Electricity and Magnetism

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CAMBRIDGE UNIVERSITY PRESS
Cambridge
New York Port Chester Melbourne Sydney

Published by the Press Syndicate of the University of Cambridge
The Pitt Building, Trumpington Street, Cambridge CB2 1RP
40 West 20th Street, New York, NY 10011-4211, USA
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

First published 1991

Printed in Great Britain at the University Press, Cambridge

British Library cataloguing in publication data
Cottingham, W. N.
Electricity and magnetism.
1. Electromagnetism
I. Title II. Greenwood, D. A.
537

Library of Congress cataloguing in publication data
Cottingham, W. N.
Electricity and magnetism/W. N. Cottingham, D. A. Greenwood.
p. cm.
ISBN 0 521 36229 6. ISBN 0 521 36803 0 (pbk.)
1. Electromagnetism. I. Greenwood, D. A. II. Title.
QC760.C674 1991
537 dc20 90-26471 CIP

ISBN 0 521 36229 6 hardback
ISBN 0 521 36803 0 paperback

This is an undergraduate textbook on the physics of electricity, magnetism, and electromagnetic fields and waves. It is written mainly with the physics student in mind, although it will also be of use to students of electrical and electronic engineering. The approach is concise but clear. The authors have assumed that the reader will be familiar with the basic phenomena; however, they have set out the theory in a completely self-contained and coherent way.

After a short mathematical prologue, the theory of electricity and magnetism, and the relationship between them, is developed. The relationship between the microscopic structure of matter and the macroscopic electric and magnetic fields is stressed throughout, and the theory is developed to the point where the reader can appreciate the beauty and coherence of the Maxwell equations which describe these fields. The theory is then applied to a wide range of topics from the properties of materials, including semiconductors and superconductors, to the generation of radiation by electric currents. A final chapter makes the connection between Maxwell's equations and the special theory of relativity. Each chapter ends with a set of problems, answers to which are also provided.

The authors have extensive experience of teaching physics to undergraduate students at the University of Bristol. The clarity of the mathematical treatment they provide will facilitate a thorough grasp of the subject, and makes this a highly attractive text.

Preface

This is an undergraduate textbook on the physics of electricity, magnetism, and electromagnetic fields and waves. It is written mainly with the physics student in mind, although it will also be of use to students of electrical and electronic engineering. We have aimed to produce a concise text which emphasises the meaning and significance of the concepts that appear in the theory, and the overall coherence and beauty of the Maxwell equations.

The theory is set out in a self-contained way, but we assume that the reader will already have some knowledge of the basic phenomena of electricity and magnetism. (At the University of Bristol there is an established tradition of demonstration experiments in the introductory first year physics lectures.) We also assume some familiarity with the mathematics of scalar and vector fields, and the properties of the ∇ operator. The basic theorems are set out for reference in the Mathematical Prologue. The Dirac δ-function is introduced in a non-rigorous way on the first page of Chapter 1, and used freely: in our experience, physics students readily accept this as an obviously useful mathematical device. A few other mathematical tools are developed in the text, as and when they are needed. To avoid impeding the flow of the main argument, the technical details of mathematical manipulations are sometimes relegated to the problems.

The relationship between the microscopic structure of matter and the macroscopic fields which are the main concern of the text is stressed from the start, albeit from a classical standpoint. Not only is this basic for an understanding of the theory, but it is important to appreciate the limitations on the theory's domain of applicability. For example, the important technology of electronic devices lies at the boundary between what is clearly microscopic and what is clearly macroscopic; an appreciation of the nature of the macroscopic fields in materials is essential to an understanding of the underlying physics.

We arrive at the Maxwell equations in Chapter 5, as abstractions from laboratory experiments. Subsequent chapters cover a wide range of applications, from the macroscopic description of the electric and magnetic properties of material media, including superconductors, to the generation of radiation by electric currents. The final chapter makes the connection between the Maxwell equations and the special theory of relativity.

We envisage our book as one to work from. The problems following each chapter illustrate and extend the text, and form an essential part of that work.

We are grateful to Bob Chambers, David Gibbs, Brian Pollard, and many other colleagues, who have helped clarify our presentation of the subject at many points. We thank Margaret James, who worked valiantly on the typescript.

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Units, constants, and formulae

SI units

	Name	Symbol	Basic units
electric current	ampere	A	A
electric charge	coulomb	C	s A
potential difference	volt	V	kg m^2 s^{-3} A^{-1}
capacitance	farad	F	kg^{-1} m^{-2} s^4 A^2
resistance	ohm	Ω	kg m^2 s^{-3} A^{-2}
magnetic field	tesla	T	kg s^{-2} A^{-1}
magnetic flux	weber	Wb	kg m^2 s^{-2} A^{-1}
inductance	henry	H	kg m^2 s^{-2} A^{-2}

$$\varepsilon_0 \approx 8.85 \times 10^{-12}\ \mathrm{kg^{-1}\ m^{-3}\ s^4\ A^2}$$

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{kg\ m\ s^{-2}\ A^{-2}}$$

$$(\mu_0 \varepsilon_0)^{-1} = c^2, \quad c = 299\,792\,458\ \mathrm{m\ s^{-1}}$$

Physical constants

Proton charge	e	1.60218×10^{-19} C
Electron mass	m_e	9.1094×10^{-31} kg
Proton mass	m_p	1.67262×10^{-27} kg
Boltzmann's constant	k_B	1.3807×10^{-23} J K^{-1}
Avogadro's number	N_A	6.0221×10^{23} mol^{-1}
Planck's constant	h	6.6261×10^{-34} J s

$$1\ \mathrm{eV} \approx 1.602\,18 \times 10^{-19}\ \mathrm{J}$$

Notation

$\mathbf{r}$, $\mathbf{k}$, etc., denote vectors (x, y, z), (k_x, k_y, k_z), and $r = |\mathbf{r}| = (x^2+y^2+z^2)^{\frac{1}{2}}$, $\hat{\mathbf{r}} = \mathbf{r}/r$, etc.
Spherical polar coordinates are denoted by (r, θ, ϕ).
Cylindrical polar coordinates are denoted by (ρ, ϕ, z), where $\rho = (x^2+y^2)^{\frac{1}{2}}$.

$$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} = \frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

Vector identities

u, v denote scalar functions; $\mathbf{F}$, $\mathbf{G}$ denote vector functions.

$$\boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}$$

$$\boldsymbol{\nabla} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = \frac{\partial u}{\partial x}\mathbf{i}+\frac{\partial u}{\partial y}\mathbf{j}+\frac{\partial u}{\partial z}\mathbf{k}$$

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}, \quad \boldsymbol{\nabla}\times\mathbf{F} = \left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{F}) = 0, \quad \boldsymbol{\nabla}\times\boldsymbol{\nabla} u = 0$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{F}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{F})-\nabla^2\mathbf{F}, \text{ where } \nabla^2\mathbf{F} = (\nabla^2 F_x, \nabla^2 F_y, \nabla^2 F_z)$$

$$\boldsymbol{\nabla}(uv) = u\boldsymbol{\nabla} v+v\boldsymbol{\nabla} u$$

$$\boldsymbol{\nabla}\cdot(u\mathbf{F}) = \mathbf{F}\cdot\boldsymbol{\nabla} u+u\boldsymbol{\nabla}\cdot\mathbf{F}$$

$$\boldsymbol{\nabla}\cdot(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot(\boldsymbol{\nabla}\times\mathbf{F})-\mathbf{F}\cdot(\boldsymbol{\nabla}\times\mathbf{G})$$

$$\boldsymbol{\nabla}\times(u\mathbf{F}) = \boldsymbol{\nabla} u\times\mathbf{F}+u\boldsymbol{\nabla}\times\mathbf{F}$$

$$\boldsymbol{\nabla}\times(\mathbf{F}\times\mathbf{G}) = \mathbf{F}(\boldsymbol{\nabla}\cdot\mathbf{G})-\mathbf{G}(\boldsymbol{\nabla}\cdot\mathbf{F})+(\mathbf{G}\cdot\boldsymbol{\nabla})\mathbf{F}-(\mathbf{F}\cdot\boldsymbol{\nabla})\mathbf{G}$$

Useful results: $\boldsymbol{\nabla}\cdot\mathbf{r} = 3$, $\boldsymbol{\nabla}(r^n) = nr^{n-1}\hat{\mathbf{r}}$
If $\mathbf{a}$ is a constant vector, $\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{r}) = \mathbf{a}$, $(\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{r} = \mathbf{a}$.

Maxwell's equations

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$$

$$\nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

In material media, these become

$$\nabla \cdot \mathbf{D} = \rho_{\text{free}}$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{\text{free}}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

where

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P},$$

$$\mathbf{H} = (\mathbf{B}/\mu_0) - \mathbf{M}$$

Glossary of symbols

$\mathbf{A}(\mathbf{r})$, $\mathbf{A}(\mathbf{r}, t)$ vector potential
$\mathbf{a}$ acceleration
a_0 Bohr radius
$\mathbf{B}$ magnetic field
$\mathbf{B}_c$, B_{c1}, B_{c2} critical magnetic fields
C capacitance
$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$ electric displacement
$\mathbf{E}$, $\mathbf{E}_{at}$ macroscopic, atomic electric field
$\mathscr{E}$ electromotive force (e.m.f.), energy
$\mathbf{F}$ force
$F(r)$ § 1.2 averaging function
$\mathscr{F}$ magnetic flux
$\mathbf{H} = (\mathbf{B}/\mu_0) - \mathbf{M}$
I electric current
$\mathbf{J}$, $\mathbf{J}_{at}$ macroscopic, atomic current density
$\mathrm{K} = 2\omega\kappa/c$ absorption coefficient
$\mathbf{k}$ wavevector, $|\mathbf{k}| = 2\pi/\lambda$
$\mathbf{L}$ angular momentum vector
L length, § 1.2 averaging length
L, L_{ij} self, mutual inductance
$d\mathbf{l}$ line element
$\mathbf{M}$ magnetisation
$\mathbf{m}$ magnetic dipole moment
$\mathbf{N}$ Poynting vector
N number per unit volume
$\hat{\mathbf{n}}$ unit normal to surface
$n = \mathrm{Re}(\sqrt{\varepsilon_r})$ refractive index
$\mathbf{P}$ polarisation
P power
$\mathbf{p}$ electric dipole moment

Q electric charge
Q_{ij} electric quadrupole tensor
R resistance, §12.7 reflection coefficient
$d\mathbf{S}$ element of surface
S surface area
T temperature, §12.7 transmission coefficient
T_c critical temperature
U field energy
V volume, potential difference
W work function
Z_0 characteristic impedance

α atomic polarisability
γ damping constant
δ skin depth
$\varepsilon_r = 1 + \chi_e$ relative permittivity (dielectric constant)
$\kappa = \mathrm{Im}(\sqrt{\varepsilon_r})$
Λ London penetration depth
λ wavelength
$\boldsymbol{\mu}$ magnetic moment of molecule
$\mu_r = 1 + \chi_r$ relative permeability
ν frequency
ρ, ρ_{el}, ρ_{at} macroscopic, electronic, atomic charge density
σ surface charge density, electrical conductivity, cross-section
τ collision time, damping time
$\Phi(\mathbf{r})$, $\Phi_{at}(\mathbf{r})$ macroscopic, atomic electrostatic potential
$\Phi(\mathbf{r}, t)$ scalar potential
Φ_0 §14.5 flux quantum
ϕ angular coordinate, phase angle
$\chi(\mathbf{r})$, $\chi(\mathbf{r}, t)$ §9.1, §16.1 gauge function
χ_e electric susceptibility
χ_m magnetic susceptibility
$d\Omega$ element of solid angle
ω angular frequency
ω_p plasma frequency

Contents

Mathematical Prologue

In the chapters which follow, we assume that you are already familiar with the basic mathematics of scalar and vector fields in three dimensions, the properties of the ∇ operator, the integral theorems which hold for these fields, and so forth. In this prologue, we remind you of some basic definitions, and outline (without proof) those mathematical theorems of which we shall make extensive use. We also establish our notation and sign conventions.

We envisage space filled with electromagnetic fields, and at any instant we describe these fields mathematically using functions which may be scalar functions of position (like the potential $\Phi(\mathbf{r})$) or vector functions of position (like the electric field $\mathbf{E}(\mathbf{r})$). We shall assume that the functions which appear in the theory are continuous, and have derivatives existing as required, except perhaps at special points or on special surfaces. Singularities in the mathematics will usually correspond to singularities in the physics. For example, the electrostatic potential of a point charge Q at the origin is $Q/4\pi\varepsilon_0 r$, and this function satisfies our conditions except at $\mathbf{r} = 0$, which is the position of the point charge.

We sometimes focus on these fields in limited regions of space, say inside a volume V enclosed by a surface S, or over a surface $S(\Gamma)$ bounded by a curve Γ.

P.1 Volume integrals

Volume integrals will often arise naturally in the theory, for example when we calculate the total charge or total energy in some volume V of space. The volume integral of a function $f(\mathbf{r})$ is defined by

$$\int_V f(\mathbf{r})\,\mathrm{d}V = \mathrm{limit}\left(\sum_i f(\mathbf{r}_i)\,\delta V_i\right), \tag{P.1}$$

where the volume V is dissected into elements δV_i, and $\mathbf{r}_i$ lies in δV_i. The linear dimensions of the δV_i go to zero in the limit.

We might for example take $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ so that

$$\int_V f(\mathbf{r})\,\mathrm{d}V = \iiint f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

or work in spherical polar coordinates, and take $\mathrm{d}V = r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$, or use any other coordinate system which is convenient for the problem under discussion. However, we wish to emphasise the *meaning* of integrals, rather than techniques for their evaluation. For practical purposes, their values can always be found to any required accuracy by direct computation of approximating sums (see the definitions P.1–P.5).

P.2 Surfaces and surface integrals

A closed surface S, or an open surface $S(\Gamma)$, can be dissected into elements, as is indicated in Fig. P.1. If the surface is smooth, and the elements are sufficiently small, each element can be approximated by an element of a plane. Two important properties can be defined for a plane surface element: the first is its area δS, and the second a unit vector $\hat{\mathbf{n}}$ that is normal to the plane. It is sometimes useful to consider the surface element as itself a vector $\delta\mathbf{S} = \hat{\mathbf{n}}\,\delta S$, of magnitude δS, pointing in the direction $\hat{\mathbf{n}}$.

The area of a surface is defined to be:

$$\text{area} = \int_S \mathrm{d}S = \text{limit}\,(\sum_i \delta S_i), \tag{P.2}$$

where the linear dimensions of the δS_i go to zero in the limit.

In a similar way we can define integrals of functions over surfaces:

$$\int_S f(\mathbf{r})\,\mathrm{d}S = \text{limit}\,(\sum_i f(\mathbf{r}_i)\,\delta S_i), \tag{P.3}$$

where $\mathbf{r}_i$ is a point in δS_i.

The case when $f(\mathbf{r}) = \mathbf{F}(\mathbf{r})\cdot\hat{\mathbf{n}}$ often occurs in physical theories, for example when we consider the flow of electric charge or energy across a surface. We may conveniently write

$$\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_S \mathbf{F}\cdot\mathrm{d}\mathbf{S}.$$

It is important to make rules which specify the direction of the unit normal $\hat{\mathbf{n}}$.

For a surface S enclosing a volume V we always take the direction of $\hat{\mathbf{n}}$ to be pointing outwards from the volume (Fig. P.1).

For a two-sided surface $S(\Gamma)$, bounded by a curve Γ, we relate the direction of $\hat{\mathbf{n}}$ to the sense in which we go round Γ by a 'right-hand' rule