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HANDBOOK OF CATEGORICAL ALGEBRA 3

Categories of Sheaves

Francis Borceux

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ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

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Categories of Sheaves

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to Bill Lawvere,
for his kindness and his genius

Preface to volume 3

This third volume of the *Handbook of categorical algebra* is neither a book on topos theory nor a book on sheaf theory. Our main concern is to study various approaches to the notion of a "set valued sheaf" and describe the structure and the properties of the corresponding categories of sheaves. Those categories are toposes, indeed, so that this book can serve also as a first introduction to the theory of toposes and, of course, to the theory of sheaves.

The crucial idea behind the notion of a sheaf is to work not just with a "plain" set of elements, but with a whole system of elements at various levels. Of course, reasonable rules are imposed concerning the interactions between the various levels: an element at some level can be restricted to all lower levels and, if a compatible family of elements is given at various individual levels, it is possible to "glue" the family into an element defined at the global level covered by the individual ones. The various notions of sheaf depend on the way the words "level", "restriction" and "covering" are defined.

The easiest examples are borrowed from topology, where the various "levels" are the open subsets of a fixed space X: for example a continuous function on X may very well be defined "at the level of the open subset $U \subseteq X$", without being the restriction of a continuous function defined on the whole of X. This notion of a sheaf on a topological space is studied as such in chapter 2 and important examples are given, including the sheaf representation of a commutative ring in terms of local rings.

It is a matter of fact that the notion of a sheaf on a topological space, even if admitting elegant topological descriptions, depends only on the structure of the lattice of open subsets, not at all on the set of points topologized by these open subsets. A "locale" is a lattice recapturing the most characteristic properties of the lattices of open subsets: we study this notion for itself in chapter 1 and develop in chapter 2 the corresponding notion of a sheaf on a locale, in parallel with the notion of a sheaf on a space.

In chapter 1, devoted to locales, we emphasize the fact that this structure fits the needs of both studying intuitionistic logic and of treating topological properties algebraically. In chapter 2, devoted to sheaves, we underline again the two approaches in terms of "models of an intuitionistic set theory" and "étale topological mappings".

In a sheaf on a locale, the elements are defined at various levels, with corresponding restriction and gluing properties along the "coverings" $u = \bigvee_{i \in I} u_i$. A "site" is even more general: it is a small category provided with a so-called Grothendieck topology, i.e. a good notion of "coverings" allowing a corresponding generalization of the notion of a sheaf in terms of "restrictions and gluings". The categories of sheaves obtained in this way are the "Grothendieck toposes", which play an important role in algebraic geometry. Studying them is the purpose of chapter 3.

Chapter 4 throws up a strong link between algebra and sheaf theory. It is proved that for "many" mathematical theories $\mathcal{T}$, it is possible to find a "generic" model M of $\mathcal{T}$ in some canonical Grothendieck topos $\mathscr{E}$; the model M is generic in the sense that all the possible models of $\mathcal{T}$ in all the possible Grothendieck toposes $\mathscr{F}$ (including the "topos" of sets) are just obtained as the inverse images of the model $M \in \mathscr{E}$ along the geometric morphisms of toposes $f\colon \mathscr{F} \longrightarrow \mathscr{E}$. This is the so-called "classifying topos theorem". Among the possible such theories $\mathcal{T}$, we find all the theories defined by colimits and finite limits, thus in particular all the usual algebraic theories.

Chapter 5 is the one which focuses on the categorical structure of the categories of sheaves. The general notion of a "topos" is exhibited and it is observed that all categories of sheaves previously studied are examples of toposes. The various exactness and completeness properties of toposes are studied.

From the very beginning of this volume, it is often observed that locales or sheaves provide situations in close connection with intuitionistic logic. This is definitely formalized in chapter 6, by showing how proofs in an arbitrary topos can be worked out just as in the category of "intuitionistic sets". In other words, we describe the internal logic of a topos, which is of intuitionistic nature, and provides a powerful tool of investigation.

Chapter 7 focuses on some classical non-intuitionistic properties of set theory which do not hold in general for toposes, like the axiom of choice or the law of excluded middle. The axiom of infinity is discussed in chapter 8: it often holds in a topos, for example in all Grothendieck toposes.

Finally we "close the circle" in chapter 9 by showing how the notion of a topos – which formalizes axiomatically the notion of a "category of sheaves" – is a very natural setting for developing the most general theory of sheaves! We introduce the notion of a "topology" in a topos and study the corresponding category of sheaves, which is again a topos.

Introduction to this handbook

My concern in writing the three volumes of this *Handbook of categorical algebra* has been to propose a directly accessible account of what – in my opinion – a Ph.D. student should ideally know of category theory before starting research on one precise topic in this domain. Of course, there are already many good books on category theory: general accounts of the state of the art as it was in the late sixties, or specialized books on more specific recent topics. If you add to this several famous original papers not covered by any book and some important but never published works, you get a mass of material which gives probably a deeper insight in the field than this *Handbook* can do. But the great number and the diversity of those excellent sources just act to convince me that an integrated presentation of the most relevant aspects of them remains a useful service to the mathematical community. This is the objective of these three volumes.

The first volume presents those basic aspects of category theory which are present as such in almost every topic of categorical algebra. This includes the general theory of limits, adjoint functors and Kan extensions, but also quite sophisticated methods (like categories of fractions or orthogonal subcategories) for constructing adjoint functors. Special attention is also devoted to some refinements of the standard notions, like Cauchy completeness, flat functors, distributors, 2-categories, bicategories, lax-functors, and so on.

The second volume presents a selection of the most famous classes of "structured categories", with the exception of toposes which appear in volume 3. The first historical example is that of abelian categories, which we follow by its natural non-additive generalizations: the regular and exact categories. Next we study various approaches to "categories of models of a theory": algebraic categories, monadic categories, locally presentable and accessible categories. We introduce also enriched category theory and devote some attention to topological categories. The volume ends with the theory of fibred categories "à la Bénabou".

The third volume is entirely devoted to the study of categories of sheaves: sheaves on a space, a locale, a site. This is the opportunity for developing the essential aspects of the theory of locales and introducing Grothendieck toposes. We relate this with the algebraic aspects of volume 2 by proving in this context the existence of a classifying topos for coherent theories. All these considerations lead naturally to the notion of an elementary topos. We study quite extensively the internal logic of toposes, including the law of excluded middle and the axiom of infinity. We conclude by showing how toposes are a natural context for defining sheaves.

Besides a technical development of the theory, many people appreciate historical notes explaining how the ideas appeared and grew. Let me tell you a story about that.

It was in July, I don't remember the year. I was participating in a summer meeting on category theory at the Isles of Thorns, in Sussex. Somebody was actually giving a talk on the history of Eilenberg and Mac Lane's collaboration in the forties, making clear what the exact contribution of the two authors was. At some point, somebody in the audience started to complain about the speaker giving credit to Eilenberg and Mac Lane for some basic aspect of their work which – he claimed – they borrowed from somebody else. A very sophisticated and animated discussion followed, which I was too ignorant to follow properly. The only things I can remember are the names of the two opponents: the speaker was Saunders Mac Lane and his opponent was Samuel Eilenberg. I was not born when they invented category theory. With my little story in mind, maybe you will forgive me for not having tried to give credit to anybody for the notions and results presented in this *Handbook*.

Let me conclude this introduction by thanking the various typists for their excellent job and my colleagues of the Louvain-la-Neuve category seminar for the fruitful discussions we had on various points of this *Handbook*. I want especially to acknowledge the numerous suggestions Enrico Vitale has made for improving the quality of my work.

Handbook of categorical algebra

Contents of the three volumes

Contents

1
Locales

A topological space is built up from points and open subsets. But in some sense, open subsets are much more essential than points. Indeed, knowing the points of the space does not give any information on the topology. On the other hand, the lattice of open subsets very often characterizes at the same time the set of points. For example in a Hausdorff space, the points of the space can be characterized as the atoms of the lattice of closed subsets, which is just the dual of the lattice of open subsets.

A locale is just a lattice which mimics the properties of the lattice of open subsets of a topological space. Locales appear very naturally when studying sheaves (see chapters 2, 3) and, even when studying sheaves on a topological space, many constructions lead to the consideration of locales which are no longer isomorphic to the lattice of open subsets of a space.

But besides generalizing nicely the notion of topological space, locales are important for a completely different reason: they satisfy all the axioms of intuitionistic propositional calculus, i.e., roughly speaking, the classical propositional calculus without the law of excluded middle. This last remark will turn out to grow in importance through the remaining chapters of this book. We shall start by making more precise this reference to intuitionistic logic.

1.1 The intuitionistic propositional calculus

Let us consider a mathematical theory, say the theory of groups, and all the formulas we can write in this theory, like

$$\forall x \ \exists y \ \ x + y = 0,$$
$$\forall x \ \forall y \ \ x + y = 0.$$

Some of the formulas are true (like the first one), some of the formulas are false (like the second one). To prove the validity of a formula, one uses of course the axioms of the theory of groups, but also the axioms and deduction rules of propositional and predicate calculus. The propositional calculus is, roughly speaking, the theory which studies the various consequences one can infer from the validity of some formulas, by combining them using the logical connectors $\wedge$ (and), $\vee$ (or), $\Rightarrow$ (implies), $\neg$ (not). The predicate calculus takes additionally into account the two quantifiers $\exists$ (there exists) and $\forall$ (for all). Clearly, the consequences we can infer from some data depend on the assumptions we accept for our propositional or predicate calculus. First of all there are axioms, which declare that some types of formulas are necessarily "true"; we write $\vdash \varphi$ to indicate the truth of φ. Next there are deduction rules, which assert that given some true formulas, some derived formula is true as well.

Definition 1.1.1 *The intuitionistic propositional calculus is the one having for axioms:*

(PC1) $\vdash \varphi \Rightarrow (\psi \Rightarrow \varphi)$;
(PC2) $\vdash \big(\varphi \Rightarrow (\psi \Rightarrow \theta)\big) \Rightarrow \big((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \theta)\big)$;
(PC3) $\vdash \varphi \Rightarrow \big(\psi \Rightarrow (\varphi \wedge \psi)\big)$;
(PC4) $\vdash (\varphi \wedge \psi) \Rightarrow \varphi$;
(PC5) $\vdash (\varphi \wedge \psi) \Rightarrow \psi$;
(PC6) $\vdash \varphi \Rightarrow (\varphi \vee \psi)$;
(PC7) $\vdash \psi \Rightarrow (\varphi \vee \psi)$;
(PC8) $\vdash (\varphi \Rightarrow \theta) \Rightarrow \Big((\psi \Rightarrow \theta) \Rightarrow \big((\varphi \vee \psi) \Rightarrow \theta\big)\Big)$;
(PC9) $\vdash (\varphi \Rightarrow \psi) \Rightarrow \big((\varphi \Rightarrow \neg\psi) \Rightarrow \neg\varphi\big)$;
(PC10) $\vdash \neg\varphi \Rightarrow (\varphi \Rightarrow \psi)$;

and for rule of deduction, the modus ponens:

if $\vdash \varphi$ and $\vdash \varphi \Rightarrow \psi$,
then $\vdash \psi$.

In this definition, φ, ψ, θ *are arbitrary formulas.*

Definition 1.1.2 *The classical propositional calculus is the one obtained from the intuitionistic propositional calculus by adding the axiom*

$$\vdash \varphi \vee \neg\varphi$$

(the so-called "law of the excluded middle").