Lecture Notes in Mathematics

Vladimir I. Gurariy
Wolfgang Lusky

# Geometry of Müntz Spaces and Related Questions

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After the completion of our book the first named author, Vladimir I. Gurariy, died. The world lost a great mathematician and I lost a close friend.

Wolfgang Lusky

# Preface

Let $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$ be an increasing sequence of non-negative numbers:

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$$

Moreover let $M(\Lambda) = \{t^{\lambda_k}\}_{k=0}^{\infty}$ be the sequence of the functions $t^{\lambda_k}$ on $[0,1]$ and let $[M(\Lambda)]_E$ be the closed linear span of $M(\Lambda)$ in a given Banach space $E$ containing $M(\Lambda)$. We call $M(\Lambda)$ a Müntz sequence and $[M(\Lambda)]_E$ a Müntz space.

In our book we shall be mainly concerned with $E = C := C[0,1]$, the Banach space of all realvalued continuous functions on $[0,1]$ endowed with the sup-norm, and $E = C_0 := C_0[0,1]$, the subspace of $C$ consisting of all those functions $f \in C$ with $f(0) = 0$. Furthermore we deal with $E = L_p = L_p[0,1]$, $1 \leq p \leq \infty$, the space of all (classes of) realvalued measurable functions $f$ on $[0,1]$ with

$$||f||_{L_p} = \left(\int_0^1 |f(t)|^p dt\right)^{1/p} < \infty \quad \text{if } 1 \leq p < \infty\,.$$

If $p = \infty$ then we take for $||f||_{L_\infty}$ the essential sup-norm instead.

We want to study geometric properties of the corresponding Müntz sequences and spaces. Let us begin with the famous Müntz theorem, [110]:

For $E = C$ or $E = L_p$, $1 \leq p < \infty$, we have

$$[M(\Lambda)]_E \neq E \text{ if and only if } \sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty\,.$$

(A proof of this fact in more generality will be given in 6.1.)

So, if $\sum_{k=1}^{\infty} 1/\lambda_k < \infty$ we obtain new Banach spaces $[M(\Lambda)]_E$. This sets the stage for the central problem we discuss in (Part II of) our book:

What kind of Banach space $[M(\Lambda)]_E$ do we obtain depending on the given $\Lambda$ if $\sum_{k=1}^{\infty} 1/\lambda_k < \infty$?

This problem is far from being solved. Here we present the known theorems and prove new results in this direction. For example, if $\Lambda$ is quasilacunary then $[M(\Lambda)]_{L_p}$ is isomorphic to $l_p$ for $1 \le p < \infty$ and $[M(\Lambda)]_C$ is isomorphic to $c_0$ (Sect. 9.1). But for non-quasilacunary $\Lambda$ this is not always the case. There are at least two different isomorphism classes for $[M(\Lambda)]_C$ (Sect. 10.2). Moreover there is a continuum of different isometry classes for $[M(\Lambda)]_C$ (Sect. 10.4). In general, $[M(\Lambda)]_E$ can be regarded as a sequence space rather than a function space. $[M(\Lambda)]_{L_p}$ is always isomorphic to a subspace of $l_p$ and $[M(\Lambda)]_C$ is isomorphic to a subspace of $c_0$ provided that the Müntz condition $\sum_k 1/\lambda_k < \infty$ and the gap condition $\inf_k(\lambda_{k+1} - \lambda_k) > 0$ are satisfied. In addition, $[M(\Lambda)]_{L_1}$ is always isomorphic to a dual Banach space (Sect. 9.1).

It is an open problem if every $[M(\Lambda)]_E$ has a basis. We discuss more general bounded approximation properties in Chap. 9. However, $[M(\Lambda)]_C$ can never have a monotone basis (Sect. 9.4). In this context it is interesting to note that $M(\Lambda)$ is always a minimal system provided that the Müntz condition holds. But $M(\Lambda)$ is never a basis or even uniformly minimal in $[M(\Lambda)]_E$ for $E = C$ or $E = L_p$ unless $\Lambda$ is lacunary (Sect. 9.3). In contrast to Müntz sequences the trigonometric system $\{z^k\}_{k=-\infty}^{\infty}$ on $\{z \in \mathbf{C} : |z| = 1\}$ is uniformly minimal and even an Auerbach system. The traditional bridge between the trigonometric system and the classical Müntz system $\{t^n\}_{n=0}^{\infty}$, the substitution by Chebyshev polynomials [12], breaks down if we go over to subsequences of $\{t^n\}_{n=0}^{\infty}$. So there is no way to relate a general Müntz sequence $\{t^{\lambda_n}\}_{n=0}^{\infty}$ to the trigonometric system.

It is even unknown in general if the finite dimensional Müntz spaces $[M(\{\lambda_0, \lambda_1, \dots, \lambda_n\})]_C$ have uniformly bounded basis constants. In Sects. 10.3 and 12.2 we discuss some special cases and related questions. In Chap. 12 we investigate phenomena which, we feel, deserve further investigation. Take a Müntz sequence $\{t^{\lambda_k}\}_{k=1}^{\infty}$, fix $n$ and put $B_m = \text{span}\{t^{\lambda_{m+1}}, \dots, t^{\lambda_{m+n}}\}$ in $C$. Then, for many different $\Lambda = \{\lambda_k\}_{k=1}^{\infty}$, the sequence of n-dimensional Banach spaces $\{B_m\}_{m=1}^{\infty}$ converges to the subspace $\text{span}\{t, t\log t, \dots, t^{n-1}\log t\}$ of $C$ with respect to the logarithm of the Banach-Mazur distance. This might be helpful for gaining further insight in the isomorphism character of $[M(\Lambda)]_C$. In Chap. 11 we treat more general classes of subspaces of $C[0,1]$ which have many common features with $[M(\Lambda)]_C$.

It is well-known that there is a close relationship between the theory of Müntz spaces and fields like approximation theory, harmonic analysis and functional analysis. The first major contribution to this theory after the seminal papers of Müntz [110] and Szász [136] was given by L. Schwartz [128] and Clarkson and Erdös [19] who established the fact that, for integer $\Lambda$, each $x(t) \in [M(\Lambda)]_C$ has an analytic continuation to the open complex unit disk. This means, for example, that $[M(\Lambda)]_C$ consists entirely of functions which are real-analytic on $]0,1[$ provided that the Müntz condition and the gap condition hold! (See Sect. 6.2.)

In our book we want to change the accent from an analytical to a more geometrical approach and attempt to put well-known and new results into the perspective of a geometrical framework. At the same time we do not pretend completeness, we rather want to put the emphasis on unsolved problems, conjectures and ideas according to the taste of the authors. Although there is a natural overlap in this book with portions from excellent books such as [12] and [22] we present this material here from our geometric point of view. It seems to be the first time that Müntz spaces are treated under strict geometric orientation.

We assume that the reader has a basic knowledge of functional analysis.

The book is divided into two parts and twelve chapters. The first part contains the preliminary material from the geometry of normed spaces which is then applied to concrete Müntz spaces in Part II and which the authors believe to be promising for further investigation.

Both parts are essentially selfcontained and can be read independently of each other. In the summary Part I we skip some of the proofs and refer to the literature instead while, as a rule, in Part II we work out the proofs in full detail.

But Part I is more comprehensive than necessary for a simple outline of the preliminaries to Part II. There we give a systematic treatise of classical Banach space notions such as opening and inclination of subspaces (in Chap. 1). Moreover we introduce the projection function and projection type of a Banach space (1.6) and discuss their relation to Banach spaces with or without bases. Here the study of dispositions of subspaces in Banach spaces plays the main role.

In Chap. 2 we deal with general sequences in Banach spaces and properties such as minimality, completeness or stability. After the introduction of basic notions such as isomorphisms and the Banach-Mazur distance in Chap. 3 we study spaces which are (almost) universal with respect to a given class of Banach spaces and similar notions for bases in Chap. 4. Finally, Chap. 5 is devoted to a discussion of approximation properties centered around the commuting bounded approximation property (CBAP).

All our Banach spaces are assumed to be real unless indicated otherwise. (But almost all proofs in the following can be taken over literally to the complex case.) If $E$ is a Banach space let $E^*$ denote its topological dual space, i.e. the space of all linear bounded functionals on $E$.

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# Contents

## Part I Subspaces and Sequences in Banach Spaces

# Part I

# Subspaces and Sequences in Banach Spaces

In the first part of our book we will be concerned with dispositional properties of a Banach space $E$, i.e. the geometry of $E$ and its subspaces. They are important tools for the analysis of $E$ and the study of phenomena such as basic decompositions, approximations etc. A special role is played by the connection between "dispositional" properties (in terms of angles, etc.) and "distancional" properties (in terms of Banach-Mazur distance) which were discovered and developed in the sixties, [43, 82, 116].

In Chap. 1 we introduce the basic notions of the subspace disposition theory (see [43]) while in Chap. 2 we deal with applications to sequences in normed spaces. We will mention some often used technical theorems in the spirit of "planimetry" or "stereometry" in normed spaces.

The emphasis of Chap. 3 lies on isomorphisms and embeddings of Banach spaces. In Chap. 4 we study Banach spaces with almost universal disposition. Finally, Chap. 5 is devoted to bounded approximation properties properties of normed spaces involving various operators of finite rank (FDD, CBAP, etc), see [99–101].

So Part I is related to questions which, besides being of independent interest, also will lead us to the study of the geometry of Müntz and Müntz-type sequences in Part II.

# 1

# Disposition of Subspaces

In this chapter we discuss how two or more subspaces in a Banach space affect each other by their position in a Banach space and we give applications in the geometry of Banach spaces.

We start with a discussion of well-known different definitions of opening and relate these notions to the inclination of subspaces. This leads, for example, to conditions for the closure of the sum of two subspaces. Finally we focus on operator theoretic aspects. We introduce projection constants and discuss the notions of load and projection function which turn out to be important tools for the analysis of a Banach space.

## 1.1 Different Definitions of the Opening of Subspaces

M. Krein, M. Krasnoselskii and D. Milman introduced in [74] the following definition of the opening of two subspaces $U$ and $V$ in a Banach space $E$:

$$\hat{\Theta}(U,V) = \max\left\{ \sup_{x\in U, ||x||=1} \rho(x,V), \sup_{y\in V, ||y||=1} \rho(y,U) \right\} .$$

(Here $\rho(\cdot,\cdot)$ denotes the distance with respect to the metric given by the norm.)

A significant part of the applications of this concept is based on the following theorem proved in [74] (see also [44]). Recall, the *density character* of a Banach space $E$ is the smallest cardinality of a dense subset of $E$.

**Theorem 1.1.1** *Assume that, for the subspaces $U$ and $V$ of $E$, one of the following conditions holds:*
*(i) The density characters of $U$ and $V$ are different.*
*(ii) One of the spaces $U$ and $V$ is infinite dimensional and the other one is finite dimensional.*
*(iii) Both spaces are finite dimensional and their dimensions do not coincide.*
*Then we have*

$$\hat{\Theta}(U,V) \geq 1/2 \, .$$

*If in addition at least one of the subspaces $U$ and $V$ is finite dimensional or $E$ is a Hilbert space then $\hat{\Theta}(U,V) = 1$.*

If $E$ is a Hilbert space and dim $U$ = dim $V$ then $\hat{\Theta}(U,V)$ can be quite small (see 1.3.1).

I. Gohberg and A. Marcus [32] changed the definition of sphere opening by introducing the spherical opening $\tilde{\Theta}$ in the following way:

$$\tilde{\Theta}(U,V) = \max\left\{ \sup_{x \in S_U} \rho(x, S_V), \sup_{y \in S_V} \rho(y, S_U) \right\} ,$$

where $S_U$ is the unit sphere in $U$, i.e. the set of all elements $x \in U$ with $||x|| = 1$. Analoguously, $S_V$ is defined. They established the following

**Theorem 1.1.2** *The set of all closed subspaces in a Banach space $E$ is a complete metric space with respect to the spherical opening $\tilde{\Theta}(U,V)$ as metric.*

It is easy to see that

$$\hat{\Theta}(U,V) \leq 1 \quad \text{and} \quad \hat{\Theta}(U,V) \leq \tilde{\Theta}(U,V) \leq 2\hat{\Theta}(U,V) \, .$$

(Use the fact that $\rho(x,V) \leq \rho(x,S_V) \leq 2\rho(x,V)$ and $\rho(x,U) \leq \rho(x,S_U) \leq 2\rho(x,U)$ for $||x|| = 1$.) Theorem 1.1.1 and, accordingly, 1.1.2 become incorrect if one replaces $\hat{\Theta}$ by $\tilde{\Theta}$ and $\tilde{\Theta}$ by $\hat{\Theta}$, resp.

**Examples.** a) Take $\mathbf{R}^2$ with the Euklidean norm and put $U = \{(x,0) \, : \, x \in \mathbf{R}\}$ and $V = \mathbf{R}^2$. Then, according to 1.1.1, $\hat{\Theta}(U,V) = 1$ but $\tilde{\Theta}(U,V) = \sqrt{2}$.
b) Now let $E = \mathbf{R}^2$ be endowed with the norm $||(x,y)|| = \max(|x|,|y|)$. Put

$$U = \left\{ \left(x, \frac{x}{4}\right) \, : \, x \in \mathbf{R} \right\}, \quad V = \left\{ \left(x, \frac{x}{2}\right) \, : \, x \in \mathbf{R} \right\}$$

and $W = \{(x,0) \, : \, x \in \mathbf{R}\}$. An elementary computation shows $\hat{\Theta}(W,U) = 1/4$, $\hat{\Theta}(W,V) = 1/2$ and $\hat{\Theta}(U,V) = 1/6$. Hence

$$\hat{\Theta}(W,U) + \hat{\Theta}(U,V) < \frac{1}{2} = \hat{\Theta}(W,V)$$

which proves that $\hat{\Theta}$ is not a metric.

Let us introduce a third definition of opening for which the statements of both theorems are correct.

**Definition 1.1.3** *[43] The* ball opening *of the subspaces $U$ and $V$ in a Banach space $E$ is defined by the following quantity*

$$\Theta(U,V) = \max\left\{ \sup_{x \in B_U} \rho(x, B_V), \sup_{y \in B_V} \rho(y, B_U) \right\} ,$$

*where $B_U$ is the unit ball in $U$, i.e. the set of all elements $x \in U$ with $||x|| \leq 1$ and likewise for $B_V$.*

We derive from the definitions

**Lemma 1.1.4** *We have*

$$(a)\ \ \hat{\Theta}(U,V) \le \Theta(U,V) \le 1 \qquad \text{and}$$
$$(b)\ \ \Theta(U,V) \le \tilde{\Theta}(U,V) \le 2\Theta(U,V).$$

*Proof.* (a) is a direct consequence of the definitions.
(b): If $||x|| = 1$ and $y \neq 0$, then we obtain

$$||x - \frac{y}{||y||}|| \le ||x - y|| + |\ 1 - ||y||\ | \le 2||x - y||\ .$$

Hence $\rho(x, S_V) \le 2\rho(x, B_V)$ and, similarly, $\rho(y, S_U) \le 2\rho(y, B_U)$. This implies $\tilde{\Theta}(U,V) \le 2\Theta(U,V)$. For arbitrary $x \in B_U$ with $x \neq 0$ we have

$$\rho(x, B_V) = ||x||\rho\left(\frac{x}{||x||}, \frac{1}{||x||}B_V\right) \le \rho\left(\frac{x}{||x||}, S_V\right)$$

and similarly $\rho(y, B_U) \le \rho(y/||y||, S_U)$. Hence

$$\Theta(U,V) \le \tilde{\Theta}(U,V) \le 2\Theta(U,V)\ .$$ □

If $E$ is a Hilbert space then $\rho(x, U)$ is the norm of the orthogonal projection of $x$ with kernel $U$. This implies that $\Theta(U,V) = \hat{\Theta}(U,V)$ for all closed subspaces of Hilbert spaces.

**Theorem 1.1.5** *For the ball opening $\Theta(U,V)$ the statements of both Theorems 1.1.1 as well as 1.1.2 are valid.*

*Proof.* The statement of Theorem 1.1.1 for $\Theta(U,V)$ follows from 1.1.1 and the fact that $\hat{\Theta} \le \Theta \le 1$. To prove the assertion of Theorem 1.1.2 for the ball opening $\Theta(U,V)$, in view of the inequalities in 1.1.4 (b), it is sufficient to observe that, again, $\Theta$ is a metric, i.e. the triangle inequality

$$\Theta(U_1, U_3) \le \Theta(U_1, U_2) + \Theta(U_2, U_3)$$

is satisfied. This follows by direct verification. □

## 1.2 Inclination

Now we discuss a related notion.

**Definition 1.2.1** *[43] Let $U$ and $V$ be subspaces of a Banach space $E$ such that $U \neq \{0\}$. The* inclination *of $U$ to $V$ is defined by*

$$(\widehat{U,V}) = \inf_{x \in U, ||x||=1} \rho(x, V)\ .$$

*If $V$ is spanned by the element $x \in E$ we will use the notation $(\widehat{U,x})$ instead of $(\widehat{U,V})$. Then we speak of the inclination of $U$ to $x$. Analoguously we define the inclination of an element to a subspace and the inclination of an element to an element of $E$.*

Let $U \cap V = \{0\}$ and let $P : U + V \to U$ be the projection with $P(u+v) = v$ for all $u \in U$ and $v \in V$. Then we easily obtain $(\widehat{U,V}) = ||P||^{-1}$. Indeed

$$||P|| = \sup_{x \in U, y \in V} \frac{||x||}{||x+y||} = \left( \inf_{x \in U, y \in V} \frac{||x+y||}{||x||} \right)^{-1} = \frac{1}{(\widehat{U,V})}.$$

(This even includes the case of unbounded $P$ where $(\widehat{U,V}) = 0$ and distinguishes inclination from many other definitions of "angle between two subspaces".)

The definition of inclination has wide applications in the theory of bases (see, for example [53]) which is mainly due to the following criterion proved by Grinblum in equivalent terms (see [35, 36]).

For a given sequence $\bar{e} = \{e_i\}_{i=1}^{\infty}$ of elements in a Banach space $E$ let us denote by $L_{i,j}$ the span of $e_i, e_{i+1}, \ldots, e_j$.

**Definition 1.2.2** *$\bar{e}$ is called* complete *in $E$ if closed span $\{e_i\}_{i=1}^{\infty} = E$.*

*$\bar{e}$ is called* basis *of $E$ if each $x \in E$ has a unique representation as $x = \sum_{i=1}^{\infty} \alpha_i e_i$ where the series converges in norm.*

For more details about these notions see Sects. 2.3–2.5

**Theorem 1.2.3** *[35] Let $\bar{e} = \{e_i\}_{i=1}^{\infty}$ be a complete system in a Banach space $E$ such that $e_k \neq 0$ for all $k$. Then the following are equivalent*
*(i) $\bar{e}$ is a basis of $E$*
*(ii) There is some $\beta > 0$ such that*

$$(\widehat{L_{1,i}, L_{i+1,j}}) \geq \beta > 0 \quad \textit{whenever } i < j$$

*(iii) There is a constant $\beta > 0$ such that, for all choices of $\alpha_k$,*

$$\beta \left\| \sum_{k=1}^{i} \alpha_k e_k \right\| \leq \left\| \sum_{k=1}^{j} \alpha_k e_k \right\| \quad \textit{whenever } i < j$$

*Proof.* The equivalence between (ii) and (iii) follows from the definition of inclination and the remark following 1.2.1
*(i) ⇒ (iii)*: Let $x \in E$, say $x = \sum_{k=1}^{\infty} \alpha_k e_k$. Put $|||x||| = \sup_n || \sum_{k=1}^{n} \alpha_k e_k ||$. Then, by assumption, $||x|| \leq |||x||| < \infty$. An elementary computation shows that $E$ is complete under $|||\cdot|||$. So the open mapping theorem yields a constant $\beta > 0$, independent of $x$, with $\beta|||x||| \leq ||x|| \leq |||x|||$. Taking $x = \sum_{k=1}^{j} \alpha_k e_k$ we obtain (iii).
*(iii) ⇒ (i)*: Using (iii) we see that

$$\left\{x \,:\, x = \sum_{k=1}^{\infty} \alpha_k e_k \text{ for some } \alpha_k \text{ with norm converging series}\right\}$$

is a closed subspace of $E$. Since $\bar{e}$ is complete we obtain that every $x \in E$ has a representation of the form $x = \sum_{k=1}^{\infty} \alpha_k e_k$. This representation is unique. Indeed if $0 = \sum_{k=1}^{\infty} \alpha_k e_k$ then (iii) implies that $\alpha_k = 0$ for all $k$. Hence $\bar{e}$ is a basis of $E$. □

The supremum of all $\beta$ in the preceding theorem will be called the *index* of the basis $\{e_i\}_{i=1}^{\infty}$ and denoted by $\gamma(\{e_i\}_{i=1}^{\infty})$.

We also want to define the index $\gamma(\{e_k\}_{k=1}^{\infty})$ for a general sequence in $E$:

$$\gamma(\{e_k\}_{k=1}^{\infty}) = \inf\{(\widehat{L_{1,i}, L_{i+1,j}}) \,:\, i < j\}\,.$$

The notion of inclination is non-symmetric, i.e. we have $(\widehat{U,V}) \neq (\widehat{V,U})$ in general. The following proposition gives the value of the "degree of non-symmetry".

**Proposition 1.2.4** *Let $U$ and $V$ be non-zero subspaces of $E$. If $(\widehat{U,V}) = \delta$ then $(\widehat{V,U}) \geq (1+\delta)^{-1}\delta$. If $E = C[0,1]$ then this inequality is sharp for any $\delta \in \,]0,1]$.*

*Proof.* Let $y \in V$, $||y|| = 1$, $x \in U$. We shall evaluate $||x+y||$. Consider two cases:

Case 1. $||x|| \leq (1+\delta)^{-1}$. Then

$$||x+y|| \geq ||y|| - ||x|| \geq 1 - \frac{1}{1+\delta} = \frac{\delta}{1+\delta}\,.$$

Case 2. $||x|| > (1+\delta)^{-1}$. Then

$$||x+y|| \geq \rho(x,V) \geq (\widehat{U,V})||x|| \geq \frac{\delta}{1+\delta}\,.$$

The elements $x \in U$ and $y \in V$ are chosen arbitrarily. Therefore, we have

$$(\widehat{V,U}) = \inf_{x\in U, y\in V, ||y||=1} ||x+y|| \geq \frac{\delta}{1+\delta}\,.$$

To show that this inequality is sharp consider the following two functions in $E = C[0,1]$: $x(t) = 1$, and $y(t) = (1-\delta) + 2\delta t$. We check directly that $(\widehat{x,y}) = \delta$ and $(\widehat{y,x}) = (1+\delta)^{-1}\delta$. Thus the theorem is proved. □

Proposition 1.2.4 and Definition 1.2.1 imply the following

**Corollary 1.2.5** *If $(\widehat{U,V}) \geq \delta$ then, for each $x \in U$ and $y \in V$, we have*

$$||x+y|| \geq \max\left(\delta||x||,\ \frac{\delta}{1+\delta}||y||\right)$$