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Asymptotic Cyclic Cohomology

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Introduction

This work is a contribution to the study of topological K-Theory and cyclic cohomology of complete normed algebras. The aim is the construction of a cohomology theory, defined by a natural chain complex, on the category of Banach algebras which

a) is the target of a Chern character from topological K-theory (resp. bivariant K-theory).

b) has nice functorial properties which faithfully reflect the properties of topological K-theory.

c) is closely related to cyclic cohomology but avoids the usual pathologies of cyclic cohomology for operator algebras.

d) is accessible to computation in sufficiently many cases.

The final goal is to establish a Grothendieck-Riemann-Roch theorem for the constructed Chern character which for commutative C^*-algebras reduces to the classical Grothendieck-Riemann-Roch formula.

In his "Noncommutative Geometry" Alain Connes has developed the framework for a large number of far reaching generalisations of the index theorems of Atiyah and Singer. To motivate the problem addressed in this book and to put it in the right context we recall some basic principles of index theory and noncommutative geometry.

The classical index theorem for an elliptic differential operator D on a compact manifold M identifies the Fredholm index of this operator with the direct image of the symbol class of the operator under the Gysin map in topological K-Theory:

$$Ind_a(D) = \pi!(\sigma(D))$$

$$\pi! : K^*(T^*M) \to K^*(pt.) \simeq \mathbb{Z}$$

In more general situations where one considers not necessarily compact manifolds (for example operators on the universal cover of a compact manifold which are invariant under deck transformations, operators on a compact manifold differentiating only along the leaves of a foliation and being elliptic on the leaves, or elliptic operators of bounded geometry on an open manifold of bounded geometry) the considered elliptic operators are not Fredholm operators anymore. Nevertheless it is still possible to associate an index invariant with them which now has to be interpreted as an element of the operator K-group of some C^*-algebra. Moreover, Kasparov and Connes proved a number of very general index theorems of the form:

$$Ind_a(D) = \pi!(\sigma(D)) \in K_0(C^* - \text{algebra})$$

The C^*-algebras occuring in this way can be of quite general type and their K-groups usually cannot be identified with the K-groups of some topological space as in the classical cases.

As far as applications are concerned, the classical index theorem, formulated and proved in the context of topological K-theory, gains its full power only after being translated into a cohomological index formula with the help of a differentiable Grothendieck-Riemann-Roch Theorem. This theorem claims that for any K-oriented map $f : X \to Y$ of smooth compact manifolds the diagram

$$\begin{array}{ccc} K^*(X) & \xrightarrow{f!} & K^*(Y) \\ {\scriptstyle ch}\downarrow & & \downarrow{\scriptstyle ch} \\ H^*_{dR}(X) & \xrightarrow[f_*(-\cup Td(f))]{} & H^*_{dR}(Y) \end{array}$$

commutes. Here

$$ch : K^* \to H^*_{dR}$$

denotes the Chern character which is given by a universal characteristic class that identifies complexified topological K-theory of a manifold with its de Rham cohomology:

$$ch : K^*(M) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\simeq} H^*_{dR}(M).$$

Under this translation the direct image in K-theory can be identified with an explicit pushforward map in cohomology. Together, the index and Grothendieck-Riemann-Roch theorem yield a formula expressing the Fredholm index of an elliptic operator D as integral over the manifold of a universal characteristic class associated to the symbol of D:

$$Ind_a(D) = \int_M \text{characteristic class}(\sigma(D))$$

To obtain index formulas from the generalized index theorems above it is necessary to develop a Grothendieck-Riemann-Roch formalism in the context of operator K-theory. This means that one looks for a (co)homology theory on the category of C^*-, Banach-, resp. abstract algebras, which is defined by a natural chain complex and carries enough additional structure to provide a commutative diagram

$$\begin{array}{ccc} K_*(A) & \xrightarrow{f!} & K_*(B) \\ {\scriptstyle ch}\downarrow & & \downarrow{\scriptstyle ch} \\ H_*(A) & \xrightarrow[?]{} & H_*(B) \end{array}$$

On the subcategory of algebras of smooth (resp. continuous) functions on compact manifolds it should correspond to the classical Grothendieck-Riemann-Roch theorem.

So the Grothendieck-Riemann-Roch problem consists of three parts:

1. Define a (co)homology theory for Banach- (C^*-) algebras which generalizes the deRham (co)homology of manifolds.

2. Construct a Chern-character from K-theory to this noncommutative deRham-(co)homology.

3. Find a cohomological pushforward map and establish a suitable Grothendieck-Riemann-Roch theorem.

After having formulated this program, Alain Connes also made the first real breakthrough concerning a solution of the problem. In his foundational paper "Non-commutative Differential Geometry" [CO] he introduced a generalization of de Rham theory in the noncommutative setting, cyclic (co)homology HC_*(resp. HC^*), which can be calculated as the (co)homology of a functorial chain complex vanishing in negative dimensions , and he constructed an algebraically defined Chern character

$$ch : K_* \to HC_* .$$

The dual Chern character pairing

$$ch : K_* \otimes HC^* \to \mathbb{C}$$

generalizes the pairing between idempotent matrices and traces in degree zero and the pairing between invertible matrices and closed one-currents on the given algebra in degree one.

Cyclic cohomology proved to be a very powerful tool in many areas of K-theory, as the large number of well known applications shows. The project of constructing characteristic classes for operator K-theory however soon faced serious difficulties. Whereas the $\mathbb{Z}/2\mathbb{Z}$-periodic version

$$HP^* := \lim_{\to k} HC^{*+2k}$$

of cyclic cohomology of the algebra of smooth functions on a manifold coincides with the deRham homology of the manifold,

$$HP^*(\mathcal{C}^\infty(M)) \simeq H_*^{dR}(M)\,,$$

the periodic cyclic cohomology of its enveloping C^*-algebra of continuous functions equals the space of Borel measures on M in even degree and vanishes in odd degree.

$$HP^*(C(M)) \simeq \begin{cases} C(M)' & * = 0 \\ 0 & * = 1 \end{cases}$$

Thus while the Chern character pairing between reduced K-theory and reduced periodic cyclic cohomology yields a perfect pairing for the Fréchet algebra $\mathcal{C}^\infty(M)$, it vanishes for its enveloping C^*-algebra $C(M)$. (Note that both algebras can be considered as equivalent as far as K-theory is concerned). This example shows how cyclic cohomology and K-theory can behave quite differently in certain situations and that the Chern character from K-theory to cyclic homology can be far from being an isomorphism.

Actually the pathological behaviour of the Chern character pairing for (stable) C^*-algebras has nothing to do with the particular structure of cyclic cohomology but is a consequence of the continuity of the Chern character as the following argument shows:

Let C_* be any cyclic theory, i.e. a functor from Banach algebras to chain complexes equipped with a Chern character $ch : K_*A \to h(C_*A)$ associating a cycle to each idempotent (resp. invertible) matrix over A. Let φ be an even cocycle for this theory (the argument for odd cocycles is similar). This cocycle yields a map (still denoted by the same letter)

$$\varphi : \{e \in A,\ e^2 = e\} \to \mathbb{C}$$

which provides the pairing of the cohomology class of φ with $K_0(A)$.

Suppose that the Chern character pairing satisfies the following conditions:
(They hold for the Chern character pairings with continuous periodic cyclic cohomology HP^* and with entire cyclic cohomology HC^*_ϵ.)

1) $\varphi(e)$ depends only on the homotopy class of e.

2) $\varphi(e) = \varphi(e') + \varphi(e'')$ if $[e] = [e'] + [e'']$ in $K_0(A)$.

3) $|\varphi(e)| \leq F(\| e \|)$ for some function F on the real half-line.

Then if A happens to be a stable C^*-algebra, the pairing $K_*A \otimes h(C^*A) \to \mathbb{C}$ equals zero:

In fact one observes that the image of the map φ, viewed as a subset of $\mathbb{C}$, is closed under addition because A is stable and condition 2) holds. On the other hand this image is bounded by conditions 1) and 3), as any idempotent in a C^*-algebra is homotopic to a projector (selfadjoint idempotent) and nonzero projectors in C^*-algebras have norm 1. So the image of φ is a bounded subset of $\mathbb{C}$ closed under addition and thus zero.

This fact is quite annoying because the generalized index theorem and the hypothetical Grothendieck-Riemann-Roch are theorems about C^*-algebras and do not hold for more general Banach or Fréchet algebras (bivariant K-theory is well behaved only for C^*-algebras). Moreover, it is just the study of the K-theory and the cohomology of C^*-algebras which is at the heart of the most important applications: in the index-theoretic approach to the Novikov-conjecture on higher signatures of manifolds, for example, one has to analyse the K-theory and cyclic cohomology of the group-C^*-algebra $C^*_{red}(\Gamma)$ of the fundamental group of the manifold under consideration. Finally another difficulty in establishing a Grothendieck-Riemann-Roch formula is that the pushforward maps of operator K-theory have no counterpart in cyclic homology.

Connes and Moscovici defined in [CM] a modified version of cyclic cohomology, called asymptotic cyclic cohomology, and pointed out that this theory should provide a nontrivial cohomology theory on the category of C^*-algebras. Our work can be viewed as attempt to realize this program. This also explains the title of the book. The initial setup of asymptotic cyclic cohomology in [CM] had to be modified in several ways and the theory we are going to develop is however not equivalent to the one originally defined by Connes and Moscovici.

Our aim is to develop a cyclic theory, called asymptotic cyclic cohomology after [CM], which is the target of a Chern character that appropriately reflects the structure and the typical properties of operator K-theory. The theory will generalize ordinary and entire cyclic cohomology providing thus a framework for the explicit construction of (geometric) cocycles and the calculation of their pairing with concrete elements of K-groups. Finally we establish a generalized Grothendieck-Riemann-Roch theorem for the Chern character from operator K-theory to stable asymptotic homology. This will be achieved by the construction of a bivariant Chern character on Kasparovs bivariant K-theory with values in bivariant stable asymptotic cyclic cohomology.

The above argument for the vanishing of the Chern character pairing gives a first hint how one has to modify cyclic cohomology to get a theory with the desired properties. Cochains should consist of densely defined and unbounded rather than of bounded functionals or, as Connes-Moscovici propose in [CM], continuous families of unbounded cochains with larger and larger domains of definition.

To realize our goal we however start from a quite different line of thought. Our point of departure is on one hand the work of Connes, Gromov and Moscovici [CGM] on almost flat bundles and of Connes and Higson [CH] on asymptotic morphisms and bivariant K-theory, and on the other hand the work of Cuntz and Quillen [CQ] on cyclic cohomology and universal algebras.

In [CH] Connes and Higson made the important observation, that K-theory becomes in a very natural way a functor on a much bigger category than the ordinary category of Banach (C^*-algebras), namely on the category with the same objects but with the larger class of so called "asymptotic morphisms" as maps. Especially they showed that every pushforward map in K-theory associated to a generalized index theorem is induced from an explicitely constructible asymptotic morphism of the C^*-algebras involved.

A (linear) asymptotic morphism of Banach algebras is a bounded, continuous family $(f_t, t > 0)$ of continuous (linear) maps $f_t : A \to B$ such that

$$\lim_{t\to\infty} f_t(aa') - f_t(a)f_t(a') = 0 \ \forall a, a' \in A$$

The deviation from multiplicativity

$$\omega(a, a') := f_t(aa') - f_t(a)f_t(a')$$

is called the curvature of f_t at (a, a').

The interest in this notion originates (among other things) from the fact, that the E-theoretic K-groups, which are a modification of Kasparov's KK-groups, can be described as groups of asymptotic morphisms.

A cohomology theory that is the target of a good Chern character on operator K-theory should certainly have the same functorial properties as K-theory itself. Cyclic (co)homology however is by no means a functor on the asymptotic category. Therefore it is no surprise that the Chern character in cyclic homology fails to be an isomorphism in general.

On the other hand Connes, Gromov and Moscovici showed in [CGM], that the pullback of a trace τ on an algebra B under a linear map $f : A \to B$ may be interpreted as an even cocycle in the cyclic bicomplex of A:

$$f^*\tau = \sum_{n=0}^{\infty} \varphi^{2n} .$$

Moreover its components (φ^{2n}) decay exponentially fast

$$|\varphi^{2n}(a^0, \ldots, a^{2n})| \leq C^{-n}$$

when evaluated on tensors with entries $a^0, \ldots, a^{2n}$ belonging to a fixed finite subset Σ of A. The constant C depends on the deviation of f from being multiplicative on Σ.

Cochains with this growth behaviour occur already in the calculations of localized analytic indices of Connes and Moscovici [CM], where the authors point out that a cyclic theory for C^*-algebras should be based on such cocycles.

Relating this to the approach to cyclic cohomology via traces on universal algebras by Cuntz and Quillen [CQ] suggests that it might be possible to pull back arbitrary cochains in the cyclic bicomplex under linear maps and that in fact every even(odd)-dimensional cocycle in the cyclic bicomplex could be obtained as the pullback of a trace (resp. a closed one-current) under a linear map.

Thus one might hope to reinterpret cyclic cohomology as being given by a chain complex that behaves functorially under linear maps and to obtain an asymptotic cyclic theory as the envelope under linear asymptotic morphisms of the ordinary cyclic theory. Cochains in this theory should be characterized by natural growth (resp. continuity) conditions as in the example above. In fact any cyclic theory which is functorial under asymptotic morphisms would possess the pushforward maps necessary to formulate a GRR theorem.

So our starting point for the construction of asymptotic cyclic cohomology will be to take ordinary cyclic theory and to extend it to a functor on the linear asymptotic category $\mathcal{C}$. (We restrict ourselves to linear asymptotic morphisms. It would have been possible to dispense with this restriction but only at the cost of making the formulas much more complicated without providing a wider range of applications.) This means the following. First we choose a natural chain complex C^* calculating cyclic cohomology, i.e. a functor

$$C^* : \text{Algebras} \to \text{Chain Complexes}$$

such that

$$H^*(C^*) \simeq HC^* .$$

Then we consider pairs (C^*_α, Φ) consisting of

a)

a functor

$$C^*_\alpha : \mathcal{C} \to \text{Chain Complexes} \ ,$$

such that the corresponding homology groups define a homotopy functor

$$HC^*_\alpha := H^*(C^*_\alpha) : \ Homot\,\mathcal{C} \to \mathbb{C} - \text{Vector Spaces}$$

b)

a morphism of functors

$$\Phi : \ C^* \to C^*_\alpha|_{Algebras}$$

on the category of algebras inducing a natural transformation

$$HC^* \to HC^*_\alpha$$

from ordinary to asymptotic cyclic cohomology.

Among all such pairs we look for a minimal one, i.e. a pair satisfying the obvious universal property. By an argument due to J.Cuntz any such cohomology theory will be Bott-periodic, so that C^*_α (and C^*) should in fact be $\mathbb{Z}/2\mathbb{Z}$-graded complexes.

In [CO] Connes introduced a natural $\mathbb{Z}/2\mathbb{Z}$-graded complex, the (b,B)-bicomplex CC_* of a unital algebra. An equivalent (but not identical) complex Ω^{PdR}_*, the periodic de Rham complex, has been constructed later on by Cuntz and Quillen [CQ]. These are both complexes of modules of formal differential forms over the given algebra and carry a natural filtration (Hodge filtration), derived from the degree filtration on differential forms. The quotient complexes with respect to the Hodge filtration successively compute the cyclic homology groups HC_* and the completed complexes $\widehat{\Omega}^{PdR}_*$ (with respect to the Hodge filtration) calculate the periodic cyclic homology HP_* of Connes. The periodic de Rham complex provides in our opinion the best choice for the complex C^* above and it is therefore Ω^{PdR}_* that will be extended to a functor on the linear asymptotic category.

The universal problem above can be solved provided that the forgetful functor

$$\text{Banach-algebras} \to \mathcal{C}$$

has a right adjoint $R_{\mathcal{C}}$. An explicit solution would then be given by

$$\widehat{\Omega}^{PdR,\alpha}_* := \widehat{\Omega}^{PdR}_* \circ R_{\mathcal{C}}$$

If one forgets the topology for the moment and looks at the problem at a purely algebraic level, there is indeed an adjoint, provided by a canonical quotient of the full tensor algebra:

$$RA := TA/(1_A - 1_{\mathbb{C}})$$

This would lead to

$$\widehat{\Omega}^{PdR,\alpha}_*(A) = \widehat{\Omega}^{PdR}_*(RA)$$

The algebras RA are of Hochschild cohomological dimension one, which makes it possible to calculate their periodic cyclic homology via the quotient complex of the periodic de Rham complex by the second step of the Hodge filtration, the so called X-complex of Cuntz-Quillen:

$$\widehat{\Omega}_*^{PdR}(RA) \xrightarrow{qis} X_*(RA)$$

where the X-complex is given by

$$X_*(A): \quad \to \quad A \quad \stackrel{d}{\to} \quad \Omega^1 A/[\Omega^1 A, A] \quad \stackrel{b}{\to} \quad A \quad \to$$

In fact, Cuntz and Quillen [CQ] showed that cyclic homology can be developed starting from the X-complex of tensor algebras (resp. quasifree algebras). Moreover one obtains in this way a very natural and advantageous viewpoint of the basic features of the theory.

A basic observation is that the tensor algebras RA are canonically filtered by powers of the ideal

$$0 \to IA \to RA \xrightarrow{mult} A \to 0$$

So although the algebra RA depends only on the underlying vector space of A, the I-adic filtration on RA makes it possible to recover the multiplicative structure of A. Remarkably, the X-complex of RA with its I-adic filtration turns out to be quasiisomorphic, as filtered complex, to the periodic de-Rham complex of A with its Hodge filtration. So whereas the complex $X_*(RA)$ is easy to manipulate algebraically it also contains all information encoded in the periodic de Rham complex of A with its Hodge filtration. Especially one recovers the periodic cyclic homology of A as the homology of the X-complex of the (algebraic) I-adic completion of RA:

$$\widehat{\Omega}_*^{PdR}(A) \xleftarrow{qis} X_*(\widehat{RA})$$

$$HP_*(A) = H_*(X_*(\widehat{RA}))$$

In fact, the I-adic completion $\widehat{RA}$ of RA is still of cohomological dimension one although quite far from being free.

The description of periodic cyclic (co)homology using the X-complex of tensor algebras exhibits the functoriality of the (uncompleted) cyclic complexes with respect to linear maps which is crucial for us but somewhat hidden if one uses Connes original cyclic (b, B)-bicomplex.

Moreover the Cuntz-Quillen approach enables one to construct product operations and homotopy operators for cyclic theories on the level of chain complexes by a uniform procedure. One tries to guess the right formulas for the periodic de Rham complex on differential forms of degree zero and one modulo error terms of higher degree. For free algebras, which are of Hochschild cohomological dimension one, the second step of the Hodge filtration is contractible, so that it becomes possible to get rid of the error terms in this case. This yields by passing to the quasiisomorphic quotient complexes a map of X-complexes of free algebras. For free algebras of the form RA one finally recovers by taking the associated graded complexes with respect to the I-adic filtration the whole periodic de Rham complex of the initial algebra A, this time with a globally defined chain map reducing to the initial formula on forms of low degree. As homotopic initial maps on forms of low degree provide homotopic global chain maps in the end, the effect of the constructed chain maps on homology is determined by their effect on ordinary cyclic homology of degree zero and one, respectively.

There is a "Cartesian square" of functors

$$\begin{array}{ccc} \text{Algebras} & \xrightarrow{\text{forget}} & \text{Algebras, linear maps} \\ {\scriptstyle R,I-adicfilt}\downarrow & & \downarrow{\scriptstyle R} \\ \text{Filtered Alg.} & \xrightarrow{\text{forget}} & \text{Algebras} \end{array}$$

on the level of morphism sets.

This shows that the I-adic filtrations on the complexes $X_*(RA)$ are never preserved by a homomorphism of tensor algebras which is induced by a linear morphism that is not multiplicative. Therefore not the degree, but only the parity of an ordinary cyclic cycle is preserved under pushforward by a linear morphism. In fact any even (odd) cocycle (in the $\mathbb{Z}$-graded setting) occurs as the linear pullback of a trace (closed one-current). This explains again why only a $\mathbb{Z}/2\mathbb{Z}$-graded theory can be defined on the linear asymptotic category.

Concerning the original aim of making cyclic cohomology functorial under linear asymptotic morphisms our goal can be described (in terms of the Cuntz-Quillen approach) as follows.

Consider the diagram

$$\begin{array}{lccccc} \text{Morphisms:} & \text{linear} & & \epsilon\text{-mult.} & & \text{mult.} \\ \text{Algebras:} & A & = & A & = & A \\ & \downarrow & & \downarrow & & \downarrow \\ \text{Algebras:} & RA & \subset & \mathcal{R}A =? & \subset & \widehat{RA} \\ & \downarrow & & \downarrow & & \downarrow \\ \text{Chain complexes:} & X_*(RA) & \subset & X_*(\mathcal{R}A) & \subset & X_*(\widehat{RA}) \end{array}$$

In the right column the Cuntz-Quillen procedure for obtaining the cyclic complex of A is described. The universal way to extend this construction to the category of algebras with linear maps as morphisms is given in the left column: one replaces the given algebra by its tensor algebra and constructs the cyclic complex of the latter algebra. The tensor algebra already being free one can directly pass to its X-complex. The complex $X_*(RA)$ cannot be interesting homologically however. It has to be contractible because every linear map is linearly homotopic to zero. Being interested in a nontrivial homology theory which is functorial under asymptotic morphisms, i.e. a functor on a "category of ϵ-multiplicative linear maps" we have to look for an intermediate theory. One has to find a topological completion of the tensor algebra RA which is not contractible but functorial under ϵ-multiplicative maps. If it is moreover of cohomological dimension one one can again take its X-complex to arrive at a reasonable theory (middle column). Such a completion is constructed as follows.

Let $f : A \to B$ be an almost multiplicative linear map of Banach algebras. Then the induced homomorphism $Rf : RA \to RB$ of tensor algebras will not preserve I-adic filtrations but the norms of the occuring "error terms" will decay exponentially fast with their I-adic valuation. This suggests the following construction: Fix a multiplicatively closed subset K of A and consider tensors over A with entries in K. Expand a given element of this subalgebra of RA in a standard basis with respect to the I-adic filtration. A weighted L^1-norm for the coefficients of such an expansion is then introduced allowing the coefficients to grow exponentially to the basis $N > 1$ with respect to the I-adic valuation. Denote the corresponding completion by $RA_{(K,N)}$. It is a Fréchet algebra and possesses the following crucial property: If $f : A \to B$ is linear with curvature uniformly bounded on $K \subset A$ by a sufficiently small constant then Rf induces a continuous homomorphism $Rf : RA_{(K,N)} \to RB_{(K',N')}$ for suitable $K' \subset B, N' > 1$. Usually f will be a linear asymptotic morphism. As the curvature of an asymptotic morphism is uniformly bounded only over compact sets, the multiplicatively closed subsets $K \subset A$ used for the construction above will always be assumed to be compact. It turns out that the algebras $RA_{(K,N)}$ are also of cohomological dimension one.

The Fréchet algebras $RA_{(K,N)}$ form an inductive system with formal inductive limit $\mathcal{R}A$. This limit could be called the topological I-adic completion of RA. It should be viewed as virtual infinitesimal thickening of A as the kernel of the projection $\pi : \mathcal{R}A \to A$ is formally topologically nilpotent (i.e. the spectrum of its elements equals zero).

We define the analytic X-complex X_*^ϵ of a Banach algebra to be the reduced X-complex of the topological I-adic completion of the tensor algebra of its unitalization. The cohomological analytic X-complex is closely related to the entire cyclic bicomplex of Connes. It turns out to be convenient to introduce also a bivariant analytic X-complex $X_\epsilon^*(-,-)$ of a pair of algebras as the Hom-complex of the associated analytic X-complexes. The bivariant analytic X-complex is a bifunctor on the category of Banach algebras and its cohomology groups are smooth homotopy bifunctors. There exists an obvious composition product

$$X_\epsilon^*(A,B) \otimes X_\epsilon^*(B,C) \to X_\epsilon^*(A,C) \ .$$

The fundamental functoriality of the locally convex algebras $RA_{(K,N)}$ under almost multiplicative linear maps implies that every linear asymptotic morphism

$$f_t : A \to B,\ t > 0$$

induces a continuous homomorphism of formal inductive limit algebras

$$\mathcal{R}f : \mathcal{R}A \to \mathcal{R}B \otimes_\pi \mathcal{O}_\infty(\mathcal{R}_+^\infty) \ .$$

Here $\mathcal{O}_\infty(\mathcal{R}_+^\infty)$ is the algebra of germs around ∞ of smooth functions on the asymptotic parameter space $\mathcal{R}_+^\infty$. This leads one to define the (cohomological) asymptotic X-complex $X_\alpha^*(A)$ of a Banach algebra A as the cohomological X-complex of the formal topological I-adic completion $\mathcal{R}A$ with coefficients in the formal inductive limit algebra $\mathcal{O}_\infty(\mathcal{R}_+^\infty)$. The bivariant asymptotic X-complex $X_\alpha^*(A,B)$ of the pair (A,B) is introduced as the complex of germs at ∞ of homomorphisms between the X-complexes of the formal topological I-adic completions $\mathcal{R}A$ and $\mathcal{R}B$ (See chapter 6). By construction any linear asymptotic morphism defines an even cocycle in the bivariant asymptotic X-complex. The composition product carries over to the asymptotic setting and turns $X_\alpha^*(-,-)$ into a bifunctor on the linear asymptotic category. Moreover bivariant asymptotic cohomology becomes a (continuous) asymptotic homotopy bifunctor.

So much for the motivation and definition of the asymptotic cyclic theory. We have to be more precise at one point however. Asymptotic morphisms do not consist of a single, but of whole families of linear maps, and one has to keep track of the chain homotopies provided by evaluation at different "parameter values" in such families.

We do this by working throughout in the category of differential graded algebras and differential graded chain complexes. The asymptotic X-complex of the universal enveloping differential graded algebra of the given algebra is large enough to contain the higher homotopy information needed. One obtains then Cartan homotopy formulas for the "change of asymptotic parameters".

There are natural maps

$$CC^* \to X_\alpha^*, \qquad CC_\epsilon^* \to X_\alpha^*$$

in the derived category yielding natural transformations

$$HP^* \to HC_\alpha^*, \qquad HC_\epsilon^* \to HC_\alpha^*$$

on cohomology.

For the algebra of complex numbers the maps on cohomology above are isomorphisms. More generally, analytic and asymptotic homology coincide:

$$HC_\alpha^*(\mathbb{C},A) \simeq HC_\epsilon^*(\mathbb{C},A)\ .$$

The corresponding cohomology groups are in general quite different however.

The well known pairings between cyclic theories and K-theory extend to a pairing $K_*\otimes HC_\alpha^* \to \mathbb{C}$. It is uniquely determined by its naturality with respect to asymptotic morphisms and by demanding that it restricts to the classical pairing between idempotents and traces (resp. invertible elements and closed one-currents) on the ordinary cyclic complex. As for a given value of the asymptotic parameter a cocycle is given by a sequence of densely defined multilinear functionals on the underlying algebra A, the pairing can be defined for this choice of parameter only for special representatives of a finite number of classes in K_*A. Taking a family of parameter values which approaches ∞ in the asymptotic parameter space allows to define the pairing on larger and larger subsets of K_*A which finally exhaust the whole K-group and yield the pairing globally. This behaviour explains why the argument at the beginning of the introduction showing the pathological nature of the Chern chracter pairing for the classical cyclic theories on stable C^*-algebras does not apply to the asymptotic theory. Indeed there is a large class of stable C^*-algebras for which the pairing of K-theory with asymptotic cohomology is nondegenerate.

The most striking new phenomenon of asymptotic cyclic theory is that inclusions of holomorphically closed subalgebras become cohomology equivalences in many cases. This often allows one to construct asymptotic cocycles on C^*-algebras by lifting well known cyclic cocycles from a suitable dense subalgebra.

Since these subalgebras are not Banach algebras anymore, we develop the theory for the slightly larger class of admissible Fréchet algebras, i.e. Fréchet algebras possessing an analogue of the open unit ball of Banach algebras. These algebras seem to provide the natural framework for our theory.

The descent principle to holomorphically closed dense subalgebras can be used to show that asymptotic cyclic cohomology is stably Morita invariant: for any C^*-algebra A the inclusion

$$A \hookrightarrow A \otimes_{C^*} \mathcal{K}(\mathcal{H})$$

induces an asymptotic (co)homology equivalence.

In particular

$$HC^*_\alpha(\mathcal{K}(\mathcal{H})) = \begin{cases} \mathbb{C} & * = 0 \\ 0 & * = 1 \end{cases}$$

in sharp contrast to the cyclic theories known so far.

In order to go further it is necessary to develop product operations. By the principles explained above we are able to construct a chain map

$$\times : X_*R(A \otimes B) \to X_*RA \widehat{\otimes} X_*RB$$

which is associative up to homotopy and yields exterior products

$$X^*_{\epsilon,\alpha}(A) \widehat{\otimes} X^*_{\epsilon,\alpha}(B) \to X^*_{\epsilon,\alpha}(A \otimes_\pi B)$$

both for analytic and asymptotic cohomology.

It behaves naturally with respect to asymptotic morphisms. Moreover, the pairing of K-theory with analytic (resp. asymptotic) cohomology is compatible with exterior products. To be precise, the compatibility of the products in K-theory resp. the cyclic theories holds only up to a factor $2\pi i$ if the involved classes are of odd dimension: the cyclic theories are a priori $\mathbb{Z}/2\mathbb{Z}$-graded, whereas the product of odd classes in K-theory has to be defined using Bott periodicity, which causes the "period" factor $2\pi i$. This makes me believe that the exterior product on cohomology coincides up to normalization constants with Connes's product. I have not investigated this point however.

The attempt to define an exterior product of bivariant X-complexes was only partially successful up to now. The main difficulty lies in the construction of a homotopy inverse of the exterior product map for the ordinary X-complexes above. (See [P], where meanwhile a natural homotopy inverse has been constructed.) At least it is possible to establish a particular consequence of a bivariant product operation, namely the existence of a slant product

$$K_*(A) \otimes HC^*_{\epsilon,\alpha}(A \otimes_\pi B) \to HC^*_{\epsilon,\alpha}(B)$$

It is constructed in such a manner that any idempotent (or invertible) matrix over A gives rise to an explicit map $X^*_{\epsilon,\alpha}(A \otimes_\pi B) \to X^*_{\epsilon,\alpha}(B)$ of chain complexes. Its homotopy class depends only on the K-theory class of the given matrix. The slant product behaves naturally with respect to asymptotic morphisms and is compatible with the exterior product. It represents a convenient tool to prove the split injectivity of the exterior product with cohomology classes in the image of the Chern character. As an application we show that the exterior (resp. slant) product with the fundamental class of the circle yields an isomorphism

$$HC^*_\alpha(S\mathbb{C}, S\mathbb{C}) \simeq HC^*_\alpha(\mathbb{C}, \mathbb{C})$$

of the bivariant asymptotic cohomology of $\mathbb{C}$ and its suspension $S\mathbb{C} = C_0(\mathbb{R})$. Extending this argument from $\mathbb{C}$ to more general admissible Fréchet algebras A by taking the exterior product with the bivariant cohomology class of the identity on A unfortunately fails: the exterior product is only defined for unital algebras and unitalization does not commute with taking tensor products (the suspension of an algebra is nonunital). In fact it seems to me to be a difficult question, whether an admissible Fréchet algebra is equivalent in asymptotic cohomology to its double suspension (this could be called a cohomological Bott periodicity theorem). In fact such a periodicity theorem would be highly desirable because it necessarily has to hold for any theory with reasonable excision properties.

At this point the E-theoretic description of Bott periodicity [CH] fortunately saves us as it realizes the bivariant Bott- resp. Dirac elements inducing the K-theoretic periodicity maps stably by (nonlinear) asymptotic morphisms. This allows to prove a stable version of cohomological periodicity: there are natural asymptotic cohomology equivalences

$$\alpha_{SA} \in HC^1_\alpha(S^2A, SA), \qquad \beta_{SA} \in HC^1_\alpha(SA, S^2A),$$

inverse to each other under the composition product.

Suspending an algebra therefore only produces a shift of its stable asymptotic cohomology groups $HC^*_\alpha(S-, S-)$, so that stable asymptotic cohomology becomes in fact a bifunctor on the stable linear asymptotic homotopy category. This opens the way to derive exactness and excision properties of stable asymptotic cohomology which make these groups quite accessible in many situations. By adapting a well known argument from stable homotopy theory, it can be shown that the long cofibre (Puppe) sequence associated to a homomorphism $f : A \to B$ of admissible Fréchet algebras induces six term exact sequences on (bivariant) stable asymptotic cohomology relating the stable cohomology groups of A and B to those of the mapping cone C_f of f. A short exact sequence

$$0 \to J \to A \xrightarrow{p} B \to 0$$

of admissible Fréchet algebras gives rise to six term exact cohomology sequences if and only if stable excision holds. This means that the inclusion of the kernel J into the cofibre C_p of the quotient map p induces a stable asymptotic (co)homology equivalence. Following an argument of Connes and Higson we show that stable excision holds for any epimorphism of separable C^*-algebras that admits a bounded linear section. This is the only place where we have to restrict ourselves to a particular class of admissible Fréchet algebras, as we need the existence of a bounded, positive, quasicentral approximate unit in the kernel J of p.

With all this machinery developed it becomes possible to extend the Chern character to the bivariant setting, i.e. to construct a transformation of bifunctors:

$$ch: KK^*(-,-) \to HC^*_\alpha(S-, S-)$$

from Kasparov's KK-theory to stable bivariant asymptotic cohomology. In principle it is given by the "composition" (see [CH])

$$KK^* \to E^* \text{ "}\to\text{" } HC^*_{\alpha,st} \; ,$$

where the "arrow" on the right hand side maps an asymptotic morphism to the corresponding bivariant asymptotic cocycle. As the asymptotic morphisms of E-theory are nonlinear however, one has to be careful in the actual construction of the bivariant Chern character. In particular, one obtains a Chern character on K-homology defined for arbitrary Fredholm modules and generalizing the constructions known so far. The Kasparov product on bivariant K-theory corresponds to the composition product on asymptotic cohomology, which is precisely the

Grothendieck-Riemann-Roch Theorem:

The diagram

$$\begin{array}{ccc} KK^*(A,B) \otimes KK^*(B,C) & \xrightarrow{\otimes} & KK^*(A,C) \\ \downarrow{\scriptstyle ch\otimes ch} & & \downarrow{\scriptstyle ch} \\ HC^*_\alpha(SA,SB) \otimes HC^*_\alpha(SB,SC) & \xrightarrow{\frac{1}{2\pi i^{**}}\otimes} & HC^*_\alpha(SA,SC) \end{array}$$

commutes. For $A = \mathbb{C}$ this yields a Grothendieck-Riemann-Roch formula as asked for in the beginning.

(The factor $2\pi i$ occurs for the same reason as in the comparison theorem of the ordinary Chern character with products). Consequently the Chern character of a KK-equivalence yields a stable asymptotic (co)homology equivalence. The bivariant Chern character becomes an isomorphism between complexified KK-theory and stable bivariant asymptotic cohomology on a class of separable C^*-algebras containing $\mathbb{C}$ and being closed under extensions with completely positive lifting and