INTRODUCTORY FUNCTIONAL ANALYSIS WITH APPLICATIONS

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PREFACE

Purpose of the book. Functional analysis plays an increasing role in the applied sciences as well as in mathematics itself. Consequently, it becomes more and more desirable to introduce the student to the field at an early stage of study. This book is intended to familiarize the reader with the basic concepts, principles and methods of functional analysis and its applications.

Since a textbook should be written for the student, I have sought to bring basic parts of the field and related practical problems within the comfortable grasp of senior undergraduate students or beginning graduate students of mathematics and physics. I hope that graduate engineering students may also profit from the presentation.

Prerequisites. The book is elementary. A background in undergraduate mathematics, in particular, linear algebra and ordinary calculus, is sufficient as a prerequisite. Measure theory is neither assumed nor discussed. No knowledge in topology is required; the few considerations involving compactness are self-contained. Complex analysis is not needed, except in one of the later sections (Sec. 7.5), which is optional, so that it can easily be omitted. Further help is given in Appendix 1, which contains simple material for review and reference.

The book should therefore be accessible to a wide spectrum of students and may also facilitate the transition between linear algebra and advanced functional analysis.

Courses. The book is suitable for a one-semester course meeting five hours per week or for a two-semester course meeting three hours per week.

The book can also be utilized for shorter courses. In fact, chapters can be omitted without destroying the continuity or making the rest of the book a torso (for details see below). For instance:

Chapters 1 to 4 or 5 makes a very short course.

Chapters 1 to 4 and 7 is a course that includes spectral theory and other topics.

Content and arrangement. Figure 1 shows that the material has been organized into five major blocks.

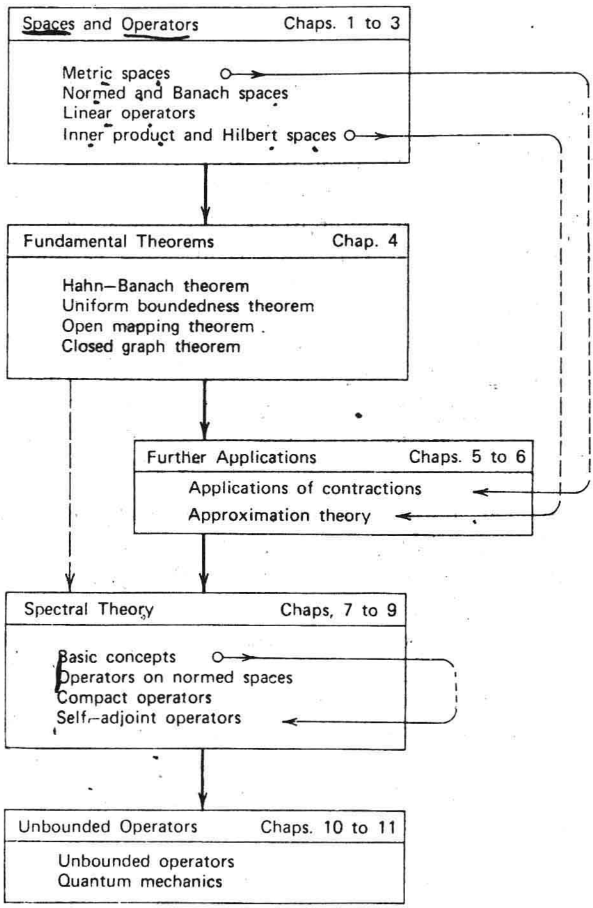

Fig. 1. Content and arrangement of material

Hilbert space theory (Chap. 3) precedes the basic theorems on normed and Banach spaces (Chap. 4) because it is simpler, contributes additional examples in Chap. 4 and, more important, gives the student a better feeling for the difficulties encountered in the transition from Hilbert spaces to general Banach spaces.

Chapters 5 and 6 can be omitted. Hence after Chap. 4 one can proceed directly to the remaining chapters (7 to 11).

Spectral theory is included in Chaps. 7 to 11. Here one has great flexibility. One may only consider Chap. 7 or Chaps. 7 and 8. Or one may focus on the basic concepts from Chap. 7 (Secs. 7.2. and 7.3) and then immediately move to Chap. 9, which deals with the spectral theory of bounded self-adjoint operators.

Applications are given at various places in the text. Chapters 5 and 6 are separate chapters on applications. They can be considered in sequence, or earlier if so desired (see Fig. 1):

Chapter 5 may be taken up immediately after Chap. 1.

Chapter 6 may be taken up immediately after Chap. 3.

Chapters 5 and 6 are optional since they are not used as a prerequisite in other chapters.

Chapter 11 is another separate chapter on applications; it deals with unbounded operators (in quantum physics), but is kept practically independent of Chap. 10.

Presentation. The material in this book has formed the basis of lecture courses and seminars for undergraduate and graduate students of mathematics, physics and engineering in this country, in Canada and in Europe. The presentation is detailed, particularly in the earlier chapters, in order to ease the way for the beginner. Less demanding proofs are often preferred over slightly shorter but more advanced ones.

In a book in which the concepts and methods are necessarily abstract, great attention should be paid to motivations. I tried to do so in the general discussion, also in carefully selecting a large number of suitable examples, which include many simple ones. I hope that this will help the student to realize that abstract concepts, ideas and techniques were often suggested by more concrete matter. The student should see that practical problems may serve as concrete models for illustrating the abstract theory, as objects for which the theory can yield concrete results and, moreover, as valuable sources of new ideas and methods in the further development of the theory.

Problems and solutions. The book contains more than 900 carefully selected problems. These are intended to help the reader in better understanding the text and developing skill and intuition in functional analysis and its applications. Some problems are very simple, to encourage the beginner. Answers to odd-numbered problems are given in Appendix 2. Actually, for many problems, Appendix 2 contains complete solutions.

The text of the book is self-contained, that is, proofs of theorems and lemmas in the text are given in the text, not in the problem set. Hence the development of the material does not depend on the problems and omission of some or all of them does not destroy the continuity of the presentation.

Reference material is included in Appendix 1, which contains some elementary facts about sets, mappings, families, etc.

References to literature consisting of books and papers are collected in Appendix 3, to help the reader in further study of the text material and some related topics. All the papers and most of the books are quoted in the text. A quotation consists of a name and a year. Here are two examples. "There are separable Banach spaces without Schauder bases; cf. P. Enflo (1973)." The reader will then find a corresponding paper listed in Appendix 3 under Enflo, P. (1973). "The theorem was generalized to complex vector spaces by H. F. Bohnenblust and A. Sobczyk (1938)." This indicates that Appendix 3 lists a paper by these authors which appeared in 1938.

Notations are explained in a list included after the table of contents.

Acknowledgments. I want to thank Professors Howard Anton (Drexel University), Helmut Florian (Technical University of Graz, Austria), Gordon E. Latta (University of Virginia), Hwang-Wen Pu (Texas A and M University), Paul V. Reichelderfer (Ohio University), Hanno Rund (University of Arizona), Donald Sherbert (University of Illinois) and Tim E. Traynor (University of Windsor) as well as many of my former and present students for helpful comments and constructive criticism.

I thank also John Wiley and Sons for their effective cooperation and great care in preparing this edition of the book.

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CONTENTS

NOTATIONS

In each line we give the number of the page on which the symbol is explained.

A^C	Complement of a set A 18, 609
A^T	Transpose of a matrix A 113
$B[a, b]$	Space of bounded functions 228
$B(A)$	Space of bounded functions 11
$BV[a, b]$	Space of functions of bounded variation 226
$B(X, Y)$	Space of bounded linear operators 118
$B(x; r)$	Open ball 18
$\tilde{B}(x; r)$	Closed ball 18
c	A sequence space 34
c_0	A sequence space 70
$\mathbf{C}$	Complex plane or the field of complex numbers 6, 51
$\mathbf{C}^n$	Unitary n-space 6
$C[a, b]$	Space of continuous functions 7
$C'[a, b]$	Space of continuously differentiable functions 110
$C(X, Y)$	Space of compact linear operators 411
$\mathfrak{D}(T)$	Domain of an operator T 83
$d(x, y)$	Distance from x to y 3
$\dim X$	Dimension of a space X 54
δ_{jk}	Kronecker delta 114
$\mathscr{E} = (E_\lambda)$	Spectral family 494
$\\|f\\|$	Norm of a bounded linear functional f 104
$\mathscr{G}(T)$	Graph of an operator T 292
I	Identity operator 84
inf	Infimum (greatest lower bound) 619
$L^p[a, b]$	A function space 62
l^p	A sequence space 11
l^∞	A sequence space 6
$L(X, Y)$	A space of linear operators 118
$M^\perp$	Annihilator of a set M 148
$\mathscr{N}(T)$	Null space of an operator T 83
0	Zero operator 84
$\varnothing$	Empty set 609

$\mathbf{R}$	Real line or the field of real numbers 5, 51
$\mathbf{R}^n$	Euclidean n-space 6
$\mathcal{R}(T)$	Range of an operator T 83
$R_\lambda(T)$	Resolvent of an operator T 370
$r_\sigma(T)$	Spectral radius of an operator T 378
$\rho(T)$	Resolvent set of an operator T 371
s	A sequence space 9
$\sigma(T)$	Spectrum of an operator T 371
$\sigma_c(T)$	Continuous spectrum of T 371
$\sigma_p(T)$	Point spectrum of T 371
$\sigma_r(T)$	Residual spectrum of T 371
span M	Span of a set M 53
sup	Supremum (least upper bound) 619
$\\|T\\|$	Norm of a bounded linear operator T 92
T^*	Hilbert-adjoint operator of T 196
$T^\times$	Adjoint operator of T 232
T^+, T^-	Positive and negative parts of T 498
T_λ^+, T_λ^-	Positive and negative parts of $T_\lambda = T - \lambda I$ 500
$T^{1/2}$	Positive square root of T 476
Var(w)	Total variation of w 225
$\xrightarrow{w}$	Weak convergence 257
X^*	Algebraic dual space of a vector space X 106
X'	Dual space of a normed space X 120
$\\|x\\|$	Norm of x 59
$\langle x, y\rangle$	Inner product of x and y 128
$x \perp y$	x is orthogonal to y 131
$Y^\perp$	Orthogonal complement of a closed subspace Y 146

CHAPTER 1
METRIC SPACES

Functional analysis is an abstract branch of mathematics that originated from classical analysis. Its development started about eighty years ago, and nowadays functional analytic methods and results are important in various fields of mathematics and its applications. The impetus came from linear algebra, linear ordinary and partial differential equations, calculus of variations, approximation theory and, in particular, linear integral equations, whose theory had the greatest effect on the development and promotion of the modern ideas. Mathematicians observed that problems from different fields often enjoy related features and properties. This fact was used for an effective unifying approach towards such problems, the unification being obtained by the omission of unessential details. Hence the advantage of such an *abstract approach* is that it concentrates on the essential facts, so that these facts become clearly visible since the investigator's attention is not disturbed by unimportant details. In this respect the abstract method is the simplest and most economical method for treating mathematical systems. Since any such abstract system will, in general, have various concrete realizations (concrete *models*), we see that the abstract method is quite versatile in its application to concrete situations. It helps to free the problem from isolation and creates relations and transitions between fields which have at first no contact with one another.

In the abstract approach, one usually starts from a set of elements satisfying certain axioms. The nature of the elements is left unspecified. This is done on purpose. The theory then consists of logical consequences which result from the axioms and are derived as theorems once and for all. This means that in this axiomatic fashion one obtains a mathematical structure whose theory is developed in an abstract way. Those general theorems can then later be applied to various special sets satisfying those axioms.

For example, in algebra this approach is used in connection with fields, rings and groups. In functional analysis we use it in connection with *abstract spaces;* these are of basic importance, and we shall consider some of them (Banach spaces, Hilbert spaces) in great detail. We shall see that in this connection the concept of a "space" is used in a very wide and surprisingly general sense. An *abstract space* will be a set of (unspecified) elements satisfying certain axioms. And by choosing different sets of axioms we shall obtain different types of abstract spaces.

The idea of using abstract spaces in a systematic fashion goes back to M. Fréchet (1906)[1] and is justified by its great success.

In this chapter we consider metric spaces. These are fundamental in functional analysis because they play a role similar to that of the real line **R** in calculus. In fact, they generalize **R** and have been created in order to provide a basis for a unified treatment of important problems from various branches of analysis.

We first define metric spaces and related concepts and illustrate them with typical examples. Special spaces of practical importance are discussed in detail. Much attention is paid to the concept of completeness, a property which a metric space may or may not have. Completeness will play a key role throughout the book.

Important concepts, brief orientation about main content

A *metric space* (cf. 1.1-1) is a set X with a *metric* on it. The metric associates with any pair of elements (*points*) of X a *distance*. The metric is defined axiomatically, the axioms being suggested by certain simple properties of the familiar distance between points on the real line **R** and the complex plane **C**. Basic examples (1.1-2 to 1.2-3) show that the concept of a metric space is remarkably general. A very important additional property which a metric space may have is *completeness* (cf. 1.4-3), which is discussed in detail in Secs. 1.5 and 1.6. Another concept of theoretical and practical interest is *separability* of a metric space (cf. 1.3-5). Separable metric spaces are simpler than nonseparable ones.

1.1 Metric Space

In calculus we study functions defined on the real line **R**. A little reflection shows that in limit processes and many other considerations we use the fact that on **R** we have available a distance function, call it d, which associates a *distance* $d(x, y) = |x - y|$ with every pair of points

[1] References are given in Appendix 3, and we shall refer to books and papers listed in Appendix 3 as is shown here.

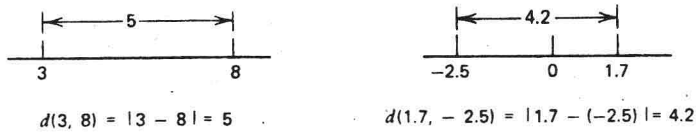

Fig. 2. Distance on **R**

$x, y \in \mathbf{R}$. Figure 2 illustrates the notation. In the plane and in "ordinary" three-dimensional space the situation is similar.

In functional analysis we shall study more general "spaces" and "functions" defined on them. We arrive at a sufficiently general and flexible concept of a "space" as follows. We replace the set of real numbers underlying **R** by an *abstract* set X (set of elements whose nature is left unspecified) and introduce on X a "distance function" which has only a few of the most fundamental properties of the distance function on **R**. But what do we mean by "most fundamental"? This question is far from being trivial. In fact, the choice and formulation of axioms in a definition always needs experience, familiarity with practical problems and a clear idea of the goal to be reached. In the present case, a development of over sixty years has led to the following concept which is basic and very useful in functional analysis and its applications.

1.1-1 Definition (Metric space, metric). A *metric space* is a pair (X, d), where X is a set and d is a *metric on* X (or *distance function on* X), that is, a function defined[2] on $X \times X$ such that for all $x, y, z \in X$ we have:

(M1) d is real-valued, finite and nonnegative.

(M2) $d(x, y) = 0$ if and only if $x = y$.

(M3) $d(x, y) = d(y, x)$ *(Symmetry)*

(M4) $d(x, y) \leq d(x, z) + d(z, y)$ **(Triangle inequality).** ∎

[2] The symbol × denotes the *Cartesian product* of sets: $A \times B$ is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence $X \times X$ is the set of all ordered pairs of elements of X.

A few related terms are as follows. X is usually called the *underlying set* of (X, d). Its elements are called *points*. For fixed x, y we call the nonnegative number $d(x, y)$ the *distance* from x to y. Properties (M1) to (M4) are the *axioms of a metric*. The name "triangle inequality" is motivated by elementary geometry as shown in Fig. 3.

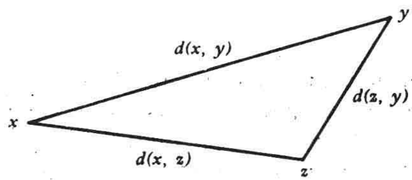

Fig. 3. Triangle inequality in the plane

From (M4) we obtain by induction the *generalized triangle inequality*

$$d(x_1, x_n) \leqq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n). \tag{1}$$

Instead of (X, d) we may simply write X if there is no danger of confusion.

A **subspace** $(Y, \tilde{d})$ of (X, d) is obtained if we take a subset $Y \subset X$ and restrict d to $Y \times Y$; thus the metric on Y is the restriction[3]

$$\tilde{d} = d|_{Y \times Y}.$$

$\tilde{d}$ is called the metric **induced** on Y by d.

We shall now list examples of metric spaces, some of which are already familiar to the reader. To prove that these are metric spaces, we must verify in each case that the axioms (M1) to (M4) are satisfied. Ordinarily, for (M4) this requires more work than for (M1) to (M3). However, in our present examples this will not be difficult, so that we can leave it to the reader (cf. the problem set). More sophisticated

[3] Appendix 1 contains a review on mappings which also includes the concept of a restriction.