HELENA RASIOWA AND ROMAN SIKORSKI

THE MATHEMATICS OF METAMATHEMATICS

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HELENA RASIOWA AND ROMAN SIKORSKI

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Die Mathematiker sind eine Art Franzosen: redet man zu ihnen, so übersetzen sie es in ihre Sprache und dann ist es alsobald ganz etwas Anderes.

J. W. Goethe *

PREFACE

The title of this book is not meant as a pun, although it may, at first sight, appear to be so.

Metamathematics is a theory which deals with formalized mathematical theories. A formalized mathematical theory is, roughly speaking, a set of certain finite sequences of symbols, called formulas and terms, and of certain simple operations performed on those sequences. The formulas and terms are substitutes—formed by means of a few simple rules—of sentences and functions in an intuitive mathematical theory. Operations on formulas correspond to elementary steps of deduction in mathematical reasonings. The formulas corresponding to the axioms of the intuitive theory play a special part; they are the axioms of the formalized theory. The formulas which can be derived from the axioms by means of the accepted operations correspond to the theorems of the theory.

The set of all formulas and the set of all terms, when considered as sets of finite sequences with operations, can in turn be the subject of mathematical investigations employing more or less advanced auxiliary methods taken from mathematics. In the early period of the development of mathematical logic, the general tendency was to use only the most elementary methods possible, excluding all infinitistic methods. The precursor of that trend was Hilbert, who believed that in this way it would be possible to prove the consistency of mathematics. However, Gödel's results exposed the fiasco of Hilbert's finitistic methods as far as consistency is concerned. The use of the finitistic methods to investigate formalized theories is perhaps natural on account of the clearly finitistic character of the notion of formalized theory. In practice, however, the restriction of methods of proof to elementary finitistic ones complicates metamathematical investigations considerably. It also prevents a full recognition of the exact nature of formalized mathematical theories from the point of view of methods and ideas of modern mathematics. The use of the more advanced infinitistic methods makes it easier to explain the mathematical structure of formalized theories. The set of all terms of a formalized theory is an algebra, and generally an algebra with infinitely many operations. The set of all formulas of a formalized theory is also an algebra—in general, it is an algebra with infinite operations. After

* „Ferneres über Mathematik und Mathematiker", s. Werke, Grosse Weimarische Ausg. Abt. II, Bd. 11 (1893), s. 102.

the natural identification of equivalent formulas, the set of all formulas becomes a lattice: a Boolean algebra, a pseudo-Boolean algebra, a topological Boolean algebra, etc., depending on the type of the logic adopted in the theory. These algebras in turn are connected with the notions of a field of sets and a topological space. From this point of view it is natural to apply in metamathematics the methods of algebra, lattice theory, set theory and topology. The total sum of mathematical methods useful in metamathematics makes up what in the title of this book has been called the mathematics of metamathematics.

By means of infinitistic methods the meaning of many basic metamathematical theorems is made clear. The theorem on the completeness of the propositional calculus is seen to be exactly the same as Stone's theorem on the representation of Boolean algebras. The Gödel theorem on the completeness of the predicate calculus is a modification of the Stone representation theorem, taking into account some infinite operations in Boolean algebras. It is surprising that the Gödel completeness theorem can be obtained, for example, as a result of the Baire theorem on sets of the first category in topological spaces, etc.

The finitistic approach of Hilbert's school is completely abandoned in this book. On the contrary, the infinitistic methods, making use of the more profound ideas of mathematics, are distinctly favoured. This brings out clearly the mathematical structure of metamathematics. It also permits a greater simplicity and clarity in the proofs of the basic metamathematical theorems and emphasizes the mathematical contents of these theorems.

The title of this book is slightly inexact since not all mathematical methods used in metamathematics are exposed in it. Namely, Gödel's method of arithmetization has been omitted. The exact title of the book should be: Algebraic, lattice-theoretical, set-theoretical and topological methods in metamathematics. The arithmetization of metamathematics differs fundamentally from these methods and leads to different problems. That is why we did not consider it proper to include that subject. As a result, we have omitted that part of metamathematics which uses arithmetization in a natural way (the decision problem, the existence of undecidable sentences, etc.) and the theory of recursive functions, now being developed by a large number of mathematicians.

It is difficult to establish exactly who was the first to use infinitistic methods in metamathematics. The close relation between classical logic and the theory of Boolean algebras has been known for a long time. It was the investigations in logic of Boole himself that led to the notion which we now call the Boolean algebra. Stone's basic result concerning the representation of Boolean algebras permitted the broad application of the theory of Boolean algebras to metamathematics. The method of treating the set of formulas or the set of equivalence classes of formulas as abstract algebras, due to Lindenbaum and Tarski, proved to be an essential research tool. It established a link between the metamathematics of the theories based on classical logic and the theory of Boolean algebras. The works of Stone and Tarski on the connection between the intuitionistic logic and pseudo-complemented lattices and the further works of McKinsey and Tarski on lattice-theoretical methods in intuitionistic and modal propositional calculi established an analogous link in the metamathematics of corresponding non-classical theories. Here another essential research approach is of chief importance: the interpretation of the formulas of propositional calculi as mappings in certain lattices. This interpretation is a generalization of the truth-table method, long used in logic. The extension of this method to the intuitionistic predicate calculus was first introduced by Mostowski for problems of non-deducibility of formulas. The method of interpretation of formulas as mappings, together with the method of identifying equivalent formulas and treating the set of equivalence classes as an abstract algebra has enabled us to give an algebraic-topological proof of the Gödel completeness theorem and of other basic theorems. The concept of the product of models modulo prime filters, introduced by Łoś and widely used by the Berkeley school, is another essential contribution to the mathematical concepts of metamathematics.

Research into infinitistic methods in metamathematics is now in full swing and is far from being completed. This book does not embrace the whole of the research conducted in this field. In particular it does not include Halmos's theory of polyadic algebras and that of cylindrical algebras worked out by Henkin, Tarski and Thompson. Neither does it deal with the theory of the languages with infinitely long formulas, which, through Hanf's latest results, has found application in mathematics itself, namely in the theory of prime filters in Boolean algebras (Tarski). Certain other applications of metamathematics to mathematics (the results of A. Robinson and others) and the general theory of models are also disregarded.

Moreover, the problems discussed in this book are far from being treated in an exhaustive manner. The purpose of this book is only to introduce the reader to the basic ideas of the infinitistic approach to metamathematics, especially into the methods directly connected with the authors' own work. The accompanying bibliography has no claims to completeness.

The book is written in an elementary way in the sense that it requires no knowledge of mathematics and metamathematics beyond that of the basic notions of set theory: operations on sets, the notions of cardinal number and the transfinite induction. It does, however, assume a certain mathematical sophistication on the part of the reader.

All the mathematical knowledge necessary to understand the infinitistic methods of metamathematics is presented in Part One (Chapters I-IV). The reader will find there a brief exposition of the elementary ideas of topology and algebra, and an exposition of a part of lattice theory. The material contained in Part One was chosen entirely from the point of view of its application to metamathematics and it represents the minimum necessary to understand Parts Two and Three, which are concerned with the metamathematics of formalized theories based on classical logic or non-classical logics. Chapters III and IV can be omitted altogether by the reader interested only in classical logic. Chapter V constitutes an intuitive introduction to the technique of formalization of mathematical theories. It also contains a general definition of logic. Chapter VI is indispensable to the understanding of all the succeeding chapters. It contains the theory of the basic research tools in metamathematics: the interpretation of formulas as mappings and the construction of algebras by identification of equivalent formulas. Chapters VII and VIII deal with classical logic, Chapters IX and X with intuitionistic logic, and Chapter XI with positive and modal logic. Chapters IX and X can be read separately (after Chapters I-VI) irrespective of Chapters VII and VIII. In order to make things easier for the reader, we often give complete proofs in Chapters IX and X even when they are similar to the proofs of analogous theorems in classical logic. Positive and modal logics, on the other hand, have been treated rather superficially in Chapter XI and the proofs of theorems analogous to those of classical and intuitionistic logic have been omitted.

This book is addressed only to mathematicians and students of mathematics interested in the logical aspects of mathematics and the mathematical aspects of logic. For this reason the material illustrating logical problems given in the introductory Chapter V is taken only from mathematics. The philosophical aspects of logic and mathematics are completely omitted as foreign to the mathematical character of the book. Only in § 1 of Chapter IX is there a short summing up of the basic ideas which led to the rise of intuitionistic logic.

The inclusion of two chapters on intuitionism is not an indication of the authors' positive attitude towards intuitionistic ideas. Intuitionism, like other non-classical logics, has no practical application in mathematics. Nevertheless many authors devote their works to intuitionistic logic. On the other hand, the mathematical mechanism of intuitionistic logic is interesting: it is amazing that vaguely defined philosophical ideas concerning the notion of existence in mathematics have led to the creation of formalized logical systems which, from the mathematical point of view, proved to be equivalent to the theory of lattices of open subsets of topological spaces. Finally, the formalization of intuitionistic logic achieved by Heyting and adopted in this book is not in agreement with the philosophical views of the founder of intuitionism, Brouwer, who opposed formalism in mathematics. Since in treating intuitionistic logic we have limited ourselves to problems which are directly connected with general algebraic, lattice-theoretical and topological methods employed in this book, we have not included the latest results of Beth and Kreisel concerning other notions of satisfiability than the algebraic notion of satisfiability which we have adopted.

We have given very little space to set-theoretical and semantic antinomies. We believe that antinomies should be relegated to the history of mathematics and that the material given to the reader should be in a form free from errors in the interpretation of the notion of set, etc. Similarly, in the theory of functions no one now uses the vague and inconsistent interpretation of the idea of function given hundreds of years ago.

The terminology employed in this book differs in several places from the terminology generally used. We have tried to unify the mathematical and metamathematical terminology in parallel problems. Thus, for example, in place of "the complete theory" we write "the maximal theory", since this notion is parallel to the notion of a maximal filter in lattice theory. Besides, the word "complete" is too often used with other meanings. We also call attention to the fact that the word "model" in this book has another more general meaning than that usually adopted in works on logic. Following Bourbaki we say "ordered set" in place of "partially ordered set".

Following Nöbeling, we use the term "topological Boolean algebra" in place of "closure algebra", which is employed by many authors. Because of its applications to intuitionistic logic, in this book the interior operation is first in importance, before the closure operation. For this reason we have adopted as the axioms of topological spaces and topological Boolean algebras the axioms dual to the well known Kuratowski axioms. Consequently, it has been difficult for us to employ the name "closure algebra" for the notions defined by the word "interior" and not by the word "closure". The name "topological Boolean algebra" has seemed more convenient also for the reason that it does not introduce any asymmetry in the duality of the basic notions of topology.

We use the same symbols $\cup, \cap, -$ to stand for set operations, lattice-theoretical operations and the corresponding propositional connectives in order to emphasize the close reciprocal relation between them.

This does not lead to misunderstanding anywhere and very much simplifies the translation of logical notions into the language of set theory and lattice theory.

Theorems within a given chapter are referred to by their numbers. Theorems quoted from other chapters have, in addition, Roman numerals specifying the chapters in which they appear. Similarly, formulas within a given section are referred to by their numbers, those from the same chapter but from other sections have the number of the paragraph added, and those from other chapters are marked by Roman numerals representing the chapter, the number of the section and that of the formula.

We wish to thank Professor A. Mostowski for his valuable advice on bibliographic matters. We also wish to thank Dr A. Białynicki-Birula, who read the manuscript of the book and whose comments helped us to improve the text in many places. We also thank Dr T. Traczyk for his help in reading the proofs.

H. Rasiowa *R. Sikorski*

Warsaw, 1962.

PART ONE

LATTICES

CHAPTER I

PRELIMINARY TOPOLOGICAL, ALGEBRAIC AND LATTICE-THEORETICAL NOTIONS

§ 1. Sets, mappings, Cartesian products. We assume that the reader is familiar with the fundamental notions from set theory [1]. We recall here only the basic notation.

We write $a \in A$ if a is an *element of a set* A, and otherwise $a \notin A$. If every element of a set A belongs to a set B, we write $A \subset B$ and we say that A is a *subset* of B. The relation $\subset$ is called *inclusion*.

The *empty set* is denoted by 0.

For any sets A, B, the symbol $A \cup B$ ($A \cap B$) will denote the *union* (the *intersection*) of A and B, i.e. the set of all elements belonging to at least one of the sets (to both the sets) A, B. More generally, $\bigcup_{t \in T} A_t$ ($\bigcap_{t \in T} A_t$) will denote the *union* (the *intersection*) of sets A_t where $t \in T$, i.e. the set of all elements belonging to at least one of the sets (to each of the sets) A_t, $t \in T$.

If $A \cap B = 0$, the sets A, B are said to be *disjoint*.

The *difference* of sets A, B, i.e. the set of all elements of A which do not belong to B, will be denoted by $A - B$.

In applications we shall often consider only subsets of a fixed set X. The set X will then be called a *space*, and the difference $X - A$ (where $A \subset X$) will be called the *complement* of A and denoted by $-A$. Hence, if $A, B \subset X$, then $A - B = A \cap -B$.

The words *mapping*, *function*, *transformation* always have the same meaning. We write

$$f\colon X \to Y$$

to indicate that f is a mapping defined on X with values in Y. The set X is then called the *domain* of f. The set Y is called the *counter-domain* of f.

Usually, if f denotes a mapping, then $f(x)$ is used to denote the value of f at a point x. Sometimes we shall also write fx and f_x instead of $f(x)$.

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(1) For a detail exposition of set theory see e.g. Fraenkel [2], Hausdorff [1], Kuratowski and Mostowski [1], Sierpiński [1], [3].

If f is a mapping of a set X into a set Y, and $A \subset X$, $B \subset Y$, then $f(A)$ denotes the set of all elements $f(x)$ where $x \in A$, and $f^{-1}(B)$ denotes the set of all elements $x \in X$ such that $f(x) \in B$. The sets $f(A)$ and $f^{-1}(B)$ are called the *image* of A and the *counter-image* of B, respectively. If $f(A) = B$, we say that f maps the set A *onto* the set B. If f is one-to-one, i.e. if $f(x_1) = f(x_2)$ implies $x_1 = x_2$, then f^{-1} denotes the mapping inverse to f, i.e. the mapping from $f(X)$ onto X, such that $f^{-1}(y) = x$ if and only if $f(x) = y$. If f: $X \to Y$ and g: $Y \to Z$, then gf: $X \to Z$ denotes the *superposition* of f and g, i.e. $gf(x) = g(f(x))$ for all $x \in X$. If f is a mapping defined on a set X, g is a mapping defined on a set $X_0 \subset X$ and

$$g(x) = f(x) \quad \text{for all} \quad x \in X_0,$$

then f is called an *extension of g over X* and g is called the *restriction of f to X_0*.

Functions defined on the set of all positive (or non-negative) integers are called *(infinite) sequences*. Functions defined on a set of integers $1, \ldots, m$ are called *finite sequences* or more precisely: *m-element sequences*. If a_n denotes the element assigned to an integer n, then the sequence is denoted by $\{a_n\}$ or—in the case of a finite sequence—by $\{a_1, \ldots, a_m\}$.

More generally, if for every t in a non-empty set T, a_t is an element of a set A, then the function which assigns the element a_t to every $t \in T$ will be denoted by $\{a_t\}_{t \in T}$ or simply by $\{a_t\}$.

The symbol $\boldsymbol{P}_{t \in T} A_t$ will denote the *Cartesian product* of sets A_t ($t \in T$), i.e. the set of all mappings $a = \{a_t\}_{t \in T}$ such that $a_t \in A_t$ for every $t \in T$. In particular, $\boldsymbol{P}_{n=1}^{m} A_n$ ($\boldsymbol{P}_{n=1}^{\infty} A_n$) will denote the set of all m-element sequences (of all infinite sequences) $\{a_n\}$ such that $a_n \in A_n$ for $n = 1, 2, \ldots, m$ (for $n = 1, 2, \ldots$). Instead of $\boldsymbol{P}_{n=1}^{m} A_n$ ($\boldsymbol{P}_{n=1}^{\infty} A_n$) we shall also write $A_1 \times \ldots \times A_m$ ($A_1 \times A_2 \times \ldots$).

If all the sets are equal,

$$A_t = A \quad \text{for every} \quad t \in T,$$

we write A^T instead of $\boldsymbol{P}_{t \in T} A_t$. In other words, A^T is the set of all mappings from T into A. We also write A^m and $A^{\aleph_0}$ instead of $A \times \ldots \times A$ (m times) and $A \times A \times \ldots$, respectively.

Mappings f: $A^m \to A$ will often be called *operations* in A.

If, for every $t \in T$, $\mathfrak{K}_t$ is a class of subsets of a set X_t, then $\overset{*}{\boldsymbol{P}}_{t \in T} \mathfrak{K}_t$ will denote the class of all sets $\boldsymbol{P}_{t \in T} A_t$ such that $A_t \in \mathfrak{K}_t$ for every $t \in T$. By definition, $\overset{*}{\boldsymbol{P}}_{t \in T} \mathfrak{K}_t$ is a class of subsets of $\boldsymbol{P}_{t \in T} X_t$.

We suppose that the reader is familiar with the notion of cardinal and ordinal numbers and of transfinite induction. The cardinal number of a set X (i.e. the power of X) will be denoted by $\overline{\overline{X}}$. The cardinal number of *enumerable* or *countable* sets (i.e. of one-to-one images of the set of all positive integers) is denoted by $\aleph_0$.

§ 2. Topological spaces (¹). A *topological space* is a set X in which, with every set $A \subset X$, there is associated a set $\mathbf{I}A \subset X$ in such a way that

$$(i_1) \qquad \mathbf{I}(A \cap B) = \mathbf{I}A \cap \mathbf{I}B ,$$

$$(i_2) \qquad \mathbf{I}A \subset A ,$$

$$(i_3) \qquad \mathbf{II}A = \mathbf{I}A ,$$

$$(i_4) \qquad \mathbf{I}X = X .$$

More precisely, a topological space is a pair $\{X, \mathbf{I}\}$ where X is a set and $\mathbf{I}$ satisfies (i_1)-(i_4).

An operation $\mathbf{I}$ satisfying (i_1)-(i_4) is called an *interior operation*. For every set $A \subset X$, the set $\mathbf{I}A$ is called the *interior* of the set A.

The set $-\mathbf{I}-A$ (i.e. the set $X-\mathbf{I}(X-A)$) is called the *closure* of A and denoted by $\mathbf{C}A$ ($A \subset X$). By definition,

$$(1) \qquad \mathbf{C}A = -\mathbf{I}-A \quad \text{and} \quad \mathbf{I}A = -\mathbf{C}-A .$$

It follows immediately from (i_1)-(i_4) that the operation $\mathbf{C}$ satisfies the conditions (²)

$$(c_1) \qquad \mathbf{C}(A \cup B) = \mathbf{C}A \cup \mathbf{C}B ,$$

$$(c_2) \qquad A \subset \mathbf{C}A ,$$

$$(c_3) \qquad \mathbf{CC}A = \mathbf{C}A ,$$

$$(c_4) \qquad \mathbf{C}0 = 0 .$$

Notice that, conversely, if with every set $A \subset X$ there is associated a set $\mathbf{C}A \subset X$ in such a way that conditions (c_1)-(c_4) are satisfied, then the operation $\mathbf{I}$ defined by the second of equations (1) satisfies conditions (i_1)-(i_4) and the first of equations (1) also holds.

Any operation $\mathbf{C}$ satisfying (c_1)-(c_4) is called a *closure operation*.

It follows immediately from (i_1) and (c_1) that, for arbitrary subsets A, B of a topological space X,

$$(2) \qquad A \subset B \quad \text{implies} \quad \mathbf{I}A \subset \mathbf{I}B \text{ and } \mathbf{C}A \subset \mathbf{C}B .$$

A subset A of a topological space X is said to be *open (closed)* in X if $A = \mathbf{I}A$ (if $A = \mathbf{C}A$). By (1), a set A is open (closed) if and only if its complement $-A$ is closed (open).

(¹) For a detail exposition of the theory of topological spaces see e.g. Kelley [1], Kuratowski [3].

(²) Axioms (c_1)-(c_4) are due to Kuratowski [1].

By (i_3) (by (c_3)) the interior (the closure) of any set A is open (closed). It follows from (i_2), (c_4) and from (c_2), (i_4) that the empty set 0 and the whole space X are both open and closed.

Observe that if B is open, then for every set A

(3) $$B \subset A \quad \text{if and only if} \quad B \subset \mathbf{I}A$$

on account of (i_2), (i_3) and (2). This implies that $\mathbf{I}A$ is the greatest open subset of A. Analogously, if B is closed, then for every set A

(4) $$A \subset B \quad \text{if and only if} \quad \mathbf{C}A \subset B$$

on account of (c_2), (c_3) and (2). This implies that $\mathbf{C}A$ is the least closed set containing A.

By (i_1) (by (c_1)) the intersection (the union) of two open (closed) sets is open (closed). By an easy induction we infer that the intersection (union) of any finite sequence of open (closed) sets is open (closed).

If A is closed (open) and B is open (closed) then $A-B$ is closed (open) since $A-B = A \cap -B$.

The union $\bigcup_{t\in T} A_t$ (the intersection $\bigcap_{t\in T} A_t$) of any number of open (closed) sets A_t is open (closed). In fact, if A_t is open, then by (2), $A_t = \mathbf{I}A_t \subset \mathbf{I}\bigcup_{t\in T} A_t$ and consequently $\bigcup_{t\in T} A_t \subset \mathbf{I}\bigcup_{t\in T} A_t$; the converse inclusion follows from (i_2). The analogous statement for closed sets can be proved similarly or can be obtained from the case of open sets just proved when we pass to complements.

A class $\boldsymbol{B}$ of open subsets of X is said to be a *basis* of X if every open subset of X is the union of some sets belonging to $\boldsymbol{B}$.

A class $\boldsymbol{B}_0$ of open subsets of X is said to be a *subbasis* of X if the class $\boldsymbol{B}$ composed of the empty set 0, the whole space X, and of all the finite intersections $B_1 \cap \ldots \cap B_n$ where $B_1, \ldots, B_n \in \boldsymbol{B}_0$, is a basis of X. Of course, if a subbasis $\boldsymbol{B}_0$ contains 0 and X and if $B_1, B_2 \in \boldsymbol{B}_0$ implies $B_1 \cap B_2 \in \boldsymbol{B}_0$, then $\boldsymbol{B}$ is a basis of X.

The following simple theorem is often used to define an interior operation $\mathbf{I}$ in any set X.

2.1. *For every class $\boldsymbol{B}_0$ of subsets of a set X there exists exactly one interior operation $\mathbf{I}$ in X such that $\boldsymbol{B}_0$ is a subbasis of the topological space $\{X, \mathbf{I}\}$.*

Let $\boldsymbol{B}$ be the class composed of the empty set 0, the whole space X and of all the finite intersections $B_1 \cap \ldots \cap B_n$ where $B_1, \ldots, B_n \in \boldsymbol{B}_0$. For every set $A \subset X$, let $\mathbf{I}A$ be the union of all sets $B \in \boldsymbol{B}$ such that $B \subset A$. Properties (i_2), (i_3), (i_4) follow immediately from the definition of $\mathbf{I}$. Property (i_1) follows from the fact that $B_1 \cap B_2 \in \boldsymbol{B}$ for $B_1, B_2 \in \boldsymbol{B}$. Observe that $\boldsymbol{B}$ is a basis of the topological space X just defined. Thus $\boldsymbol{B}_0$ is a subbasis of X.

Conversely, if $\mathbf{I}$ is an interior operation in X such that $\boldsymbol{B}_0$ is a subbasis of $\{X, \mathbf{I}\}$, then the class $\boldsymbol{B}$ defined above is a basis. Consequently, the open set $\mathbf{I}A$ is the union of all sets $B \in \boldsymbol{B}$ such that $B \subset \mathbf{I}A$. But sets in $\boldsymbol{B}$ are open, and for open B the inclusion $B \subset \mathbf{I}A$ holds if and only if $B \subset A$ (see (3)). Thus $\mathbf{I}A$ is the union of all sets $B \in \boldsymbol{B}$ such that $B \subset A$, i.e. the interior operation $\mathbf{I}$ coincides with the interior operation defined in the first part of the proof of 2.1. This proves the uniqueness of $\mathbf{I}$.

For instance, let X be the set of all real numbers. Take as $\boldsymbol{B}_0$ the class of all intervals $a < \xi < b$, the empty set included. By 2.1 there exists only one interior operation $\mathbf{I}$ in X such that $\boldsymbol{B}_0$ is a subbasis. We shall always consider the set of all real numbers as a topological space with the interior operation $\mathbf{I}$. Observe that $\boldsymbol{B}_0$ is also a basis of X. A set A is open in X if and only if it is a finite or enumerable union of open intervals or if it is empty. Note that the enumerable class of all intervals $a < \xi < b$ with rational a and b (the empty set included) is also a basis for X.

Let X be any topological space. For arbitrary sets $A, B \subset X$ we have, by (i_1) and (1),

$$\mathbf{I}(A \cup B) \cap \mathbf{I}-B \subset \mathbf{I}A\,,$$

that is,

$$\mathbf{I}(A \cup B) \subset \mathbf{I}A \cup \mathbf{C}B\,. \tag{5}$$

Consequently, by (1) and (i_3),

$$\mathbf{I}(A \cup B) = \mathbf{II}(A \cup B) \subset \mathbf{I}(\mathbf{I}A \cup \mathbf{C}B)\,.$$

Replacing A and B in (5) by $\mathbf{C}B$ and $\mathbf{I}A$ respectively, we obtain

$$\mathbf{I}(\mathbf{I}A \cup \mathbf{C}B) \subset \mathbf{IC}B \cup \mathbf{CI}A\,.$$

Consequently

$$\mathbf{I}(A \cup B) \subset \mathbf{CI}A \cup \mathbf{IC}B\,. \tag{6}$$

A subset A of a topological space X is said to be *dense* provided $\mathbf{C}A = X$. A set A is said to be a *boundary* set provided its complement $-A$ is dense, i.e. $\mathbf{I}A = 0$.

A set A is said to be *nowhere dense* provided its closure $\mathbf{C}A$ is a boundary set, i.e. $\mathbf{IC}A = 0$.

If a set A is dense and $A \subset B$, then B is also dense, by (2). Consequently, each subset of a boundary set is a boundary set. Each subset of a nowhere dense set is nowhere dense. Each nowhere dense set is a boundary set. A closed set is nowhere dense if and only if it is a boundary set.

A is a boundary set if and only if it does not contain a non-empty open set B (i.e. if $B - A \neq 0$ for every open set $B \neq 0$). In fact, if B is open and $B \subset A$, then $B = \mathbf{I}B \subset \mathbf{I}A$; hence, if $B \neq 0$, then $\mathbf{I}A \neq 0$, i.e. A is not a boundary set. Conversely, if A is not a boundary set, then $B = \mathbf{I}A$ is an open non-empty subset of A.

A set A is nowhere dense if and only if $\mathbf{C}A$ contains no open non-empty set, i.e. if $B - \mathbf{C}A \neq 0$ for every open set $B \neq 0$.

For instance, for every set A, the set $A - \mathbf{I}A$ is a boundary set. In fact, $\mathbf{I}(A - \mathbf{I}A) \subset A - \mathbf{I}A$ by (i_2). On the other hand, $\mathbf{I}(A - \mathbf{I}A) \subset \mathbf{I}A$ by (2). Hence $\mathbf{I}(A - \mathbf{I}A) = 0$.

Since $\mathbf{C}A - A = (-A) - \mathbf{I}(-A)$, we infer that, for every set A, the set $\mathbf{C}A - A$ is a boundary set. Hence, for every open set A, the set

$$\mathbf{C}A - A$$

is nowhere dense since it is a closed boundary set.

It follows immediately from (6) that the union of a boundary set and a nowhere dense set is boundary. Hence, by (c_1), the union of two (and consequently of an arbitrary finite number of) nowhere dense sets is nowhere dense.

A set A is said to be of the *first category* if it is the union of a sequence of nowhere dense sets. The union of any sequence of sets of the first category is also of the first category. Every subset of a set of the first category is also of the first category.

A topological space X is said to be *compact* if, for every indexed set $\{A_t\}_{t \in T}$ of open subsets, the equation $X = \bigcup_{t \in T} A_t$ implies the existence of a finite set $T_0 \subset T$ such that $X = \bigcup_{t \in T_0} A_t$. Replacing open sets by their complements we infer that a topological space X is compact if and only if, for any indexed set $\{B_t\}_{t \in T}$ of closed subsets, the equation $\bigcap_{t \in T} B_t = 0$ implies $\bigcap_{t \in T_0} B_t = 0$ for a finite set T_0 (in other words: if the intersection of every finite number of closed subsets B_t is non-empty, then the intersection of all B_t is non-empty).

If X is a compact space, the A_t are open subsets and B is a closed subset, then the inclusion $B \subset \bigcup_{t \in T} A_t$ implies $B \subset \bigcup_{t \in T_0} A_t$ for a finite set $T_0 \subset T$. In fact, we have $X = (-B) \cup \bigcup_{t \in T} A_t$. Therefore, by compactness, $X = (-B) \cup \bigcup_{t \in T_0} A_t$ for a finite set $T_0 \subset T$, i.e. $B \subset \bigcup_{t \in T_0} A_t$.

By passing to complements we infer from the last remark that if X is a compact space, A_t are closed subsets and B is an open subset, then the inclusion $\bigcap_{t \in T} A_t \subset B$ implies $\bigcap_{t \in T_0} A_t \subset B$ for a finite set $T_0 \subset T$.

A topological space X is said to be a T_0-*space* if, for every pair of distinct points x, y, there exists an open set containing exactly one of them.

A topological space X is said to be a T_1-*space* if, for every pair of distinct points x, y, there is an open set A such that $x \in A$ and $y \notin A$. A topological space X is a T_1-space if and only if every one-element