local coordinates z_i centered at ∞_i such that $f^*z = z_i^{n_i}$, where $D = \sum n_i \infty_i$ is the divisor $f^{-1}(\infty)$ and z^{-1} is a local coordinate at infinity on P^1.

REMARK: Since $D = \sum_{i=1}^k n_i \infty_i$ has degree n we have $\sum_{i=1}^k n_i = n$. So we have naturally associated to the geometric datum (1) a partition of n, and hence a Heisenberg algebra $\mathcal{H}^{\underline{n}}$.

Now we use the covering map f to to get a sheaf $f_*\mathcal{L}$ on P^1, which is the sheaf of germs of holomorphic sections of a rank n bundle on P^1. Similarly we get, from the trivialisation ϕ and the log coordinates z_i, a trivialisation of $f_*\mathcal{L}$. We will denote this by $f_*\phi$. So we have a map

$$\{f : X \to P^1, \mathcal{L}, \phi\} \mapsto \{f_*\mathcal{L}, f_*\phi\} \leftrightarrow W \in \mathrm{Gr}^{(n)}.$$

(More generally, if $n = mk$ we can consider a m-fold cover of P^1 by some curve X and a rank k bundle over X to obtain a point of $\mathrm{Gr}^{(n)}$ by pushing forward to P^1).

§5. Reconstruction of the Curve.

We generalize in this section the method of Segal and Wilson [**3**, sect. 6]. Suppose the point $W \in \mathrm{Gr}^{(n)}$ comes from the geometric data $\{f : X \to P^1, \mathcal{L}, \phi\}$ as in the previous section . We want to reconstruct the data from W alone. Note that

$$\begin{aligned} W &= f_*\mathcal{L}(U_0)|_{S^1}, \\ &= \mathcal{L}(f^{-1}(U_0))|_{f^{-1}(S^1)} \end{aligned}$$

Define

$$W^{\mathrm{alg}} = \{h \in W | h = \sum_{-\infty}^{N} h_i z^i, h_i \in C^n\}.$$

This can be identified with meromorphic sections of L which have only poles at the divisor $\bar{D} = \sum \infty_i$. Define then the *stabilizer algebra* of W to be

$$\mathcal{S}_W = \{\psi \in \Lambda gl(n,\mathbf{C})^{\text{alg}} \mid \psi W^{\text{alg}} \subset W^{\text{alg}}\}.$$

It is clear that $\mathcal{S}_W$ contains the pushforward of all meromorphic functions on X with only poles at $\bar{D}$. In fact if we define $\mathcal{S}_W^{\underline{n}} = \mathcal{S}_W \cap \mathcal{H}^{\underline{n}}$ we have

PROPOSITION: $f_* H^0(X - \bar{D}, \mathcal{O}_X) = \mathcal{S}_W^{\underline{n}}$.

So we have found a commutative subring $\mathcal{S}_W^{\underline{n}}$ of the noncommutative ring $\mathcal{S}_W$ and we are in a position to do algebraic geometry.

PROPOSITION: $\mathcal{S}_W^{\underline{n}}$ admits a filtration

$$\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \ldots \subseteq \mathcal{S}_W^{\underline{n}},$$

such that we can reconstruct the curve X from the associated graded rings

$$\mathcal{R}_X = \bigoplus_{i=0}^{\infty} \mathcal{S}_i, \quad \mathcal{R}_D = \bigoplus_{i=0}^{\infty} \mathcal{S}_i / \mathcal{S}_{i-1},$$

by

$$X = \text{Proj}\mathcal{R}_X, \quad D = \text{Proj}\mathcal{R}_D.$$

Here Proj$R$ denotes the scheme associated to a graded ring R; the point set of this scheme consists of the homogeneous prime ideals not containing the irrelevant ideal (see e.g., Hartshorne [8]).

§6. Flows on the Grassmannian and Isospectral Curves.

In general one can define a flow on the Grassmannian by

$$W \in \text{Gr}^{(n)} \mapsto W(t) = e^{tp} W, p \in \Lambda gl(n,\mathbf{C}).$$

Under these flows the stabilizer algebra transforms by conjugation: $\mathcal{S}_W^{\underline{n}}(t) = e^{tp} \mathcal{S}_W^{\underline{n}} e^{-tp}$. So if we want to fix $\mathcal{S}_W^{\underline{n}}$ for fixed partition $\underline{n}$ and all $W \in \text{Gr}^{(n)}$ we have to take $p \in \mathcal{H}^{\underline{n}}$. If W comes from the geometric data $\{f : X \to P^1, \mathcal{L}, \phi\}$ as before the curve X will be "isospectral". More generally for generators $p_i^{\underline{n}}$ of the positive part of $\mathcal{H}^{\underline{n}}$ we can define $W(t_1, t_2, \ldots) = e^{\sum t_i p_i^{\underline{n}}} W$ and these commuting flows will preserve $f : X \to P^1$ and induce a linear flow on the Jacobian of X. As usual this can also be expressed in terms of matricial Lax equations, the solution of which can be expressed in terms of θ-functions. See e.g., Adams, Harnad, Hurtubise [9].

§7. Remarks and questions.

a. In representation theory not only the Heisenberg algebras $\mathcal{H}^{\underline{n}}$ but also the so-called translation group $\mathcal{T}^{\underline{n}}$ centralizing $\mathcal{H}^{\underline{n}}$ is important. $\mathcal{T}^{\underline{n}}$ is the discrete group generated by matrices $T = \mathrm{diag}(z^{l_1}1_{n_1}, \dots, z^{l_k}1_{n_k})$, with $\sum l_i = 0$. (Note that for the principal partition $\underline{n} = n$ the group $\mathcal{T}^{\underline{n}}$ is trivial). This group acts on the Grassmannian and preserves $\mathcal{S}_W^{\underline{n}}$. The action on the geometric data $\{f : X \to P^1, \mathcal{L}, \phi\}$ is simple to describe: the trivialisation and the covering are fixed and the sheaf $\mathcal{L}$ maps to $T \cdot \mathcal{L} = \mathcal{L} \otimes \mathcal{O}(\sum l_i \infty_i)$. In the simplest non trivial case of $n = 2$ and partition $\underline{2} = (1,1)$ the translation group leads to the Toda lattice (see Flaschka [10], Bergvelt, ten Kroode [11]).

b. One can start with an arbitrary $W \in \mathrm{Gr}^{(n)}$ and consider for all partitions $\underline{n}$ the various commutative rings $\mathcal{S}_W^{\underline{n}}$ and the corresponding schemes $X_{\underline{n}}$. It is unclear what the relation between these is. Even more ambitiously one can ask the question if it is possible to associate to the whole non-commutative ring $\mathcal{S}_W$ some geometric object (in some non commutative geometry).

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