AN ELEMENTARY COURSE

IN THE

INTEGRAL CALCULUS

BY

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MURRAY'S INTEG. CALC.

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PREFACE

THIS book has been written primarily for use at Cornell University and similar institutions. In this university the classes in calculus are composed mainly of students in engineering for whom an elementary course in the Integral Calculus is prescribed for the third term of the first year. Their purpose in taking the course is to acquire facility in performing easy integrations and the power of making the simple applications which arise in practical work. While the requirements of this special class of students have been kept in mind, care has also been taken to make the book suitable for any one beginning the study of this branch of mathematics. The volume contains little more than can be mastered by a student of average ability in a few months, and an effort has been made to present the subject-matter, which is of an elementary character, in a simple manner.

The object of the first two chapters is to give the student a clear idea of what the Integral Calculus is, and of the uses to which it may be applied. As this introduction is somewhat longer than is usual in elementary works on the calculus, some teachers may, perhaps, prefer to postpone the reading of several of the articles until the student has had a certain amount of practice in the processes of integration. It is believed, however, that a careful study of Chapters I., II., will arouse the student's interest and quicken his understanding of the subject. There may be some difference of opinion also as to whether the beginner snould be introduced to the subject through Chapter I. or through Chapter II. The decision of this question will depend upon the point of view of the individual teacher. So far as the remaining portion of the book is concerned it is a matter of indifference which of these chapters is taken first; and, with slight modifications, they can be interchanged. In Chapter III. the fundamental rules and methods of integration are explained. Since it has been deemed advisable to introduce practical applications as early as possible, Chapter IV. is devoted to the determination of plane areas and of volumes of solids of revolution. The subject of Integral Curves, which is of especial importance to the engineer, is treated in Chapter XII.

Many of the examples are original. Others, especially some of those given in the practical applications, by reason of their nature and importance, are common to all elementary courses on calculus. In several instances, examples of particular interest have been drawn from other works.

A list of lessons suggested for a short course of eleven or twelve weeks is given on page viii. This list has been arranged so that four lessons and a review will be a week's work.

It is hardly possible to name all the sources from which the writer of an elementary work may have obtained suggestions and ideas. I am especially conscious, however, of my indebtedness to the treatises of De Morgan, Williamson, Edwards, Stegemann and Kiepert, and Lamb.

To my colleagues in the department of mathematics at Cornell University, I am under obligations for many valuable criticisms and suggestions. Both the arrangement and the contents have been influenced in a large measure by our conferences and discussions. As originally projected, the volume was to have been written in collaboration with Dr. Hutchinson, but circumstances prevented the carrying-out of this plan.

Chapters V., VI., in part, and Articles 28, 73, in their entirety, have been contributed by him. My colleagues have aided me also in correcting the proofs.

From Professor I. P. Church of the College of Civil Engineering and Professor W. F. Durand of the Sibley College of Mechanical Engineering, I have received valuable suggestions for making the book useful to engineering students. Professor Durand kindly placed at my disposal, with other notes, his article on "Integral Curves" in the Sibley Journal of Engineering, Vol. XI., No. 4; and Chapter XII. is, with slight changes, a reproduction of that article. I take this opportunity of thanking Mr. A. T. Bruegel, Instructor in the kinematics of machinery, and Mr. Murray Macneill, Fellow in mathematics in this university, the former for the interest and care taken by him in drawing the figures, the latter for his assistance in verifying examples and reading proof sheets.

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LIST OF LESSONS SUGGESTED FOR A SHORT COURSE

[REVIEWS TO FOLLOW EVERY FOURTH LESSON]

1. Arts. 1, 2, 3.
2. Arts. 4, 5, 6, 7.
3. Arts. 8, 9, 10, 11.
4. Arts. 12, 17, 18, to Ex. 9.
5. Arts. 13, 18, Exs. 10–12, 19.
6. Arts. 14, 20.
7. Arts. 15, 16, 21.
8. Arts. 22, 23, Exs. 1–23, odd examples.
9. Art. 23, Exs. 24–32, 24, 25, Exs. 1–8, page 55.
10. Pages 56, 57, even or odd examples, Exs. 9–41, Exs. 42–47.
11. Arts. 26, 27, Exs. 1, 2, page 68.
12. Arts. 28, 29, Exs. 3, 4, page 68.
13. Exs. 5–14, page 68.
14. Art. 30.
15. Arts. 31, 32.
16. Page 76, Exs. 1–15.
17. Pages 76, 77, Exs. 16–26.
18. Arts. 33, 34.
19. Arts. 35, 36. Selected examples.
20. Arts. 37, 38, 39, 40.
21. Arts. 42, 43.
22. Art. 45.
23. Selected examples, pages 98, 99.
24. Arts. 46, 47, 48 (*a*).
25. Arts. 48 (*d*), 49, 50 (*a*), (*c*).
26. Art. 51 (*a*).
27. Arts. 52, 53.
28. Arts. 58, 59, 60.
29. Arts. 61, 62.
30. Arts. 64, 65, 66.
31. Arts. 67, 68, 69.
32. Arts. 63, 70. Selected examples, pages 164, 165.
33. Art. 71. Selected examples, page 165.
34. Art. 72. Selected examples, page 166.
35. Art. 73.
36. Art. 74. Selected examples, pages 164, 165.
37. Art. 75, Ex. 9, page 164.
38. Art. 76. Selected examples, Art. 77, Ex. 1.
39. Selected examples, pages 162–166.
40. Arts. 78, 79. Selected examples.
41. Art. 80. Selected examples.
42. Arts. 81, 82, 84, 85.
43. Arts. 86, 87, 88.
44. Arts. 89, 91, 92, 94, 95.

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INTEGRAL CALCULUS

CHAPTER I

INTEGRATION A PROCESS OF SUMMATION

1. Uses of the integral calculus. Definition and sign of integration. The integral calculus can be used for two purposes, namely:

(*a*) To find the sum of an infinitely large number of infinitesimals of the form $f(x)\,dx$;

(*b*) To find the function whose differential or whose differential coefficient is given; that is, to find an anti-differential or an anti-derivative.

The integral calculus was invented in the course of an endeavor to calculate the plane area bounded by curves. The area was supposed to be divided into an infinitely great number of infinitesimal parts, each part being called *an element* of the area; and the sum of these parts was the area required. The process of finding this sum was called *integration*, a name which implies the combination of the small areas into a whole, and hence the sum itself was called the whole or *the integral.*

From the point of view of the first of the purposes just indicated, integration may be defined as *a process of summation.* In many of the applications of the integral calculus, and, in particular, in the larger number of those made by engineers, this is the definition to be taken. On the other hand, however, in many problems it is not a sum, but merely an anti-differential, that is required. For this purpose, integration may be defined as *an operation which is the inverse of differentiation* It may at once be stated that in the course of making a summation by means of the integral calculus it will be necessary to find the anti-differential of some function; and it may also be said at this point, that the anti-differential can be shown to be the result of making a summation. Each of the above definitions of integration can be derived from the other. These statements will be found verified in Arts. 4, 11, 13.

In the differential calculus, the letter d is used as the symbol of differentiation, and $d\,f(x)$ is read "the differential of $f(x)$." In the integral calculus the symbol of integration is* $\int$, and $\int f(x)\,dx$ is read "the integral of $f(x)\,dx$." The signs d and $\int$ are signs of operations; but they also indicate the results of the operations of differentiation and integration respectively on the functions that are written after them.

The principal aims of this book are: (1) to explain how summations of infinitesimals of the form $f(x)\,dx$ may be made; (2) to show how the anti-differentials of some particular functions may be obtained.

2. Illustrations of the summation of infinitesimals. Two simple illustrations of the summation of an infinite number of infinitely small quantities will now be given. They will help to familiarize the student with a certain geometrical principle and with the fundamental theorem of the integral calculus, which are set forth in Arts. 3, 4. The method employed in these particular instances is identical with that used in the general case which follows them.

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* This is merely the long *S*, which was used as a sign of summation by the earlier writers, and meant "the sum of." The sign $\int$ was first employed in 1675, and is due to Gottfried Wilhelm Leibniz (1646–1716), who invented the differential calculus independently of Newton. The word *integral* appeared first in a solution of James Bernoulli (1654–1705), which was published in the *Acta Eruditorum*, Leipzig, in 1690. Leibniz had called the integral calculus *calculus summatorius*, but in 1696 the term *calculus integralis* was agreed upon between Leibniz and John Bernoulli (1667–1748). See Cajori, *History of Mathematics*, pp. 221, 237.

(*a*) Find the area between the line whose equation is $y = mx$, the x-axis, and the ordinates for which $x = a$, $x = b$.

Let OL be the line $y = mx$; let OA be equal to a, and OB to b, and draw the ordinates AP, BQ. It is required to find the area of $APQB$. Divide the segment AB into n parts, each equal to

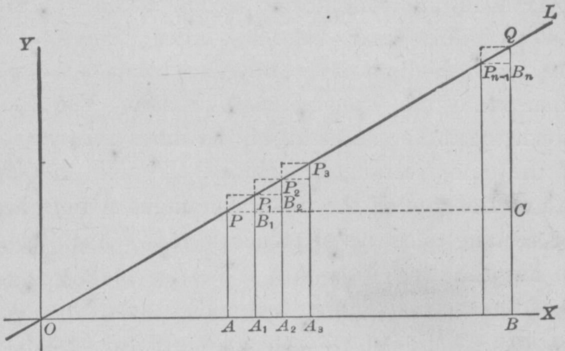

FIG. 1.

Δx; and at the points of section A_1, A_2, $\cdots$, erect ordinates A_1P_1, A_2P_2, $\cdots$, which meet OL in P_1, P_2, $\cdots$. Through P, P_1, P_2, $\cdots$, Q, draw lines parallel to the axis of x and intersecting the nearest ordinate on each side, as shown in Fig. 1, and produce PB_1 to meet BQ in C.

It will first be shown that the area $APQB$ is the limit of the sum of the areas of the rectangles PA_1, P_1A_2, $\cdots$, when n, the number of equal divisions of AB, approaches infinity, or, what is the same thing, when Δx approaches zero. The area $APQB$ is greater than the sum of the "inner" rectangles PA_1, P_1A_2, $\cdots$; and it is less than the sum of the "outer" rectangles AP_1, A_1P_2, $\cdots$. The difference between the sum of the inner rectangles and the sum of the outer rectangles is equal to the sum of the small rectangles PP_1, P_1P_2, $\cdots$.

The latter sum is equal to

$$B_1P_1\,\Delta x + B_2P_2\,\Delta x + \cdots + B_nQ\,\Delta x;$$

that is, to $(B_1P_1 + B_2P_2 + \cdots + B_nQ)\,\Delta x$, which is $CQ\,\Delta x$.

This may be briefly expressed,

$$\Sigma AP_1 - \Sigma PA_1 = \Sigma PP_1$$
$$= CQ\,\Delta x.$$

When Δx is an infinitesimal, the second member of this equation is also an infinitesimal of the first order; therefore, when Δx is infinitely small the limit of the difference between the total areas of the inner and of the outer rectangles is zero. The area $APQB$ lies between the total area of all of the inner and the total area of all of the outer rectangles. Hence, the area $APQB$ is the limit both of the sum of the inner rectangles and of the sum of the outer rectangles as Δx approaches zero. Each elementary rectangle has the area $y\,\Delta x$, that is $mx\,\Delta x$, since $y = mx$. The altitudes of the successive inner rectangles, going from A towards B, are ma, $m(a+\Delta x)$, $m(a+2\,\Delta x)$, $\cdots$, $m(a+(n-1)\,\Delta x)$. Hence,

$$\text{Area } APQB = \text{limit}_{\Delta x \doteq 0} m\{a\,\Delta x + (a+\Delta x)\,\Delta x + (a + 2\,\Delta x)\,\Delta x + \cdots$$
$$+ (a + \overline{n-1}\,\Delta x)\,\Delta x\}\text{*}$$
$$= \text{limit}_{\Delta x \doteq 0} m\{a + (a+\Delta x) + (a + 2\,\Delta x) + \cdots$$
$$+ (a + \overline{n-1}\,\Delta x)\}\,\Delta x.$$

Addition of the arithmetic series in brackets gives

$$\text{Area } APQB = \text{limit}_{\Delta x \doteq 0} \frac{mn\,\Delta x}{2}\{2\,a + (n-1)\,\Delta x\}$$
$$= \text{limit}_{\Delta x \doteq 0} \frac{m(b-a)}{2}\{b + a - \Delta x\}, \text{ since } n\,\Delta x = b - a,$$
$$= m\left(\frac{b^2}{2} - \frac{a^2}{2}\right).$$

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* The symbol $\Delta x \doteq 0$ means "when Δx approaches zero as a limit." It is due to the late Professor Oliver of Cornell University.

In this example the element of area is obtained by taking a rectangle of altitude y, that is, mx, and width Δx, and then letting Δx become infinitesimal.

The expression $n \doteq \infty$ may be used instead of $\Delta x \doteq 0$, since

$$\Delta x = \frac{b-a}{n}.$$

It may be noted in passing, that if the anti-differential of $mx\,dx$, namely $\frac{mx^2}{2}$, be taken, and b and a be substituted in turn for x, the difference between the resulting values will be the expression obtained above.

(*b*) Find the area between the parabola $y = x^2$, the x-axis, and the ordinates for which $x = a$, $x = b$.

Let Q_1OQ be the parabola $y = x^2$; let OA be equal to a, and OB to b. Draw the ordinates AP, BQ. It is required to find the area $APQB$. Divide the segment AB into n parts each equal to

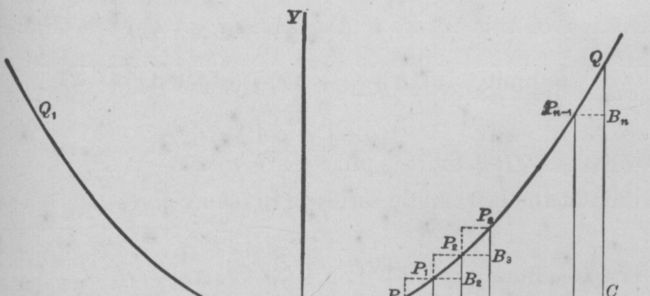

Fig. 2.

Δx, and at the points of division A_1, A_2, $\cdots$, erect ordinates A_1P_1, A_2P_2, $\cdots$. Through P, P_1, P_2, $\cdots$, draw lines parallel to the axis of x and intersecting the nearest ordinates on each side, as in Fig. 2. It can be shown, in the same way as in the previous illustration, that the area $APQB$ is equal to the limit of the sum of